The Essential Components of Coincident Points for Weakly Inward and Outward Set-Valued Mappings

G. Isac
Department of Mathematics and Computer Sciences
Royal Military College of Canada, Kingston, Ontario, Canada K7K 5L0
isac-g@rmc.ca

G. X.-Z. Yuan*
Department of Mathematics
The University of Queensland, Brisbane, Australia 4072
xzy@maths.uq.edu.au

(Received and accepted January 1998)

Abstract—In this paper, by the concept of essential components of coincident points for set-valued mappings, we study the existence of essential components of both coincident and fixed points for non-self upper semicontinuous set-valued mappings in normed spaces. These results include corresponding results in the literature as special cases. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Inward (respectively, outward) set-valued mappings, Upper semicontinuous, Essential component, Coincident points, Fixed points.

1. INTRODUCTION

The study of the stability for both coincident and fixed points is one of the most important issues in nonlinear analysis. Since Kinoshita [1] introduced the notion of essential components for fixed points of single-valued continuous mappings, he proved that for any continuous mapping of the Hilbert cube into itself, there exists at least one essential component of its fixed points. Some new progress has been given by Luo et al. [2]. Recently Yu and Xiang [3] also established the existence of essential components for Ky Fan points and equilibrium points.

In this paper, by the concept of essential components of coincident points for set-valued mappings and following the ideas from Luo et al. [2] and Yu and Xiang [3], our aim is to prove the existence of essential components of both coincident and fixed points for non-self upper semicontinuous set-valued mappings in normed spaces. Our results include corresponding results in the literature as special cases.

2. PRELIMINARY

Let X be a nonempty set; we denote by $2^X$ the family of all subsets of X. If X and Y are two...
Hausdorff topological spaces and $T : Y \to 2^X$ is a set-valued mapping, then

1. $T$ is upper semicontinuous at $y_0 \in Y$, if for each open set $U$ in $X$ with $U \supseteq T(y_0)$, there exists an open neighborhood $O(y_0)$ of $y_0$ such that $U \supseteq T(y)$ for any $y \in O(y_0)$; and

2. $T$ is upper semicontinuous (in short, USC) on $Y$, if $T$ is upper semicontinuous at every point $y_0 \in Y$.

If $A$ is a subset of a metric space $(E, d)$ and $a$ is a given positive number, we denote by $O(A, a) := \{x \in E : d(x, A) < a\}$ the open neighborhood of $A$ with radius $a$ in $E$; and also we use $Bd(X)$ to denote the boundary of $X$ in $E$.

Let $X$ be a nonempty subset of a topological vector space $E$. Then a nonself set-valued mapping $F : X \to 2^E$ is said to be a weakly inward (respectively, outward) mapping if for each $x \in Bd(X)$, $F(x) \cap I_X(x) \neq \emptyset$ (respectively, $F(x) \cap O_X(x) \neq \emptyset$), where $I_X(x)$ (respectively, $O_X(x)$) is the so-called inward (respectively, outward) set of $X$ at $x$ and defined as $I_X(x) := \cup_{\lambda > 0} \lambda(X - x)$ (respectively, $O_X(x) := \cup_{\lambda < 0} \lambda(X - x)$).

Throughout this paper, let $X$ be a nonempty compact convex subset of a normed linear space and let $S$ be the set of all USC set-valued mappings with nonempty closed convex values from $X$ to $X$. Then for any $f, g \in S$, we can define a metric $\rho_1$ by $\rho_1(f, g) := \sup_{x \in X} h(f(x), g(x))$ for each $f, g \in S$, where $h$ is the Hausdorff metric defined on $X$ through the norm $\| \cdot \|$ of $E$. Clearly, $(S, \rho_1)$ is a complete metric space by Lemma 3 of [4] and Theorem 4.3.9 of [5].

In what follows, we set

$$Y := \{(f, g) \in S \times S : \text{where } f - g \text{ is a weakly inward mapping}\}.$$  

Then we can show that indeed $(Y, \rho)$ is a complete metric space, where the metric $\rho$ is defined by $\rho((f, g), (f', g')) := \rho_1(f, f') + \rho_1(g, g')$ for each $(f, g), (f', g') \in Y$, i.e., we have the following result.

Lemma 1. The subspace $Y \subset S \times S$ is closed.

Proof. For the convenience of readers, we include its proof as follows. Suppose $\{y_\alpha\} = \{(f_\alpha, g_\alpha)\} \subset Y$ is a net with $y_\alpha \to y = (f, g) \in S \times S$. Since $(f_\alpha, g_\alpha) \in Y$, for any $x \in Bd(X)$, we have $(f_\alpha(x) - g_\alpha(x)) \cap \bar{I}_X(x) \neq \emptyset$. Thus there are $u_\alpha \in f_\alpha(x)$ and $v_\alpha \in g_\alpha(x)$ such that $u_\alpha - v_\alpha \in I_X(x) \neq \emptyset$. Noting that $f_\alpha \to f, g_\alpha \to g$, and both $f(x)$ and $g(x)$ are compact, it is easy to verify that $\{u_\alpha\} \to u_0 \in f(x)$ and $\{v_\alpha\} \to v_0 \in g(x)$. Indeed, if $u_0 \notin f(x_0)$, there exists a $\delta > 0$ such that $O(u_0, \delta) \cap O(f(x_0), \delta) = \emptyset$. Since $\lim_\alpha u_\alpha = u_0$ and $\lim_\alpha f_\alpha = f$, there exists $\alpha_0$ such that for each $\alpha \geq \alpha_0$, we have that $u_\alpha \in O(u_0, \delta)$ and $f_\alpha(u) \in O(f(u_0), \delta)$ for all $u \in X$ and thus $u_\alpha \in f_\alpha(x) \subset O(f(x), \delta)$. This implies that $u_\alpha \in O(u_0, \delta) \cap O(f(x), \delta) = \emptyset$ for all $\alpha \geq \alpha_0$, which is a contradiction. Therefore, we must have $u_0 \in f(x)$, and by the same reason, $v_0 \in f(x)$.

Next we want to show that $u_0 - v_0 \in \bar{I}_X(x) \neq \emptyset$. Note that for each $\alpha$, as $u_\alpha - v_\alpha \in \bar{I}_X(x) \neq \emptyset$, and noting that $\bar{I}_X(x) \neq \emptyset$ is closed, it follows that

$$u_0 - v_0 = \lim_\alpha (u_\alpha - v_\alpha) = \lim_\alpha \lambda_\alpha(x_\alpha - x) = \lambda_0(x_0 - x) \in \bar{I}_X(x) \neq \emptyset.$$  

Therefore, for any $x \in Bd(X)$, it follows that $(f(x) - g(x)) \cap \bar{I}_X(x) \neq \emptyset$, which means the set $Y$ is a closed subset of $S \times S$.

For any $y = (f, g) \in Y$, we define a set-valued mapping $F : Y \to 2^X$ by

$$F(y) := \{x \in X : f(x) \cap g(x) \neq \emptyset\}$$  

for each $y = (f, g) \in Y$. Then Lemma 2 below tells us that $F$ is a set-valued mapping from $Y$ to $X$ with nonempty values.

Lemma 2. Let $X$ be a nonempty compact convex subset of a normed space $E$ and let two set-valued mappings $f$ and $g : X \to 2^X$ be upper semicontinuous with nonempty and convex and
compact valued. For any \( x \in \text{Bd}(X) \), the set \( f(x) - g(x) \cap \overline{T_X(x)} \neq \emptyset \). Then there exists \( x^* \in X \) such that \( f(x^*) \cap g(x^*) \neq \emptyset \).

**Proof.** It is a special case of Theorem 5 of [6].

For each \( y \in Y \), the component of a point \( x \in F(y) \) is the union of all connected subsets of \( F(y) \) which contain the point \( x \). From [7], we know that components are connected closed subsets of \( F(y) \) and thus they are also compact as \( F(y) \) is compact. It is also easy to see that the components of two distinct points of \( F(y) \) either coincide or are disjoint, so that all components constitute a decomposition of \( F(y) \) into connected pairwise disjoint compact subsets, i.e.,

\[
F(y) = \bigcup_{\alpha \in \Lambda} F_{\alpha}(y),
\]

where \( \Lambda \) is an index set, for any \( \alpha \in \Lambda \), \( F_{\alpha}(y) \) is a nonempty connected compact, and for any \( \alpha, \beta \in \Lambda (\alpha \neq \beta) \), \( F_{\alpha}(y) \cap F_{\beta}(y) = \emptyset \). In order to study the stability of coincident points for set-valued mappings, we first recall the following definition (e.g., see [1,2]).

**Definition 1.** For \( y \in Y \), suppose the set \( F(y) = \bigcup_{\alpha \in \Lambda} F_{\alpha}(y) \). Then a component \( F_{\alpha}(y) \) for some \( \alpha \in \Lambda \) is said to be an essential component of \( y \) if for each open set \( O \) containing \( F_{\alpha}(y) \), there exists \( \delta > 0 \) such that for any \( y' \in Y \) with \( \rho(y, y') < \delta \), \( F(y') \cap O \neq \emptyset \).

The definition above means that even though we could not expect the continuity for all coincident points of a pair of set-valued mappings in general, however, there is a case that maybe some of coincident points enjoy the continuous stability. In the rest of this paper, we will show the existence of such nice coincident points for USC set-valued mappings.

In order to establish our existence theorem of essential components for coincident points of USC set-valued mappings in \( Y \), we first need the following result which is a generalization of Theorem 2 in [2] to normed spaces.

**Lemma 3.** Let \( X \) be a nonempty compact convex set in a normed space \((E, \| \cdot \|)\). Then the set-valued mapping \( F : Y \rightarrow 2^X \) is upper semicontinuous with nonempty compact valued.

**Proof.** For each \( y = (f, g) \in Y \), we first show that \( F(y) \subset X \) compact. As \( X \) is compact, it suffices to show that \( F(y) \) is closed. Suppose \( \{x_{\alpha}\} \subset F(y) \) is a net such that \( x_{\alpha} \rightarrow x_0 \in X \). Since \( x_{\alpha} \in F(y) \), it follows that \( f(x_{\alpha}) \cap g(x_{\alpha}) \neq \emptyset \). Suppose now \( f(x_0) \cap g(x_0) = \emptyset \). As both \( f(x_0) \) and \( g(x_0) \) are compact, there exists \( \delta > 0 \) such that \( O(f(x_0), \delta) \cap O(g(x_0), \delta) = \emptyset \). By the upper semicontinuity of \( f \) and \( g \) and \( x_{\alpha} \rightarrow x_0 \), there is \( \alpha_0 \) such that for any \( \alpha > \alpha_0 \), \( f(x_{\alpha}) \subset O(f(x_0), \delta) \) and \( g(x_{\alpha}) \subset O(g(x_0), \delta) \), which contradicts with our assumption that \( f(x_{\alpha}) \cap g(x_{\alpha}) = \emptyset \). Thus, \( x_0 \in F(y) \) and \( F(y) \) is compact. By a fact that \( X \) is compact, in order to show that \( F \) is USC, it suffices to show that the graph of \( F \), i.e., the set \( \text{Graph}_F := \{(y, x) \in Y \times X : x \in F(y), y \in Y\} \) is closed in \( Y \times X \). Suppose \( \{(y_\alpha, x_\alpha)\} \) is a net in \( \text{Graph}_F \) such that \( (y_\alpha, x_\alpha) \rightarrow (y_0, x_0) \). We denote by \( y_\alpha := (f_\alpha, g_\alpha) \). Then \( x_\alpha \in F(y_\alpha) \) and \( f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) \neq \emptyset \). If \( f_\alpha(x_0) \cap g_\alpha(x_0) = \emptyset \), since \( f_\alpha \) and \( g_\alpha \) are both compact, there exists a positive number \( b \in (0, 1) \) such that \( O(f_\alpha(x_0), b) \cap O(g_\alpha(x_0), b) = \emptyset \). Since \( f_\alpha \rightarrow f_0 \) and \( g_\alpha \rightarrow g_0 \), there exists \( \alpha_1 \) such that for each \( \alpha \geq \alpha_1 \), we have that \( f_\alpha(u) \subset O(f_\alpha(u), b/2) \) and \( g_\alpha(u) \subset O(g_\alpha(u), b/2) \) for all \( u \in X \). Note that \( f_0 \) and \( g_0 \) are upper semicontinuous at \( x_0 \) and \( x_\alpha \rightarrow x_0 \), there exists \( \alpha_2 \geq \alpha_1 \) such that \( O(f_\alpha(x_0), b) \cap O(g_\alpha(x_0), b) = \emptyset \). Since \( f_\alpha \rightarrow f_0 \) and \( g_\alpha \rightarrow g_0 \), there exists \( \alpha_1 \) such that for each \( \alpha \geq \alpha_1 \), we have that \( f_\alpha(x_\alpha) \subset O(f_\alpha(x_0), b/2) \) and \( g_\alpha(x_\alpha) \subset O(g_\alpha(x_0), b/2) \). Thus, for all \( \alpha \geq \alpha_2 \), we have that \( f_\alpha(x_\alpha) \subset O(f_\alpha(x_0), b) \) and \( g_\alpha(x_\alpha) \subset O(g_\alpha(x_0), b) \) for all \( \alpha \geq \alpha_2 \). Since \( O(f_\alpha(x_0), b) \cap O(g_\alpha(x_0), b) = \emptyset \), it follows that \( f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) = \emptyset \), which is a contradiction. Hence, we must also have \( f_\alpha(x_\alpha) \cap g_\alpha(x_\alpha) \neq \emptyset \). Therefore, the graph of \( F \) is closed and the mapping \( F \) is upper semicontinuous with nonempty compact values.

We recall that for given nonempty subsets \( A \) and \( B \) of a metric space \( E \), the Hausdorff metric \( h \) between \( A \) and \( B \) is defined by \( h(A, B) := \inf \{ \varepsilon : A \subset O(B, \varepsilon) \text{ and } B \subset O(A, \varepsilon) \} \). Then we have the following simple fact (see also Lemma 2 of [2]).
**Lemma 4.** Let \( A, B, \) and \( F \) be nonempty convex and bounded subsets of a normed linear space \( E \). Then \( h(A, \lambda B + \mu C) \leq \lambda h(A, B) + \mu h(A, C) \), where \( h \) is the Hausdorff metric defined on \( E \), \( \lambda \geq 0 \) and \( \mu \geq 0 \) with \( \lambda + \mu = 1 \).

**Proof.** By the definition of Hausdorff metric \( h(A, B) \), it suffices to prove that for any given \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) with \( B \subset O(A, \epsilon_1) \) and \( A \subset O(B, \epsilon_1) \), and \( C \subset O(A, \epsilon_2) \) and \( A \subset O(C, \epsilon_2) \), we have that \( A \subset O(\lambda B + \mu C, \lambda \epsilon_1 + \mu \epsilon_2) \) and \( \lambda B + \mu C \subset O(A, \lambda \epsilon_1 + \mu \epsilon_2) \). For any \( a \in A \), as \( A \subset O(B, \epsilon_1) \) and \( A \subset O(C, \epsilon_2) \), there exist \( b \in B \) and \( c \in C \) such that \( d(a, b) < \epsilon_1 \) and \( d(a, c) < \epsilon_2 \). Note that \( \lambda + \mu = 1 \), it follows that

\[
d(a, \lambda b + \mu c) = \|a - \lambda b - \mu c\| \leq \lambda \|a - b\| + \mu \|a - c\| \leq \lambda \epsilon_1 + \mu \epsilon_2,
\]

which implies that \( A \subset O(\lambda B + \mu C, \lambda \epsilon_1 + \mu \epsilon_2) \). By the convexity of \( B \) and \( C \) and the similar argument used above, we can also verify that \( \lambda B + \mu C \subset O(A, \lambda \epsilon_1 + \mu \epsilon_2) \) and thus, the proof is completed.

We now have the main result of this paper as follows.

**Theorem 5.** Let \( X \) be a nonempty compact convex subset of a normed space \( E \). For any \( y \in Y \), there exists at least one essential connected component of \( F(y) \).

**Proof.** For any given \( y \in Y \), as Luo et al. did in [2], suppose that \( F(y) \) is decomposed as follows:

\[
F(y) = \bigcup_{\alpha \in \Lambda} F_{\alpha}(y),
\]

where \( \Lambda \) is an index set, for any \( \alpha \in \Lambda \), \( F_{\alpha}(y) \) is a connected compact and for any \( \alpha, \beta \in \Lambda (\alpha \neq \beta) \), \( F_{\alpha}(x) \cap F_{\beta}(x) = \emptyset \). We shall prove that there exists at least one essential component of \( F(y) \).

Let us suppose otherwise there is not any essential connected component. Then for any \( \alpha \in \Lambda \), there exists an open set \( O_{\alpha} \supset F_{\alpha}(y) \) such that for any \( \epsilon > 0 \), there is \( y_{\alpha} \in Y \) with \( \rho(y, y_{\alpha}) < \epsilon \) such that \( F(y_{\alpha}) \cap O_{\alpha} = \emptyset \). As \( F(y) \) is compact, there exist two open and finite coverings \( \{V_1\} \) and \( \{W_1\} \) which satisfy the following conditions (e.g., see [1]):

1. \( \overline{W}_i \subset V_i \);
2. \( V_i \cap V_j = \emptyset \) for each \( i \neq j \); and
3. \( V_i \) contains at least one \( F_{\alpha_i}(y) \) with \( O_{\alpha_i} \supset V_i \supset F_{\alpha_i}(y) \).

By Lemma 3, \( F \) is upper semicontinuous at \( y \) and note that \( \bigcup_{i=1}^{n} W_i \supset F(y) \) and \( \bigcup_{i=1}^{n} W_i \) is open, then there exists a \( \delta > 0 \) such that \( \bigcup_{i=1}^{n} W_i \supset F(y') \) for any \( y' \in Y \) with \( \rho(y, y') < \delta \). Thus there is \( y_{\alpha_i} \in Y \) with \( \rho(y, y_{\alpha_i}) < \delta \) such that \( F(y_{\alpha_i}) \cap O_{\alpha_i} = \emptyset \).

Let \( y = (f, g) \) and \( y_{\alpha_i} = (f_{\alpha_i}, g_{\alpha_i}) \), where \( i = 1, \ldots, n \). We define two set-valued mappings

\[
f^*(x) = \begin{cases} f(x), & \text{if } x \in X \setminus \bigcup_{i=1}^{n} V_i, \\ f_i^*(x), & \text{if } x \in V_i \text{ for some } i = 1, 2, \ldots, n, \end{cases}
\]

and

\[
g^*(x) = \begin{cases} g(x), & \text{if } x \in X \setminus \bigcup_{i=1}^{n} V_i, \\ g_i^*(x), & \text{if } x \in V_i \text{ for some } i = 1, 2, \ldots, n, \end{cases}
\]

where \( f_i^*, g_i^* : X \to 2^X \) are set-valued mappings defined by

\[
f_i^*(x) := \lambda_i(x)f(x) + \mu_i(x)f_{\alpha_i}(x)
\]

and

\[
g_i^*(x) := \lambda_i(x)g(x) + \mu_i(x)g_{\alpha_i}(x)
\]
for each $x \in X$; and $\lambda_i$ and $\mu_i$ are two functions from $X$ to $\mathbb{R}$ defined by

$$\lambda_i(x) := \frac{d(x, W_i)}{d(x, W_i) + d(x, X \setminus \bigcup_{i=1}^{n} V_i)}$$

and

$$\mu_i(x) := \frac{d(x, X \setminus \bigcup_{i=1}^{n} V_i)}{d(x, W_i) + d(x, X \setminus \bigcup_{i=1}^{n} V_i)}$$

for each $x \in X$. It is clear that $\lambda_i$ and $\mu_i$ are continuous. Noting that $f, g, f_\alpha$, and $g_\alpha$ are upper semicontinuous with compact values, it follows by Theorems 7.3.11, 7.3.14, and 7.3.15 of [5] that $f^*_i$ and $g^*_i$ are upper semicontinuous. Second, for each given $x \in X$, if $x \in X \setminus \bigcup_{i=1}^{n} V_i$, then $f(x) = f^*_i(x)$ and $g(x) = g^*_i(x)$ if $x \in X \setminus \bigcup_{i=1}^{n} V_i$ for all $i = 1, 2, \ldots, n$; and $f^*_i(x) := f_\alpha(x)$ and $g^*_i(x) := g_\alpha(x)$ if $x \in W_i$. Next we show that both $f^*$ and $g^*$ are upper semicontinuous on $X$. Without loss of generality, let $G$ be any given open subset of $E^*$ and suppose there exists $x_0 \in X$ such that $f^*(x_0) \subset G$. If there exists $i \in \{1, 2, \ldots, n\}$, by the upper semicontinuity of $f^*_i$, there exists a nonempty open neighbourhood $N(x_0)$ of $x_0$ in $V_i$ such that $f^*_i(u) \subset G$.

As $V_i$ is open in $X$, we may assume that $N(x_0)$ is also an open neighbourhood of $X$ and thus $f^*$ is upper semicontinuous at $x_0 \in V_i$ by the definition of upper semicontinuity. In the case $x_0 \in X \setminus \bigcup_{i=1}^{n} V_i$, by the fact that $f^*(x_0) = f^*_i(x_0) = f(x)$ for all $i = 1, 2, \ldots, n$, and the upper semicontinuity of $f$ and $f_i$, it follows that for each $i = 1, 2, \ldots, n$, there exists a nonempty open neighbourhood $V_i(x_0)$ of $x_0$ in $X$ such that $f^*_i(u) \subset G$ for all $u \in V_i(x_0)$ (as $f^*_i(x_0) \subset G$). Since $f$ is also upper semicontinuous, there exists a nonempty open neighbourhood $V_0(x_0)$ of $x_0$ in $X$ such that $f(u) \subset G$ for each $u \in V_0(x_0)$. Let $N(x_0) := \bigcap_{i=0}^{n} V_i(x_0)$. Then $N(x_0)$ is a nonempty open neighborhood of $x_0$ in $X$. Moreover, it is easy to see that $f^*(u) \subset G$ for each $u \in N(x_0)$. Indeed, if $u \in N(x_0) \cap (X \setminus \bigcup_{i=1}^{n} V_i)$, then $f^*(u) = f(u) \subset G$; in the case $u \in N(x_0) \cap V_i$ for some $i = 1, \ldots, n$, then $f^*(u) = f^*_i(u) \subset G$. Therefore, we have shown that for each open neighbourhood $G$ of $E^*$ with $f^*(x_0) \subset G$ for some $x_0 \in X$, there exists a nonempty open neighbourhood $N(x_0)$ of $x_0$ in $X$ such that $f^*(u) \subset G$ for each $u \in N(x_0)$. This means $f^*$ is upper semicontinuous. By the same reason, it follows that the mapping $g^*: X \rightarrow 2^X$ is also upper semicontinuous. Moreover, both $f^*$ and $g^*$ take nonempty closed and convex values.

Next we want to show that for any $x \in \text{Bd}(X)$, $f^* - g^* \cap \overline{I_X(x)} \neq \emptyset$. First, if $x \in \text{Bd}(X)$ and $x \in X \setminus \bigcup_{i=1}^{n} V_i$, it follows that $f^*(x) - g^*(x) = f(x) - g(x)$, and then we have that

$$(f^*(x) - g^*(x)) \cap \overline{I_X(x)} \neq \emptyset = (f(x) - g(x)) \cap \overline{I_X(x)} \neq \emptyset.$$  

Second, if $x \in \text{Bd}(X)$ and $x \in W_i$, it follows that $f^*(x) - g^*(x) = f_\alpha(x) - g_\alpha(x)$ and then we have that

$$(f^*(x) - g^*(x)) \cap \overline{I_X(x)} \neq \emptyset = (f_\alpha(x) - g_\alpha(x)) \cap \overline{I_X(x)} \neq \emptyset.$$  

Finally, if $x \in \text{Bd}(X)$ and $x \in V_i \setminus W_i$, it follows that

$$f^*(x) - g^*(x) = (\lambda_i(x)f(x) + \mu_i f_\alpha(x)) - (\lambda_i(x)g(x) + \mu_i g_\alpha(x))$$

$$= \lambda_i(x)(f(x) - g(x)) + \mu_i(x)(f_\alpha(x) - g_\alpha(x)).$$

Since $X$ is convex, we have that $\overline{I_X(x)} \neq \emptyset$ is convex. Noting that $(f(x) - g(x)) \cap \overline{I_X(x)} \neq \emptyset$, $(f_\alpha(x) - g_\alpha(x)) \cap \overline{I_X(x)} \neq \emptyset$, and $\lambda_i(x) + \mu_i(x) = 1$, it follows that

$$f^*(x) - g^*(x) = (\lambda_i(x)(f(x) - g(x)) + \mu_i(x)(f_\alpha(x) - g_\alpha(x))) \cap \overline{I_X(x)} \neq \emptyset.$$  

Hence, for any $x \in \text{Bd}(X)$, we show that

$$(f^*(x) - g^*(x)) \cap \overline{I_X(x)} \neq \emptyset,$$
which means that $y^* = (f^*, g^*) \in Y$, so that $F(y^*) \neq \emptyset$. Noting that $\rho(y, y_\alpha) < \delta$ for $i = 1, 2, \ldots, n$, it follows by Lemma 4 that

$$h(f(x), \lambda_i(x)f(x) + \mu_i(x)f_\alpha(x)) \leq h(f(x), f_\alpha(x))$$

and

$$h(g(x), \lambda_i(x)g(x) + \mu_i(x)g_\alpha(x)) \leq h(g(x), g_\alpha(x)).$$

Therefore, $\rho(y, y^*) < \delta$ and $F(y^*) \subseteq \bigcup_{i=1}^{n} W_i$. Note that for any $x_0 \in F(y^*)$, there is an index $i_0$ such that $x_0 \in W_i$, and hence, $x_0 \in W_i \subseteq W_{i_0} \subseteq O_{\alpha_{i_0}}$. Therefore, $f^*(x_0) = f_{\alpha_{i_0}}(x_0), g^*(x_0) = g_{\alpha_{i_0}}(x_0)$, and $x_0 \in F(y_{i_0})$. This contradicts our assumption that $F(y_{i_0}) \cap O_{\alpha_{i_0}} = \emptyset$. Hence, there exists at least one essential connected component of $F(y)$, completing the proof.

REMARK 1. If we set

$$Y_1 := \{(f, g) \in S \times S : f - g \text{ is a weakly outward mapping}\},$$

then we can also show that indeed $(Y_1, \rho)$ is a complete metric space, where the metric $\rho$ is defined by $\rho((f, g), (f', g')) := \rho_1(f, f') + \rho_1(g, g')$ for each $(f, g), (f', g') \in Y$. Moreover, the same conclusions in Lemmas 2, 3, and 4 hold. By the same proof as used in Theorem 5, we have the following existence of essential components of coincident points for weakly outward USC set-valued mappings in normed spaces.

**Theorem 6.** Let $X$ be a nonempty compact subset of a normed space $E$. For any $y \in Y_1$, there exists at least one essential connected component of $F(y)$.

**Proof.** The conclusion follows by the same proof used in Theorem 5.

**Remark 2.** For any $f \in S$, if $g$ is the identity mapping, i.e., $g(x) = I(x) := x$ for all $x \in X$, then for any $x \in \text{Bd}(X)$, the boundary condition becomes

$$(f(x) - g(x)) \cap \overline{I_X(x)} = (f(x) - x) \cap \overline{I_X(x)} \neq \emptyset. \quad (*)$$

Note that for each $f \in S, f(x) \in X$ for each $x \in X$. This boundary condition $(*)$ is automatically satisfied for each $f$ from the spaces $Y_2$ defined as follows:

$$Y_2 := \{(f, I) \times S \times S : f \in S \text{ and } I \text{ is the identity mapping in } X\}.$$ 

As a special case of Theorem 5, we have the following existence of essential components for fixed points of USC set-valued mappings in normed spaces.

**Theorem 7.** Let $X$ be a nonempty compact and convex subset of a normed space $E$. Then the set of fixed points for $f \in Y_2$ has at least one essential component.

**Remark 3.** Theorems 5 and 6 show the corresponding results of Luo et al. [2] hold in normed spaces for weakly inward or outward set-valued mappings in normed spaces instead of Banach spaces. Theorem 6 is also a set-valued version of corresponding Theorem 3 of [1].

**REFERENCES**