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Note

Feasible reductions to Kolmogorov–Loveland stochastic sequences[☆]

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Abstract

For every binary sequence A , there is an infinite binary sequence S such that $A \leq_{tt}^p S$ and S is stochastic in the sense of Kolmogorov and Loveland. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the mid-1960s, Martin-Löf [23] used the general theory of algorithms to formulate the first successful definition of the randomness of individual binary sequences. Subsequent definitions, using a variety of conceptual approaches, were introduced by Levin [17], Schnorr [24, 25], Chaitin [6–8], Solovay [28], and Shen' [26]. Each of these definitions was shown to be equivalent to Martin-Löf's, in the sense that a binary sequence R is algorithmically random according to the given definition if and only if R is algorithmically random according to Martin-Löf's definition.

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In the present note, all “sequences” are infinite binary sequences, and the term “random” means “algorithmically random in the sense of Martin-Löf”. A precise definition of algorithmic randomness appears in Section 2.

One of the most useful and intuitive properties of random sequences is their stochasticity. This is the fact that if a subsequence A of a random sequence R is chosen according to an “admissible selection rule”, then the limiting frequency of 1’s in the subsequence A is exactly $\frac{1}{2}$. The broadest class of admissible selection rules that has been studied in this context is the class of *Kolmogorov–Loveland selection rules* [13, 14, 19, 20]. These algorithmic rules (which are described in Section 2) are more general than earlier selection rules proposed by von Mises [31], Wald [32], and Church [9] in two respects. First, given a sequence S , a Kolmogorov–Loveland selection rule may choose bits from S in whatever order arises from the rule’s interaction with S ; this order need not agree with the order of appearance of these bits in S . Second, a Kolmogorov–Loveland selection rule is a partial recursive rule that may succeed in choosing a sequence of distinct bits from one sequence, yet fail to choose such a sequence from another.

It is easy to see that every random sequence R is *Kolmogorov–Loveland stochastic*. This means that, for every sequence A of distinct bits of R that is chosen according to a Kolmogorov–Loveland selection rule, the limiting frequency of 1’s in A is $\frac{1}{2}$. In the late 1980s, Shen’ [27] proved that the converse does not hold, thereby solving a problem that had been open for some 20 years. (See [15, 29, 18] for more detailed histories of this problem and the role of stochasticity in the foundations of probability theory.) Thus, the random sequences form a proper subset of the set of all Kolmogorov–Loveland stochastic sequences.

This note refines the method of Shen’ [27] in order to establish a stronger, more quantitative separation between randomness and Komogorov–Loveland stochasticity.

Kučera [16] and Gács [10] have proven that for every sequence A there is a random sequence R such that A is Turing reducible to R . However, it is well known that this does not hold for truth-table reducibility (Turing reducibility with computable running time). In fact, Juedes et al. [12] have noted that, in the sense of Baire category, *almost every* sequence A has the property that A is not reducible to *any* random sequence in *any* computable running time.

In contrast with this fact, the main theorem of the present note (Theorem 3.5) states that, for *every* sequence A , there is a Kolmogorov–Loveland stochastic sequence S such that A is *feasibly* reducible to S . In fact, A can be reduced to S by a polynomial-time truth-table reduction. The proof of this result uses a relativization of a method of van Lambalgen [30] and Shen’ [27], together with a simple encoding of the sequence A into a “nearly uniform” probability measure on the set of all sequences.

It follows immediately from the main theorem that there are sequences S that are Kolmogorov–Loveland stochastic, but also strongly deep in the sense of Bennett [1]. Such sequences S are computationally “very far from random” [1, 12].

The main theorem also implies that the class RAND of all random oracles cannot be replaced by the class KL-STOCH of all Kolmogorov–Loveland stochastic oracles in

some known characterizations of complexity classes. As just one example, a “folklore” result states that $P(\text{RAND}) \cap \text{REC} = \text{BPP}$, that is, that a recursive language is \leq_T^P -reducible to some random language if and only if it is probabilistically decidable with bounded error in polynomial time [1, 3]. In contrast, the main theorem immediately implies that $P(\text{KL-STOCH})$ contains every language. See [22, 2–4] for other known characterizations using random oracles that, by the main theorem, cannot be extended to Kolmogorov–Loveland stochastic oracles.

2. Notation and preliminaries

We write $\{0, 1\}^*$ for the set of all (finite, binary) strings, and we write $|x|$ for the length of a string x . The empty string, λ , is the unique string of length 0. The standard enumeration of $\{0, 1\}^*$ is the sequence $s_0 = \lambda$, $s_1 = 0$, $s_2 = 1$, $s_3 = 00, \dots$, ordered first by length and then lexicographically.

The complement of a language A is $A^c = \{0, 1\}^* - A$, and the symmetric difference of languages A and B is $A \triangle B = (A - B) \cup (B - A)$.

The Boolean value of a condition ϕ is $\llbracket \phi \rrbracket = \text{if } \phi \text{ then } 1 \text{ else } 0$.

We work in the Cantor space \mathbf{C} , consisting of all languages $A \subseteq \{0, 1\}^*$. We identify each language A with its characteristic sequence, which is the (infinite, binary) sequence A whose n th bit is $\llbracket s_n \in A \rrbracket$ for each $n \in \mathbb{N}$. (The leftmost bit of A is the 0th bit.) Relying on this identification, we also consider \mathbf{C} to be the set of all sequences.

A string w is a prefix of a sequence A , and we write $w \sqsubseteq A$, if there is a sequence B such that $A = wB$. For each string $w \in \{0, 1\}^*$, the cylinder generated by w is the set

$$\mathbf{C}_w = \{A \in \mathbf{C} \mid w \sqsubseteq A\}.$$

Note that $\mathbf{C}_\lambda = \mathbf{C}$.

Let D be a discrete domain such as \mathbb{N} , $\{0, 1\}^*$, or $\mathbb{N} \times \{0, 1\}^*$. A function $f: D \rightarrow \mathbb{R}$ is computable if there is a total recursive function $\hat{f}: \mathbb{N} \times D \rightarrow \mathbb{Q}$ such that, for all $r \in \mathbb{N}$ and $x \in D$, $|\hat{f}(r, x) - f(x)| \leq 2^{-r}$. A function $f: D \rightarrow \mathbb{R}$ is lower semicomputable if there is a total recursive function $\hat{f}: \mathbb{N} \times D \rightarrow \mathbb{Q}$ such that (i) for all $r \in \mathbb{N}$ and $x \in D$, $\hat{f}(r, x) \leq \hat{f}(r + 1, x)$, and (ii) for all $x \in D$, $\lim_{r \rightarrow \infty} \hat{f}(r, x) = f(x)$. A sequence $(\beta_0, \beta_1, \dots)$ of real numbers converges computably to a limit $\beta \in \mathbb{R}$ if there is a total recursive function $m: \mathbb{N} \rightarrow \mathbb{N}$, called a modulus of convergence, such that, for all $r \in \mathbb{N}$ and $i \geq m(r)$, $|\beta_i - \beta| \leq 2^{-r}$. Similarly, a series $\sum_{n=0}^\infty \alpha_n$ of nonnegative reals α_n is computably convergent if there is a total recursive function (modulus of convergence) $m: \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $r \in \mathbb{N}$, $\sum_{n=m(r)}^\infty \alpha_n \leq 2^{-r}$.

A bias sequence is a sequence $\vec{\beta} = (\beta_0, \beta_1, \dots)$ of real numbers (biases) $\beta_i \in [0, 1]$. A bias sequence $\vec{\beta}$ determines the coin-toss probability measure $\mu^{\vec{\beta}}$ on Cantor space, which corresponds to a random experiment in which a language $A \in \mathbf{C}$ is chosen probabilistically as follows. For each string s_i , we toss a special coin whose probability is β_i of coming up heads, in which case $s_i \in A$, and $1 - \beta_i$ of coming up tails, in which

case $s_i \notin A$. The coin tosses are independent of one another. In the special case where $\vec{\beta} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, $\mu^{\vec{\beta}}$ is the uniform probability measure on \mathbf{C} .

As noted in the introduction, there are several equivalent definitions of algorithmic randomness. The definition in terms of martingales, introduced by Schnorr [24], is most convenient for our purposes here. Given a bias sequence $\vec{\beta}$, a $\vec{\beta}$ -martingale is a function $d : \{0, 1\}^* \rightarrow [0, \infty)$ such that for all $w \in \{0, 1\}^*$,

$$d(w) = (1 - \beta_{|w|})d(w0) + \beta_{|w|}d(w1).$$

(The reader is referred to [30, 5] for discussion of this definition and its motivation.) The success set of a $\vec{\beta}$ -martingale d is

$$S^\infty[d] = \{A \in \mathbf{C} \mid (\forall k)(\exists w \sqsubseteq A) d(w) \geq k\}.$$

The unitary success set of a $\vec{\beta}$ -martingale d is

$$S^1[d] = \bigcup_{d(w) \geq 1} C_w.$$

A sequence $R \in \mathbf{C}$ is (algorithmically) $\vec{\beta}$ -random, and we write $R \in \text{RAND}_{\vec{\beta}}$, if there is no lower semicomputable $\vec{\beta}$ -martingale d such that $R \in S^\infty[d]$. A sequence $R \in \mathbf{C}$ is rec- $\vec{\beta}$ -random, and we write $R \in \text{RAND}_{\vec{\beta}}(\text{rec})$, if there is no computable $\vec{\beta}$ -martingale d such that $R \in S^\infty[d]$. It is immediate from the definitions that $\text{RAND}_{\vec{\beta}} \subseteq \text{RAND}_{\vec{\beta}}(\text{rec})$. When the probability measure is uniform, that is, $\vec{\beta} = (\frac{1}{2}, \frac{1}{2}, \dots)$, we omit $\vec{\beta}$ from the notation. Thus RAND is the set of all random sequences, and $\text{RAND}(\text{rec})$ is the set of all rec-random sequences.

It is a straightforward matter to relativize the computability or lower semicomputability of a martingale d to an arbitrary oracle $A \in \mathbf{C}$ and thereby to define the class $\text{RAND}_{\vec{\beta}}^A$, consisting of all sequences that are $\vec{\beta}$ -random relative to A , and the class $\text{RAND}_{\vec{\beta}}^A(\text{rec})$, consisting of all sequences that are rec- $\vec{\beta}$ -random relative to A .

The following property of rec- $\vec{\beta}$ -random sequences (relative to an oracle A) is an easy extension of a special case of the resource-bounded Borel–Cantelli lemma of [21].

Lemma 2.1. *Let $A \in \mathbf{C}$ and let $\vec{\beta}$ be a bias sequence that is computable relative to A . Let d_0, d_1, d_2, \dots be a sequence of $\vec{\beta}$ -martingales with the following two properties:*

- (i) *The function $(n, w) \mapsto d_n(w)$ is computable relative to A .*
- (ii) *The series $\sum_{n=0}^\infty d_n(\lambda)$ is computably convergent relative to A .*

If $R \in \text{RAND}_{\vec{\beta}}^A(\text{rec})$, then there are only finitely many n for which $R \in S^1[d_n]$.

Proof. Assume the hypothesis, and let $J = \{n \mid R \in S^1[d_n]\}$. Let $m : \mathbb{N} \rightarrow \mathbb{N}$ be a modulus for the convergence of $\sum_{n=0}^\infty d_n(\lambda)$ that is total recursive relative to A , and define the function $d : \{0, 1\}^* \rightarrow [0, \infty)$ by

$$d(w) = \sum_{r=0}^\infty 2^r \sum_{n=m(2r)}^\infty d_n(w).$$

It is easily checked that d is a $\vec{\beta}$ -martingale that is computable relative to A . Since $R \in \text{RAND}_{\vec{\beta}}^A(\text{rec})$, it follows that there is a constant $c \in \mathbb{N}$ such that, for all $w \sqsubseteq R$, $d(w) < 2^c$.

Now, let $n_0 \in J$. Fix a prefix $w \sqsubseteq R$ such that $d_{n_0}(w) \geq 1$. Then we have

$$2^c d_{n_0}(w) \geq 2^c > d(w) \geq 2^c \sum_{n=m(2c)}^{\infty} d_n(w),$$

so $n_0 < m(2c)$. Thus J is finite. \square

The notion of Kolmogorov–Loveland stochasticity was defined in [13, 14, 19, 20]; detailed discussions may be found in [29, 15, 18]. A sequence is *Kolmogorov–Loveland stochastic* if any subsequence chosen by a Kolmogorov–Loveland selection rule possesses frequency stability, that is, if the proportion of 1’s in initial segments tends toward a limit of $\frac{1}{2}$. A *Kolmogorov–Loveland selection rule* is a pair of partial recursive functions that, operating on the history of what has been observed, choose the index of the next bit of the sequence to examine and determine (in advance of examination) whether or not that bit will be included in the subsequence. The standard intuition is described elegantly in [27]:

Let us imagine that the members of a sequence are written on cards which lie on an (infinitely long) table (we do not see what is on a card unless we turn it). The [selection rule] is an algorithm that says which card must be turned next and whether it must be turned only for information or [is to be] selected into the subsequence.

We write KL-STOCH for the set of all Kolmogorov–Loveland stochastic sequences.

We also make use of the following large deviation result.

Lemma 2.2 (Chernoff bound, Hagerup and Rüb [11]). *Let $p \in [0, 1]$, let X_1, \dots, X_n be independent 0/1-valued random variables such that each $P[X_i = 1] = p$, and let $S = X_1 + \dots + X_n$. Then:*

1. *For all $0 \leq \varepsilon \leq 1$, $P[S \geq (1 + \varepsilon)np] \leq e^{-(\varepsilon^2 np)/3}$.*
2. *For all $0 \leq \varepsilon < 1$, $P[S \leq (1 - \varepsilon)np] \leq e^{-(\varepsilon^2 np)/2}$.*

3. Result

In this section we prove that every sequence is feasibly reducible to some Kolmogorov–Loveland stochastic sequence. Our proof makes essential use of the following lemma, which is a straightforward relativization of Lemma 2 of [27].

Lemma 3.1 (Shen’ [27]). *Let $A \in \mathbf{C}$, and let $\vec{\beta}$ be a sequence of biases such that*

- (i) *$\vec{\beta}$ is computable relative to A , and*
- (ii) *$\vec{\beta}$ converges computably to $\frac{1}{2}$ relative to A .*

Then $\text{RAND}_{\vec{\beta}}^A \subseteq \text{KL-STOCH}^A$.

Given a sequence $A \in \mathbf{C}$, the following construction defines a sequence of biases $\vec{\beta}^A$, a function $F^A : \mathbf{C} \rightarrow \mathbf{C}$, and some auxiliary notation.

Construction 3.2. We use the functions

$$l, q, r : \mathbb{N} \rightarrow \mathbb{N},$$

where

$$l(n) = |s_n| = \lfloor \log(n+1) \rfloor,$$

$$q(n) = 12(l(n) + 3)^3,$$

and

$$r(n) = \sum_{m=0}^{n-1} q(m).$$

We also use the function

$$\varepsilon : \mathbb{N} \rightarrow [0, 1]$$

where

$$\varepsilon(n) = \frac{1}{l(n) + 3}.$$

For $A \in \mathbf{C}$ and $i \in \mathbb{N}$, define the bias $\beta_i^A \in [0, 1]$ by

$$\beta_i^A = \frac{1}{2}(1 + 3\varepsilon(n)\llbracket s_n \in A \rrbracket),$$

where n is the unique natural number for which $r(n) \leq i < r(n) + q(n)$, and let

$$\vec{\beta}^A = (\beta_0^A, \beta_1^A, \dots).$$

(Note that the bit $\llbracket s_n \in A \rrbracket$ has been encoded into each of $q(n)$ different positions in the bias sequence $\vec{\beta}^A$.) For each $A \in \mathbf{C}$ and $n \in \mathbb{N}$, define the random variable

$$\rho_n^A : \mathbf{C} \rightarrow [0, 1]$$

by

$$\rho_n^A(S) = \frac{1}{q(n)} |\{i \in \mathbb{N} \mid r(n) \leq i < r(n) + q(n) \text{ and } s_i \in S\}|,$$

where the argument S is chosen probabilistically according to the bias sequence $\vec{\beta}^A$. Finally, for each $A \in \mathbf{C}$, define the function

$$F^A : \mathbf{C} \rightarrow \mathbf{C}$$

by

$$F^A(S) = \{s_n \mid \rho_n^A(S) \geq \frac{1}{2}(1 + \varepsilon(n))\}.$$

Although the probability distribution of the random variable ρ_n^A depends on the sequence A , the actual value $\rho_n^A(S)$ does not depend on A . Moreover, we have the following:

Observation 3.3. For all $A, S \in \mathbf{C}$,

$$F^A(S) \leq_{\text{tt}}^P S.$$

Proof. The bit $\llbracket s_n \in F^A(S) \rrbracket$ is easily computed from the $q(n)$ bits

$$\llbracket s_{r(n)} \in S \rrbracket, \llbracket s_{r(n)+1} \in S \rrbracket, \dots, \llbracket s_{r(n)+q(n)-1} \in S \rrbracket,$$

and $q(n)$ is polynomial in $|s_n|$. \square

Intuitively, if S is chosen according to the bias sequence $\vec{\beta}^A$, then we expect $\rho_n^A(S)$ to be approximately $\frac{1}{2}(1 + 3\varepsilon(n))$ if $s_n \in A$, and approximately $\frac{1}{2}$ if $s_n \notin A$. Since $\frac{1}{2} < \frac{1}{2}(1 + \varepsilon(n)) < \frac{1}{2}(1 + 3\varepsilon(n))$, we thus expect that $\llbracket s_n \in F^A(S) \rrbracket$ will usually agree with $\llbracket s_n \in A \rrbracket$. The following lemma formalizes this intuition.

Lemma 3.4. For all $A \in \mathbf{C}$ and $S \in \text{RAND}_{\vec{\beta}^A}^A(\text{rec})$,

$$|A \triangle F^A(S)| < \infty.$$

Proof. Let $A \in \mathbf{C}$ and $S \in \text{RAND}_{\vec{\beta}^A}^A(\text{rec})$. For convenience, we write $\vec{\beta} = \vec{\beta}^A$, $\rho_n = \rho_n^A$, and $F = F^A$. For each $n \in \mathbb{N}$, define the event

$$\mathcal{E}_n = \{B \in \mathbf{C} \mid s_n \in A \triangle F(B)\}$$

and the function

$$d_n : \{0, 1\}^* \rightarrow [0, 1]$$

by

$$d_n(w) = P(\mathcal{E}_n \mid \mathbf{C}_w),$$

where the conditional probability refers (as do all subsequent probabilities in this proof) to the coin-toss probability measure $\mu^{\vec{\beta}}$. It is routine to check that each d_n is a $\vec{\beta}$ -martingale, and that the function $(n, w) \mapsto d_n(w)$ is computable relative to A .

By the Chernoff bound (Lemma 2.2), for each $s_n \in A$, we have

$$\begin{aligned} P[s_n \notin F(S)] &= P[\rho_n < \frac{1}{2}(1 + \varepsilon(n))] \\ &\leq P[\rho_n < \frac{1}{2}(1 + 3\varepsilon(n))(1 - \varepsilon(n))] \\ &\leq e^{-\frac{\varepsilon(n)^2 q(n)(1+3\varepsilon(n))}{4}} \\ &< e^{-\frac{\varepsilon(n)^2 q(n)}{4}} \\ &\leq (n + 1)^{-3}. \end{aligned}$$

Similarly, for each $s_n \in A^c$, we have

$$\begin{aligned} P[s_n \in F(S)] &= P[\rho_n \geq \frac{1}{2}(1 + \varepsilon(n))] \\ &\leq e^{-\frac{\varepsilon(n)^2 q(n)}{6}} \\ &\leq (n + 1)^{-2}. \end{aligned}$$

Hence, for each $n \in \mathbb{N}$,

$$d_n(\lambda) = P(\mathcal{E}_n) \leq (n + 1)^{-3} + (n + 1)^{-2}.$$

Since the series

$$\sum_{n=0}^{\infty} [(n + 1)^{-3} + (n + 1)^{-2}]$$

is computably convergent, it follows by Lemma 2.1 that the set

$$J = \{s_n \mid S \in S^1[d_n]\}$$

is finite. Since the definition of d_n implies that $A \triangle F(S) \subseteq J$, this completes the proof of the lemma. \square

Our main result is now easily established.

Theorem 3.5 (Main Theorem). *For each $A \in \mathbf{C}$, there is a sequence $S \in \text{KL-STOCH}$ such that $A \leq_{\text{tt}}^P S$.*

Proof. Let $A \in \mathbf{C}$, and let $S \in \text{RAND}_{\beta^A}^A$. By Lemma 3.1, $S \in \text{KL-STOCH}^A \subseteq \text{KL-STOCH}$. Also, since $\text{RAND}_{\beta^A}^A \subseteq \text{RAND}_{\beta^A}^A(\text{rec})$, Lemma 3.4 tells us that $|A \triangle F^A(S)| < \infty$, whence $A \leq_{\text{tt}}^P F^A(S)$. It follows by Observation 3.3 that $A \leq_{\text{tt}}^P S$. \square

As noted in the introduction, Theorem 3.5 exhibits a strong, quantitative separation of RAND from KL-STOCH, since Juedes et al. [12] have shown that only a meager (that is, negligibly small in the sense of Baire category) set of sequences have the property of being reducible to some sequence in RAND in some computable running time.

Bennett [1] has introduced the notion of *computational depth*, which measures the “value” of information in terms of the amount of “computational work” that has been “added to its organization”. In the case of infinite binary sequences, Bennett has defined both *strong depth* and *weak depth*, and shown that, in a technical, computational sense, strongly deep sequences are “very far from random”. (See [1, 18, 12] for definitions and discussion of computational depth.)

The proof of Shen’ [27] exhibits a sequence that is Kolmogorov–Loveland stochastic and not random. That sequence, however, is random relative to a computable probability

measure and so is not even weakly deep in the sense of Bennett. Nevertheless, we now note that a Kolmogorov–Loveland stochastic sequence may be strongly deep.

Corollary 3.6. *There exist Kolmogorov–Loveland stochastic sequences that are strongly deep.*

Proof. Let K be the diagonal halting problem. By Theorem 3.5, let S be a Kolmogorov–Loveland stochastic sequence such that $K \leq_{\Pi}^P S$. Bennett [1] has shown that K is strongly deep and, by his deterministic slow growth law, that strongly deep sequences are only \leq_{Π} -reducible to sequences that are themselves strongly deep. (Proofs of these results also appear in [12].) Hence, S is strongly deep. \square

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References

- [1] C.H. Bennett, Logical Depth and Physical Complexity, in: R. Herken (Ed.), *The Universal Turing Machine: A Half-Century Survey*, Oxford University Press, Oxford, 1988, pp. 227–257.
- [2] R.V. Book, On languages reducible to algorithmically random languages, *SIAM J. Comput.* 23 (1994) 1275–1282.
- [3] R.V. Book, J.H. Lutz, K.W. Wagner, An observation on probability versus randomness with applications to complexity classes, *Math. Systems Theory* 27 (1994) 201–209.
- [4] R.V. Book, H. Vollmer, K.W. Wagner, On type-2 probabilistic quantifiers, in: *Proc. 23rd Internat. Coll. Automata, Languages, and Programming*, Springer, Berlin, 1996, pp. 369–380.
- [5] J.M. Breutzmann, J.H. Lutz, Equivalence of measures of complexity classes, *SIAM J. Comput.*, to appear.
- [6] G.J. Chaitin, A theory of program size formally identical to information theory, *J. Assoc. Comput. Mach.* 22 (1975) 329–340.
- [7] G.J. Chaitin, *Algorithmic Information Theory*, Cambridge University Press, New York, 1987.
- [8] G.J. Chaitin, Incompleteness theorems for random reals, *Adv. Appl. Math.* 8 (1987) 119–146.
- [9] A. Church, On the concept of a random sequence, *Bull. Amer. Math. Soc.* 46 (1940) 130–135.
- [10] P. Gács, Every sequence is reducible to a random one, *Inform. Control* 70 (1986) 186–192.
- [11] T. Hagerup, C. Rüb, A guided tour of Chernoff bounds, *Inform. Processing Lett.* 33 (1990) 305–308.
- [12] D.W. Juedes, J.I. Lathrop, J.H. Lutz, Computational depth and reducibility, *Theoret. Comp. Sci.* 132 (1994) 37–70.
- [13] A.N. Kolmogorov, On tables of random numbers, *Sankhyā, Ser. A* 25 (1963) 369–376.
- [14] A.N. Kolmogorov, Combinatorial foundations of information theory and calculus of probabilities, *Russian Math. Surveys* 38 (1983) 29–40.
- [15] A.N. Kolmogorov, V.A. Uspenskii, Algorithms and randomness, translated in *Theory Probab. Appl.* 32 (1987) 389–412.
- [16] A. Kučera, Measure, Π_1^0 -classes, and complete extensions of PA, in: *Recursion Theory Week*, vol. 1141 of *Lecture Notes in Mathematics*, Springer, Berlin, 1985, pp. 245–259.
- [17] L.A. Levin, On the notion of a random sequence, *Sov. Math. Doklady* 14 (1973) 1413–1416.
- [18] M. Li, P.M.B. Vitányi, *An Introduction to Kolmogorov Complexity and its Applications*, Springer, Berlin, 1993.

- [19] D.W. Loveland, The Kleene hierarchy classification of recursively random sequences, *Trans. Amer. Math. Soc.* 125 (1966) 497–510.
- [20] D.W. Loveland, A new interpretation of von Mises' concept of a random sequence, *Z. Math. Logik Grundlagen Math.* 12 (1966) 279–294.
- [21] J.H. Lutz, Almost everywhere high nonuniform complexity, *J. Comput. System Sci.* 44 (1992) 220–258.
- [22] J.H. Lutz, On independent random oracles, *Theoret. Comput. Sci.* 92 (1992) 301–307.
- [23] P. Martin-Löf, On the definition of random sequences, *Inform. Control* 9 (1966) 602–619.
- [24] C.P. Schnorr, A unified approach to the definition of random sequences, *Math. Systems Theory* 5 (1971) 246–258.
- [25] C.P. Schnorr, Process complexity and effective random tests, *J. Comput. System Sci.* 7 (1973) 376–388.
- [26] A.Kh. Shen', The frequency approach to the definition of a random sequence, *Semiotika Inform.* 18 (1982) 14–42. (In Russian).
- [27] A.Kh. Shen', On relations between different algorithmic definitions of randomness, *Sov. Math. Doklady* 38 (1989) 316–319.
- [28] R.M. Solovay, 1975, reported in [8].
- [29] V.A. Uspenskii, A.L. Semenov, A.Kh. Shen', Can an individual sequence of zeros and ones be random?, *Russian Math. Surveys* 45 (1990) 121–189.
- [30] M. van Lambalgen, *Random Sequences*, Ph.D. Thesis, Department of Mathematics, University of Amsterdam, 1987.
- [31] R. von Mises, *Grundlagen der Wahrscheinlichkeitsrechnung*, *Math. Z.* 5 (1919) 52–99.
- [32] A. Wald, Die Widerspruchsfreiheit des Kollektivbegriffs in der Wahrscheinlichkeitsrechnung, *Ergebnisse eines Math. Koll.* 8 (1938) 38–72.