JOURNAL OF ALGEBRA **204,** 116–128 (1998)
ARTICLE NO. JA977361

Fat Points, Inverse Systems, and Piecewise Polynomial Functions

Anthony V. Geramita*

Queens University, Kingston, Ontario, Canada, and Universita' di Genova,

and

Henry K. Schenck[†]

Department of Mathematics, Cornell University, Ithaca, New York 14853
E-mail: schenck@math.cornell.edu

Communicated by D. A. Buchsbaum

Received February 7, 1997

We explore the connection between ideals of fat points (which correspond to subschemes of $\mathbb{P}^n$ obtained by intersecting (mixed) powers of ideals of points), and piecewise polynomial functions (splines) on a d-dimensional simplicial complex Δ embedded in $\mathbf{R}^d$. Using the inverse system approach introduced by Macaulay [11], we give a complete characterization of the free resolutions possible for ideals in $k[x, y]$ generated by powers of homogeneous linear forms (we allow the powers to differ). We show how ideals generated by powers of homogeneous linear forms are related to the question of determining, for some fixed Δ, the dimension of the vector space of splines on Δ of degree less than or equal to k. We use this relationship and the results above to derive a formula which gives the number of planar (mixed) splines in sufficiently high degree. © 1998 Academic Press

1. INTRODUCTION

In [10], Iarrobino observed that there is a relationship between splines and fat points. In this section, we give a quick overview of the relationship between fat points and ideals generated by powers of homogeneous linear

*Partially supported by the Natural Sciences and Engineering Research Council of Canada.
[†]Partially supported by the National Security Agency through Grant MDA 904-95-H-1035.

0021-8693/98 $25.00

forms, and then discuss how ideals of the latter form are related to splines. Good references for the first relationship are Geramita [8], Iarrobino [10], or Macaulay [11]; sources for the latter are Schenck [12], or Schenck and Stillman [13, 14].

Let $P_i = [p_{i0} : p_{i1} : \cdots : p_{in}] \in \mathbb{P}^n$, $I(P_i) = \wp_i \subseteq R = k[x_0, \ldots, x_n]$, and $L_{P_i} = \sum_{j=0}^{n} p_{i_j} y_j$. A fat points ideal is an ideal of the form $I = \cap_{i=1}^{m} \wp_i^{\alpha_i}$, $\alpha_i \geq 1$. Let $S = k[y_0, \ldots, y_n]$, and define an action of R on S by partial differentiation, i.e., $x_j \cdot y_i = \partial(y_i)/\partial(y_j)$. This makes S into a graded R-module. Since I is a submodule of R, it acts on S, and we can ask what elements of S are annihilated by this action. The set of such elements is denoted by I^{-1}. In Section 2, we will see that for $j \gg 0$, $(I^{-1})_j = (L_{P_1}^{j-\alpha_1+1}, \ldots, L_{P_m}^{j-\alpha_m+1})_j$, and that $\dim(I^{-1})_j = \dim(R/I)_j$. In other words, fat points ideals are strongly related to ideals generated by powers of homogeneous linear forms.

Now suppose that Δ is a d-dimensional simplicial complex, embedded in $\mathbf{R}^d$. A fundamental problem in geometric modelling and approximation theory is determining the dimension of the space of piecewise polynomial functions on Δ (imagine a polynomial supported on each maximal simplex), which meet with prescribed order of smoothness across shared $d-1$ faces. In [12], a chain complex of modules on Δ was constructed, such that the module of homogeneous splines appeared as the top homology module. The modules which appear in that chain complex are direct sums of quotients of R by ideals generated by powers of homogeneous linear forms, i.e., the ideals described above.

Thus, there is a strong connection between splines and fat points ideals. The setup of this paper is as follows. In Section 2, we give a more detailed discussion of inverse systems and fat points. For the case where $n = 1$, we use the relationship between fat points ideals and ideals generated by powers of homogeneous linear forms to completely describe possible Hilbert functions for the latter ideals. This, in turn, allows us to actually write the free resolutions which are possible for such ideals. In Section 3 we review the spline problem which was sketched above. In Section 4, we combine the results of the previous two sections to derive a formula for the number of planar mixed splines in high degree.

2. FAT POINTS AND INVERSE SYSTEMS

Let $\mathbb{X} = \{P_1, \ldots, P_s\} \subseteq \mathbb{P}^n$ be a set of distinct points in projective n-space and let $I = I_{\mathbb{X}} = \wp_1 \cap \cdots \cap \wp_s$, where $\wp_i = I(P_i) \subseteq R = k[x_0, \ldots, x_n]$.

Let $S = k[y_0, \ldots, y_n]$. We think of S both as a ring, isomorphic to R, and as an R-module where the action $R_i \times S_j \to S_{j-i}$ is that given either by partial differentiation or contraction.

Let P_i and L_{P_i} be as defined in Section 1, and $I^{-1} = \{f \in S \mid I \cdot f = 0\}$. Then I^{-1} may be described as follows:

PROPOSITION 2.1.

$$(I^{-1})_t = \langle L_{P_1}^t, \ldots, L_{P_s}^t \rangle \subseteq S_t$$

and

$$\dim_k (I^{-1})_t = H(R/I, t) = \dim_k \frac{R_t}{I_t}.$$

Proof. The proof follows from the following more general theorem, given below. ∎

THEOREM 2.2 (Ensalem and Iarrobino [6]). *Keep the notation as above, but let I be an ideal of fat points, i.e.,* $I = \wp_1^{n_1+1} \cap \cdots \cap \wp_s^{n_s+1}$. *Then we have*

$$(I^{-1})_j = \left\{ \begin{array}{ll} S_j & \textit{for } j \leq \max\{n_i\} \\ L_{P_1}^{j-n_1} S_{n_1} + \cdots + L_{P_s}^{j-n_s} S_{n_s} & \textit{for } j \geq \max\{n_i + 1\} \end{array} \right\}$$

and

$$\dim_k (I^{-1})_j = \dim_k (R/I, j) = H(R/I, j).$$

Proof. See [8, p. 22]. ∎

Of course, the vector space

$$L_{P_1}^{j-n_1} S_{n_1} + \cdots + L_{P_s}^{j-n_s} S_{n_s} \subseteq S_j$$

is nothing more than the jth graded piece of the ideal in S generated by

$$L_{P_1}^{j-n_1}, \ldots, L_{P_s}^{j-n_s}.$$

If we fix $n_1, \ldots, n_s$ and let j vary, we obtain, from an ideal of fat points, information about the size of pieces of infinitely many different ideals generated by powers of s homogeneous linear forms.

On the other hand, if we let j and the n_i vary in such a way that the numbers $j - n_i = t_i$ are fixed for each i, then we obtain information about the sizes of the various graded pieces of the ideal generated by $L_1^{t_1}, \ldots, L_s^{t_s}$ in an infinite family of ideals of fat points. For more details on this correspondence, see the discussion in Section 3 of [8].

COROLLARY 2.3. *Let $L_1, \dots, L_s$ be any s pairwise linearly independent homogeneous linear forms in $S = k[y_0, y_1]$, $0 < \alpha_1 \leq \cdots \leq \alpha_s$ be integers, and let $J = (L_1^{\alpha_1}, \dots, L_s^{\alpha_s})$. Then, for each integer t, the vector space J_t has the maximum dimension possible, i.e.,*

$$\dim_k J_t = \min\left\{ t + 1, \sum_{i=1}^{s} \max\{t - \alpha_i + 1, 0\} \right\}.$$

Proof. By Theorem 2.2, given an integer $t \geq 0$,

$$\dim_k J_t = \dim_k (R/I, t),$$

where

$$I = \wp_1^{t - \alpha_1 + 1} \cap \cdots \cap \wp_s^{t - \alpha_s + 1}$$

and $\wp_1, \dots, \wp_s$ are the ideals of the points corresponding to $L_1, \dots, L_s$ (here we use the convention that $\wp^r = R$ if $r \leq 0$). Now I is a principal ideal generated by a form F of degree d_t, where

$$d_t = \sum_{i=1}^{s} \max\{t - \alpha_i + 1, 0\}.$$

Hence,

$$\dim_k J_t = H(R/I, t) = \min(t + 1, d_t),$$

as we wanted to show. ∎

This last result is surprising in that it does not depend on the homogeneous linear forms chosen (save only that they be pairwise linearly independent). A small piece of this result, namely that the vector space $\langle L_1^t, \dots, L_s^t \rangle$ has dimension $\min\{t + 1, s\}$ (for pairwise linearly independent L_i in $k[y_0, y_1]$) can easily be deduced using properties of the van der Monde matrix. In the pure power case (where the α_i are all equal), this result specializes to a result which appears in [14].

The corollary above is also useful for determining a minimal set of generators for an ideal generated by powers of homogeneous linear forms in $k[y_0, y_1]$.

EXAMPLE 2.4. Let $L_1, \dots, L_5$ be pairwise linearly independent homogeneous linear forms in $k[y_0, y_1]$ and let $J = (L_1^4, L_2^6, L_3^7, L_4^7, L_5^9)$. From Corollary 2.3 we see that if $J' = (L_1^4, L_2^6, L_3^7, L_4^7)$ then $\dim_k (J')_9 = \min\{10, (9 - 4 + 1) + (9 - 6 + 1) + (9 - 7 + 1) + (9 - 7 + 1)\} = \min\{10, 14\} = 10$, and so $L_5^9 \in J'$ and thus $J = J'$. Using the same reasoning, we can show that $L_2^6 \notin (L_1^4)$, $L_3^7 \notin (L_1^4, L_2^6)$, and $L_4^7 \notin (L_1^4, L_2^6, L_3^7)$.

So the generators of J' are a minimal set of generators. We formalize this procedure as follows:

COROLLARY 2.5. *Let* $0 < \alpha_1 \leq \alpha_2 \cdots \leq \alpha_t$ *and let* $J = (L_1^{\alpha_1}, \ldots, L_t^{\alpha_t})$. *Then for* $m \geq 2$,

$$L_{m+1}^{\alpha_{m+1}} \notin \left(L_1^{\alpha_1}, \ldots, L_m^{\alpha_m}\right) \Leftrightarrow \alpha_{m+1} \leq \frac{\sum_{i=1}^{m} \alpha_i - m}{m-1}.$$

Proof. Let $J_m = (L_1^{\alpha_1}, \ldots, L_m^{\alpha_m})$. Then $L_{m+1}^{\alpha_{m+1}} \notin J_m$ if and only if $(J_m)_{\alpha_{m+1}} \neq (J_{m+1})_{\alpha_{m+1}}$. By Corollary 2.3,

$$\dim_k (J_m)_{\alpha_{m+1}} = \min\left\{\alpha_{m+1} + 1, \sum_{i=1}^{m} (\alpha_{m+1} - \alpha_i + 1)\right\},$$

$$\dim_k (J_{m+1})_{\alpha_{m+1}} = \min\left\{\alpha_{m+1} + 1, \sum_{i=1}^{m+1} (\alpha_{m+1} - \alpha_i + 1)\right\}.$$

Hence, $(J_m)_{\alpha_{m+1}} \neq (J_{m+1})_{\alpha_{m+1}}$ if and only if

$$\alpha_{m+1} + 1 > \sum_{i=1}^{m} (\alpha_{m+1} - \alpha_i + 1),$$

which simplifies to the above condition. ∎

Henceforth, when we write $J = (L_1^{\alpha_1}, \ldots, L_t^{\alpha_t})$, we shall assume that the *exponent vector* $\alpha = (\alpha_1, \ldots, \alpha_t)$ *of* J satisfies the conditions of the previously corollary. In other words, the homogeneous linear forms are a minimal generating set for J, i.e., for each integer $m \in 2 \ldots t-1$, $\alpha_{m+1} \leq (\sum_{i=1}^{m} \alpha_i - m)/(m-1)$ (since we are assuming that the L_i are pairwise linearly independent, and that J is not principal, we will always need the first two forms). By Corollary 2.3, we also have the following:

THEOREM 2.6. Let $J = (L_1^{\alpha_1}, \ldots, L_t^{\alpha_t})$ where the exponent vector of J corresponds to a minimal generating set of J, and d_i is as given in Corollary 2.3. Then

$$H(S/J, i) = \max\{0, i + 1 - d_i\}.$$

From this theorem we can easily calculate the socle degree of S/J. In fact, the least integer Ω for which $H(S/J, \Omega) = 0$ is the least integer p such that $p + 1 - d_p \leq 0$; equivalently $p < \sum_{i=1}^{t} \max\{p - \alpha_i + 1, 0\}$. Thus, $d_{\Omega-1} \leq \Omega - 1$ and $\Omega < d_\Omega$; the socle degree of S/J is $\Omega - 1$. Since all the minimal generators of J occur in degree at most one greater than the socle degree of S/J, we obtain that $\Omega \geq \alpha_i$ for all i.

It is easy to give a direct formula for Ω in terms of the α_i, i.e.,

$$\Omega = \left\lfloor \frac{\sum_{i=1}^{t} \alpha_i - t}{t-1} \right\rfloor + 1.$$

With this information on the Hilbert function we can now write a minimal free resolution for the ideal $J \subseteq k[x, y]$.

THEOREM 2.7. *Let J be an ideal minimally generated by $(L_1^{\alpha_1}, \dots, L_t^{\alpha_t})$, so that $\Omega - 1$ is the socle degree of S/J. Then J has resolution*

$$0 \to S(-\Omega - 1)^a \oplus S(-\Omega)^{t-1-a} \to \oplus_{i=1}^{t} S(-\alpha_i) \to J \to 0,$$

where

$$a = H(S/J, \Omega - 1) = \sum_{i=1}^{t} \alpha_i + (1-t) \cdot \Omega.$$

Proof. By the Hilbert Syzygy Theorem (see [5]), J has projective dimension one (we are assuming that J is not principal), so has a resolution

$$0 \to F_1 \to F_0 \to J \to 0.$$

We have already described the minimal generators of J, i.e., we know $F_0 = \oplus_{i=1}^{t} S(-\alpha_i)$, so we need only show that F_1 is as stated.

In Corollary 2.3 we saw that J grows as "quickly as possible." Hence, there are no syzygies on J before degree Ω. Now, since the socle degree of S/J is $\Omega - 1$ and J is of codimension two, the highest shift in the minimal resolution of S/J is $\Omega - 1 + 2 = \Omega + 1$ and it appears with multiplicity $H(S/J, \Omega - 1) = a$. Applying Theorem 2.6 to compute a, and then comparing the ranks of F_0, F_1, and J, we obtain the desired resolution. ∎

Since J_t has the maximum dimension possible, the Hilbert Series of S/J is the expected Hilbert Series for an ideal generated by generic binary forms of degree $\alpha_1, \dots, \alpha_s$. This is the series

$$\left| \prod_i (1 - z^{\alpha_i})/(1-z)^2 \right|,$$

where if $\sum c_i z^i$ is a series with integer coefficients we let $|\sum c_i z^i| = \sum d_i z^i$ with $d_i = c_i$ if $c_0, \dots, c_i > 0$ and $d_i = 0$ if $c_j \leq 0$ for some $j \leq i$.

In [7], Fröberg made a conjecture about the expected Hilbert series of an ideal generated by a generic set of forms in $k[x_1, \dots, x_n]$, and proved the conjecture for $n = 2$. Theorem 2.6 gives another proof for the case

$n = 2$, which is somewhat stronger since it shows that, in this case, we can choose powers of linear forms as the generic forms. (We are grateful to the referee for this remark.)

EXAMPLE 2.8. We continue with the previous example, in which J was minimally generated by $(L_1^4, L_2^6, L_3^7, L_4^7)$, and $\Omega = \lfloor(4+6+7+7-4)/3\rfloor + 1 = 7$. By Theorem 2.6, $a = 3$ so $t - 1 - a = 0$, hence the resolution is

$$0 \to S(-8)^3 \to S(-4) \oplus S(-6) \oplus S(-7)^2 \to J \to 0.$$

Notice that $t - 1 - a$ is zero; this occurs if and only if $t - 1$ divides $\sum \alpha_i$.

3. SPLINES ON A SIMPLICIAL COMPLEX

Let Δ be a d-dimensional simplicial complex embedded in $\mathbf{R}^d$, such that Δ and all its links are pseudomanifolds (basically, one should visualize Δ as triangulating a manifold). A *spline* on Δ is a piecewise polynomial function (imagine a polynomial supported on each maximal simplex), such that two polynomials (f, g) supported on d-simplices which share a common $d-1$ face τ meet with some desired order of smoothness along that face. If we let L_τ denote the linear form vanishing on τ, then the algebraic formulation of C^r smoothness across the face τ is that L_τ^{r+1} divides $f - g$.

In [2], Billera introduced the use of homological algebra in the study of splines; other references are [3, 4, 12–14]. In this paper, we consider *mixed splines*, which are splines where the order of smoothness may differ on the various $d-1$ faces. Let Δ_i^0 be the set of interior i faces of Δ (all d-dimensional faces are considered interior), $f_i^0 = |\Delta_i^0|$, and let α be a vector of length f_{d-1}^0, where α_i denotes the desired order of smoothness across the ith interior $d-1$ face. The set of splines of degree at most k (i.e., each individual polynomial is of degree at most k) is a vector space, which we will denote $C_k^\alpha(\Delta)$.

It turns out that a good way to study the dimension of this vector space is to embed Δ in the hyperplane $x_{d+1} = 1 \subseteq \mathbf{R}^{d+1}$, and form the cone $\hat{\Delta}$ over Δ, with vertex at the origin. If we let $C_k^\alpha(\hat{\Delta})$ be the set of splines on $\hat{\Delta}$ of degree *exactly* k, then there is a vector space isomorphism between $C_k^\alpha(\hat{\Delta})$ and $C_k^r(\Delta)$. If we form the abelian group $\oplus_{k \geq 0} C_k^\alpha(\hat{\Delta})$, then this can be viewed as a graded module, denoted $C^\alpha(\hat{\Delta})$, over the polynomial ring R in $d+1$ variables, so the dimension of $C_k^\alpha(\Delta)$ is the dimension of $C^\alpha(\hat{\Delta})$ in degree exactly k.

All this information can be encoded by defining a chain complex of modules, and computing the homology modules. First, for τ an interior

$d-1$ face of Δ, let l_τ denote the homogeneous linear form vanishing on $\hat{\tau}$ (this is just the homogenization of L_τ). For any interior face γ, define $\mathscr{I}(\gamma) = \sum_{\gamma \subseteq \tau_i \in \Delta^0_{d-1}} l_{\tau_i}^{\alpha_i+1}$; $\mathscr{I}(\gamma)$ is the ideal generated by the (mixed) powers of homogeneous linear forms which define hyperplanes incident to $\hat{\gamma}$. Let ∂_i be the relative (modulo $\partial\Delta$) simplicial boundary map, and let $\mathscr{R}$ be the chain complex defined by $\mathscr{R}_i = R^{f_i^0}$. So the homology of the $\mathscr{R}$ is just relative (modulo $\partial\Delta$) simplicial homology, with coefficients in R. Notice that ∂_i also gives us a differential on the quotient of $\mathscr{R}$ by $\mathscr{I}$, in particular, we have a chain complex $\mathscr{R}/\mathscr{I}$:

$$\cdots \to \bigoplus_{\alpha \in \Delta^0_{i+1}} \mathscr{R}/\mathscr{I}(\alpha) \xrightarrow{\partial_{i+1}} \bigoplus_{\beta \in \Delta^0_{i}} \mathscr{R}/\mathscr{I}(\beta) \xrightarrow{\partial_{i}} \bigoplus_{\gamma \in \Delta^0_{i-1}} \mathscr{R}/\mathscr{I}(\gamma) \xrightarrow{\partial_{i-1}} \cdots.$$

In [12] (where the α_i are all equal), it is shown that the top homology module of this complex is precisely the module $C^\alpha(\hat{\Delta})$. This is also easy to verify for the case where α is mixed. If we can understand the modules in the complex and the lower homology modules of this complex, then the Euler characteristic equation will allow us to understand $C^\alpha(\hat{\Delta})$.

The connection between fat points and piecewise polynomial functions is now completely clear, because understanding the modules in the chain complex is equivalent to understanding ideals of the form considered in Section 1. In particular, for cases where the lower homology modules vanish, if we understand modules of the form $\mathscr{R}/\mathscr{I}(\tau)$, then we will be able to give a complete answer to the question of describing the dimension of $C^\alpha(\hat{\Delta})_k$.

4. THE PLANAR CASE

In [12], which considered the pure power case, localization techniques were used to prove that for all $i < d$, $H_i(\mathscr{R}/\mathscr{I})$ has dimension at most $i-1$ as an $\mathscr{R} = \mathbf{R}[x_1, \dots, x_{d+1}]$ module. It is easy to check that these techniques also work for the case where α is mixed.

We now specialize to the case where Δ is embedded in $\mathbf{R}^2$, i.e., Δ is a planar simplicial complex (this is perhaps the most studied case in spline theory). The aforementioned result implies that $H_1(\mathscr{R}/\mathscr{I})$ is a zero-dimensional $\mathscr{R} = \mathbf{R}[x, y, z]$ module (i.e., is a finite dimensional graded vector space), so vanishes in sufficiently high degree.

THEOREM 4.1. *$H_1(\mathscr{R}/\mathscr{I})$ is an $\mathscr{R}$-module of finite length.*

Proof. See [12] and the remarks above. ∎

There is a short exact sequence of complexes

$$0 \to \mathscr{I} \to \mathscr{R} \to \mathscr{R}/\mathscr{I} \to 0,$$

which gives rise to a long exact sequence of homology modules. Modulo the image of ∂_1, every vertex is equivalent to a vertex on the boundary, and thus $H_0(\mathscr{R})$ vanishes, which also forces $H_0(\mathscr{R}/\mathscr{I})$ to vanish, by the long exact sequence in homology.

THEOREM 4.2. *With notation as above, if* $k \gg 0$, *then*

$$\dim_{\mathbf{R}} C^{\alpha}(\hat{\Delta})_k = \dim_{\mathbf{R}} \sum_{i=0}^{2} (-1)^i \bigoplus_{\beta \in \Delta^0_{2-i}} \mathscr{R}/\mathscr{I}(\beta)_k.$$

Proof. The Euler characteristic equation [15, p. 172]

$$\chi(H(\mathscr{R}/\mathscr{I})) = \chi(\mathscr{R}/\mathscr{I})$$

implies that

$$\dim_{\mathbf{R}} H_2(\mathscr{R}/\mathscr{I})_k = \dim_{\mathbf{R}} \sum_{i=0}^{2} (-1)^i \bigoplus_{\beta \in \Delta^0_{2-i}} \mathscr{R}/\mathscr{I}(\beta)_k$$
$$+ \dim_{\mathbf{R}} \sum_{i=0}^{1} (-1)^i H_{1-i}(\mathscr{R}/\mathscr{I})_k.$$

Since $H_2(\mathscr{R}/\mathscr{I})$ is the homogeneous spline module $C^{\alpha}(\hat{\Delta})$, the above formula, coupled with Theorem 4.1 and the fact that $H_0(\mathscr{R}/\mathscr{I}) = 0$ concludes the proof. ∎

Since we know

$$\dim_{\mathbf{R}} \bigoplus_{\sigma \in \Delta^0_2} \mathscr{R}_k = f^0_2 \cdot \binom{k+2}{2}$$

and

$$\dim_{\mathbf{R}} \bigoplus_{\tau \in \Delta^0_1} \mathscr{R}/\mathscr{I}(\tau)_k = \sum_{i=1}^{f^0_1} \left[\binom{k+2}{2} - \binom{k+2-\alpha_i-1}{2} \right],$$

we need only determine

$$\dim_{\mathbf{R}} \bigoplus_{\gamma \in \Delta^0_0} \mathscr{R}/\mathscr{I}(\gamma)_k.$$

Observe that for any vertex γ_i, we may translate γ_i to the origin, hence may assume that the linear forms in $\mathscr{I}(\gamma_i)$ involve *only* the variables x, y. Thus,

$$\mathscr{R}/\mathscr{I}(\gamma_i) \simeq \mathbf{R}[z] \otimes_{\mathbf{R}} \mathbf{R}[x, y]/\mathscr{I}(\gamma_i).$$

Let $\beta_j = \alpha_j + 1$, and suppose $L_1^{\beta_1}, \dots, L_{t_i}^{\beta_{t_i}}$ is a minimal generating set for $\mathscr{I}(\gamma_i)$, with $\beta^i = (\beta_1, \dots, \beta_{t_i})$ the corresponding exponent vector for $\mathscr{I}(\gamma_i)$. If we rewrite the resolution given in Theorem 2.7, letting Ω_i and a_i denote the values for Ω and a at γ_i, and defining $b_i = t_i - 1 - a_i$, then a free resolution for $\mathscr{R}/\mathscr{I}(\gamma_i)$ is given by

$$0 \to \mathscr{R}(-\Omega_i - 1)^{a_i} \oplus \mathscr{R}(-\Omega_i)^{b_i} \to \oplus_{j=1}^{t_i} \mathscr{R}(-\beta_j) \to \mathscr{R} \to \mathscr{R}/\mathscr{I}(\gamma_i) \to 0.$$

From the additivity of the Hilbert polynomial, we obtain that

$$\dim_{\mathbf{R}} \bigoplus_{\gamma_i \in \Delta_0^0} \mathscr{R}/\mathscr{I}(\gamma_i)_k$$

is equal to

$$\sum_{i=1}^{f_0^0} \left[\binom{k+2}{2} - \sum_{\beta_j \in \beta^i} \binom{k+2-\beta_j}{2} + b_i \cdot \binom{k+2-\Omega_i}{2} \right.$$
$$\left. + a_i \cdot \binom{k+2-\Omega_i-1}{2} \right].$$

THEOREM 4.3. *Let Δ be a planar simplicial complex, satisfying the conditions of Section 3. Then for $k \gg 0$, $\dim_{\mathbf{R}} C^{\alpha}(\hat{\Delta})_k$ is given by*

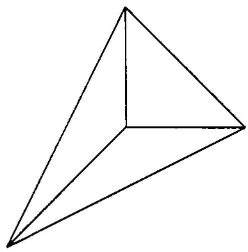
$$\left(f_2^0 - f_1^0 + f_0^0\right) \cdot \binom{k+2}{2} + \sum_{i=1}^{f_1^0} \binom{k+2-\alpha_i-1}{2}$$
$$- \sum_{i=1}^{f_0^0} \left[\sum_{\beta_j \in \beta^i} \binom{k+2-\beta_j}{2} - b_i \right.$$
$$\left. \cdot \binom{k+2-\Omega_i}{2} - \alpha_i \cdot \binom{k+2-\Omega_i-1}{2} \right],$$

where β^i is a minimal generating set for $\mathscr{I}(\gamma_i)$.

Proof. We have determined the Hilbert polynomial for each module in the chain complex $\mathscr{R}/\mathscr{I}$, so we may apply Theorem 4.2. ∎

Notice that Δ is a topological disk (i.e., $H_1(\mathscr{R}) = 0$), then we can simplify the term $(f_2^0 - f_1^0 + f_0^0) \cdot \binom{k+2}{2}$ to $\binom{k+2}{2}$, by Euler's equation. Also, it is worth mentioning that in the case where the α_i are all equal, this is the same bound given by Alfred and Schumaker [1].

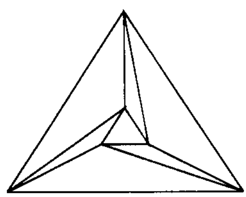
EXAMPLE 4.4. Let Δ be the planar simplicial complex given below.



Suppose the vertex locations are $(0,0)$, $(1,0)$, $(0,1)$, $(-1,-1)$, and that $\alpha = (1,2,3)$. There is only one interior vertex γ (the origin); $\mathscr{I}(\gamma) = (x^2, y^3, (x-y)^4)$, and is minimally generated by (x^2, y^3). $\Omega = 4$, and $a = 1$, $b = 0$. By Theorem 4.3, for $k \gg 0$

$$\dim_{\mathbf{R}} C^{\alpha}(\hat{\Delta})_k = \binom{k+2}{2} + \binom{k-2}{2} + \binom{k-3}{2}.$$

EXAMPLE 4.5. We conclude with a more complicated example.



Suppose $\alpha_i = 2$ on the three edges of the center triangle, $\alpha_i = 3$ on the six edges which connect the interior vertices to boundary vertices. For each of the three interior vertices, $\mathscr{I}(\gamma)$ is of the form $(l_1^3, l_2^3, l_3^4, l_4^4)$, where the l_i are pairwise linearly independent homogeneous linear forms. By Corollary 2.5, $\mathscr{I}(\gamma)$ is minimally generated by (l_1^3, l_2^3, l_3^4), so $\Omega = 4$, and $a = 2$,

$b = 0$. Thus, by Theorem 4.3,

$$\dim_{\mathbf{R}} C^{\alpha}(\hat{\Delta})_k = \binom{k+2}{2} - 3 \cdot \binom{k-1}{2} + 3 \cdot \binom{k-2}{2} + 6 \cdot \binom{k-3}{2}, \quad k \gg 0.$$

It is possible to calculate $C^{\alpha}(\hat{\Delta})$ as the kernel of a certain matrix (see Billera and Rose [3]). It is worth doing this, to see that the above examples are indeed correct. Care must be exercised when considering for which values $k \gg 0$ the theorem holds. We know by Theorem 4.1 that $H_1(\mathscr{R}/\mathscr{I})$ vanishes in high degree, but this degree has not been specified.

For the three dimensional version of this problem, there are two important aspects to consider. First, we will want to analyze ideals generated by powers of linear forms in *three* variables; this case is much harder than the case of two variables, since I will never be principal (which is what made the planar case so nice). Second, although the module $H_1(\mathscr{R}/\mathscr{I})$ will still vanish in high degree, this will not be the case for the module $H_2(\mathscr{R}/\mathscr{I})$; in fact, the latter module is often one-dimensional (recall that for $d = 3$, the spline module is $H_3(\mathscr{R}/\mathscr{I})$). So it will also be necessary to analyze the behavior of $H_2(\mathscr{R}/\mathscr{I})$.

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