Applications of Thom's Transversality Theory and Brouwer Degree Theory to Economics*

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I. INTRODUCTION

Recently, techniques of differential topology and global analysis were introduced into the economics literature by Debreu [6] and Smale [20], [21]. The tools of differential topology enables us to investigate the local uniqueness and continuity of the economic equilibria as well as the existence problem. The existence problem has been extensively studied during the last 20 years (see Arrow and Hahn [2] for a comprehensive survey). The mathematical tools for the solution were provided by algebraic topology in the form of fixed point theorems. In this differential framework, one can also show that the equilibrium varies in a continuous and unique manner with respect to changes in the economic data of the model. Debreu [6] investigated these equilibrium properties for classical pure exchange economies with a finite number of agents and a finite number of consumption goods. His analysis is restricted to finite dimensional spaces in the sense that an economy is specified by a point of finite dimensional commodity space. Smale [20] extended this finite dimensional case to the case of allowing each agent's utility function to vary arbitrarily for the same type model as Debreu [6].

In this paper we consider an economic equilibrium model with externalities where each agent's utility function depends on the state of the economy which is

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specified by the allocations of each agent and also on a price system. This includes, as a special case, the Veblen–Scitovsky price influenced equilibrium models studied recently by Arrow and Hahn [2] and Kalman and Lin [11]. McKenzie [13] was the first to prove explicitly the existence of equilibrium where each consumer's preferences and each firm's production depends on the allocation of resources among other consumers and firms. However, this did not include price influenced economies. Arrow and Hahn [2] and Laffont and Laroque [12] also study the existence of equilibrium for a MacKenzie type model. More recently, in the economics literature, Mas–Collel [15] and Shafer and Sonnenschein [19] prove the existence of pure exchange equilibrium with externalities without requiring complete or transitive preferences. Mantel [14] applies the same idea to a model with the presence of a complex tax structure and public goods.

Our approach, which differs from those of the above works on externalities, uses tools of differential topology to study the structure of the equilibria set. In particular, in addition to proving existence of equilibrium with externalities under certain assumptions (as in [13], [2], [12], [15], [19]), we also obtain local uniqueness and continuity of this equilibrium which are new results. In other words, under certain assumptions for "almost all" economies with externalities, there exists a finite number of equilibria which are stable.1

Section II presents the basic model. In Section III we prove local uniqueness of equilibria for "almost all" economies using transversality theory [1] of differential topology without requiring any convexity assumptions on preferences. We also obtain continuity of equilibria with respect to the economic data of the model. Finally, in Section IV we prove existence of equilibrium for this model using degree theory [16] of differential topology.

II. THE MODEL

We consider a space of economies with \( l \) commodities and \( n \) agents. Let \( P = \{z \in \mathbb{R}^l : z_i > 0, i = 1, \ldots, l\} \) be the commodity space where \( x^h \in P \) is the consumption bundle of agent \( h \) (\( h = 1, \ldots, n \)). Let \( S = \{p \in P : \sum_{i=1}^l p_i = 1\} \) be the price space.

A state of an economy is a pair \((x, p) \in P^n \times S\) where \( x = (x^1, \ldots, x^n) \). We denote the product space \( P^n \times S \) as the state space. For each agent \( h \), \( x^h \) is the choice variable and \((\tilde{x}^h, p) = (x^1, \ldots, x^{h-1}, x^{h+1}, \ldots, x^n, p) \in P^{n-1} \times S\) is the parameter vector which influences his/her decision making. The preferences of agent \( h \) can be represented by a real-valued function defined on the state space.

1 Recently, Fuchs and Laroque [8] obtained local uniqueness and stability of equilibria for a MacKenzie type model using a demand function approach. We study utility functions directly and do not require well defined demand functions and our methods of proofs differ.
i.e., \( u^h: P^n \times S \to R \). In other words, we allow the preference of each agent to depend not only on his own consumptions but also on the consumptions of others and the price systems. We assume \( u^h \in C^1(P^n \times S, R) \) and for every parameter vector \((\tilde{x}^h, p) \in P^{n-1} \times S\), \( u^h(\cdot, \tilde{x}^h, p) \in C^2(P, R) \).\(^2\) For a given parameter vector \((\tilde{x}^h, p)\), we assume further that \( u^h(\cdot, \tilde{x}^h, p) \) satisfies a monotonicity assumption independent of the parameters \((\tilde{x}^h, p)\). We let \( \tilde{x}^h \in P \) be the resource endowment of agent \( h \). Denote \( \mathbf{u} = (u^1, \ldots, u^n) \) and \( \tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n) \in P^n \). An \textit{economy} \( E \) is a list of utility functions (which depend on the state) and resource endowments i.e., \( E = (u, \tilde{x}) \). Formally, we assume that each agent \( h \) in the economy \( E \) satisfies

\[\text{(A.1. Boundary Condition). For any parameter vector } (\tilde{x}^h, p) \in P^n \times S, \quad u^h(\cdot, \tilde{x}^h, p)^{-1}(c) \subseteq P \text{ for every } c \in R, \text{ and}\]

\[\text{(A.2. Monotonicity). } D_nh^h(x, p) \text{ is the derivative with respect to the } h\text{th coordinate of vector } x.\]

Let \( \mathcal{U} = \{u^h \in C^1(P^n \times S, R); u^h(\cdot, \tilde{x}^h, p) \in C^2(P, R) \text{ for every } (\tilde{x}^h, p), \text{ and satisfies (A.1), (A.2)}\} \), \( \mathcal{U} \) is called the \textit{space of utility functions} for every agent \( h \). For a special case of \( \mathcal{U} \), we also consider a subspace of utility functions which possess a convexity property with respect to an agent’s own consumptions, i.e.,

\[\mathcal{U}_o = \{u^h \in \mathcal{U}(P^n \times S, R); D^n_h^2 u^h(x, p) \mid \{v \in R^n; v \cdot D^n_h u^h(x, p) = 0\} \text{ is negative definite for each } (x, p) \in P^n \times S\},\]

where \( D^n_h^2 u^h(x, p) \) is a bilinear symmetric form of \( u^h(x, p) \) with respect to \( x^h \). Since the endowment as well as its distribution of each agent are also allowed to vary in the commodity space \( P \), the economic characteristics of our model are completely specified by the product space \((\mathcal{U} \times P)^n\). Let \( \delta = (\mathcal{U} \times P)^n \) be the \textit{space of economies} and an economy \( E = (u, \tilde{x}) \in \delta \). In particular, \( \delta_0 = (\mathcal{U}_o \times P)^n \) is a space of convex economies. Clearly, \( \delta \) and \( \delta_0 \) are infinite dimensional spaces. We shall consider two different topologies on \( \delta \) for different purposes. For dealing with “generic” properties as we do in the next section, the most useful topology on \( \delta \), which we call the “Whitney” topology is defined by the product of the induced Whitney \( C^1 \) topology on \( \mathcal{U}^n \) and the induced usual topology on \( P^n \), provided the space \( C^2(P, R) \) is endowed with the Whitney \( C^2 \) topology. Toward proving the existence theorem, the “compact-open” topology on \( \delta \) is then defined by replacing the Whitney \( C^1 \) topology and Whitney \( C^2 \)

\(^2\) \( C^k(X, Y) \) denotes the space of \( k \) times continuously differentiable functions from a topological space \( X \) to a topological space \( Y \).

\(^3\) \( \overline{u^h(\cdot, \tilde{x}^h, p)^{-1}(c)} \) means the closure of the indifference surface \( u^h(\cdot, \tilde{x}^h, p)^{-1}(c) \) in \( P \). Loosely speaking, A.1 states that the indifference surfaces never intersect the boundary of commodity space and A.2 claims that every commodity is desired by every agent for any parameter vector.
topology by the $C^1$ compact-open topology and the $C^2$ compact-open topology on $C^1(P^n \times S, R)$ and $C^2(P, R)$, respectively.\footnote{For a definition of Whitney $C^k$ topology and $C^k$ compact-open topology on $C^k(X, Y)$, see Hirsch \cite{9} and Smale \cite{20}.}

For any economy $E = (u, \bar{x}) \in \mathcal{E}$, the budget set of agent $h$ at a prevailing price system $p \in S$ is denoted as usual by $B^h(p, \bar{x}^h) = \{x^h \in P: p \cdot x^h = p \cdot \bar{x}^h\}$. Now we are in a position to define two concepts of equilibrium. For every economy $E \in \mathcal{E}$, a classical equilibrium is a state $(x, p)$ with $\sum_{h=1}^{n} x^h = \sum_{h=1}^{n} \bar{x}^h$ and $x^h$ is a maximal point of $u^h(\cdot, \bar{x}^h, p)$ restricted to the budget set $B^h(p, \bar{x}^h)$ for every $h$. Given $(\bar{x}^h, p)$, a maximal point of $u^h(\cdot, \bar{x}^h, p)$ restricted to $B^h(p, \bar{x}^h)$ is also a critical point\footnote{If $f: X \rightarrow Y$ is class $C^1$, a point $x \in X$ is a regular point of $f$ if $Df(x): T_xX \rightarrow T_yY$ is surjective with $y = f(x)$ where $Df(x)$ represents the derivative of the map $f$ computed at $x$, which is a linear map from the tangent space of $X$ at $x$ to the tangent space of $Y$ at $y$, denoted by $T_xX$ and $T_yY$, respectively. If $Df(x)$ is not surjective, $x$ is a critical point of $f$. $y$ is called a regular value if every $x \in f^{-1}(y)$ is a regular point. $y$ is a critical value if at least one $x \in f^{-1}(y)$ is a critical point.} of it. For a given parameter vector $(\bar{x}^h, p)$, the condition for $x^h$ to be a critical point of $u^h(\cdot, \bar{x}^h, p)$ restricted to $B^h(p, \bar{x}^h)$ can be written as $D_h u^h(x, p) = \lambda^h p$ where $\lambda^h$ is the Lagrangian multiplier of $h$. To avoid $\lambda^h$ in the model, we substitute $\lambda^h = \| D_h u^h(x, p) \|$ where

$$\| D_h u^h(x, p) \| = \sum_{i=1}^{l} \frac{\partial u^h}{\partial x^h_i} (x, p).$$

It is obvious that $\lambda^h > 0$ by A.2. We formally define the set of classical equilibria for $E \in \mathcal{E}$ as

$$W(E) = \left\{ (x, p) \in P^n \times S: u^h(x, p) \text{ is maximized, } px^h = p\bar{x}^h, h = 1, \ldots, n, \text{ and } \sum_{h=1}^{n} x^h = \sum_{h=1}^{n} \bar{x}^h \right\},$$

and the set of extended equilibria\footnote{This concept was first used by Smale in \cite{20} for a different model. From an economic viewpoint, this concept is of little interest. However, it is useful since it can be used as a tool to derive economically interesting results for properties of classical equilibria.} for $E \in \mathcal{E}$ as

$$\Phi(E) = \left\{ (x, p) \in P^n \times S: D_h u^h(x, p) = \| D_h u^h(x, p) \| \cdot p, px^h = p\bar{x}^h, h = 1, \ldots, n, \text{ and } \sum_{h=1}^{n} x^h = \sum_{h=1}^{n} \bar{x}^h \right\}.$$
Since the condition $px^n = \rho \bar{x}^n$ can be obtained from $px^h = \rho \bar{x}^h$, $h = 1, \ldots, n - 1$, and $\sum_{h=1}^{n} x^h = \sum_{h=1}^{n} \bar{x}^h$, the set of extended equilibria for every $E \in \mathcal{E}$ can be rewritten as

$$\Phi(E) = \left\{(x, p) \in P^\times \times S : D_h u^h(x, p) = D_h u^h(x, p) \cdot p, h = 1, \ldots, n, px^h = \rho \bar{x}^h \right\}.$$  

For every economy $E = (u, \bar{x}) \in \mathcal{E}$, we define a map $\psi_E : P^n \times S \to I^n \times R^{n+1}$ by

$$\psi_E(x, p) = \left(D_h u^h(x, p) - D_h u^h(x, p) \cdot p, h = 1, \ldots, n, px^h - \rho \bar{x}^h \right),$$

where

$$I = \left\{z \in \mathbb{R}^l : \sum_{i=1}^{l} z_i = 0 \right\}.$$  

Obviously, $\psi_E \in C^1(P^n \times S, I^n \times R^{n+1})$ since for every $h$, $u^h \in C^1(P^n \times S, R)$ and $u^h(-, \bar{x}^h, p) \in C^2(P, R)$ for every $(\bar{x}^h, p)$. By definition of $\Phi(E)$, we have $\Phi(E) = \psi^{-1}_E(0)$ and $W(E) \subset \psi^{-1}_E(0)$ for every $E \in \mathcal{E}$. That is, if $(x, p)$ is a classical equilibrium, it is an extended equilibrium, and the $C^1$ map $\psi_E$ vanishes at $(x, p)$. It is clear that $W(E) = \psi^{-1}_E(0)$ for every convex economy, i.e., $E \in \mathcal{E}_0$. Furthermore, $\Phi(E)$ is closed in $P^n \times S$ since $\Phi(E) = \psi^{-1}_E(0)$ and $\psi_E$ is $C^1$. By the boundary condition A.1 and monotonicity assumption A.2, we have the following

**Proposition 1.** $\Phi(E)$ is a compact subset in $P^n \times S$ for every $E \in \mathcal{E}$.  

### III. Local Uniqueness and Continuity of Equilibria

In this section we prove local uniqueness and continuity of extended and classical equilibria for "almost all" economies in $\mathcal{E}$, which is defined by a transversality condition on $\psi_E$ below. Actually, we apply the concept of transversality only in the very special sense. That is, $f \in C^1(X, Y)$ is transversal to $y$ denoted $f \not\perp y$ if either $y \neq f(x)$ for all $x$ or $Df(x) \left[T_x X \right] = T_y Y$ for all $x \in f^{-1}(y)$, which is to say that $y$ is a regular value of $f$. In fact, $f$ is regular if and only if $f \not\perp y$ for every $y \in Y$. We need a few more definitions. An element $E \in \mathcal{E}$ is called a regular economy if and only if the associated map $\psi_E$ is transversal to the
origin, i.e., $\psi_E \not\in 0$. Moreover, the space of regular economies is denoted by the set $\mathcal{R} = \{E \in C: \psi_E \not\in 0\}$ and the space of convex regular economies is $\mathcal{R}_0 = \mathcal{R} \cap \mathcal{R}_0$. By a theorem of differential topology (for instance, [1], p. 45), $\psi_E^{-1}(0) = \Phi(E)$ is a $C^1$ submanifold of $P^n \times S$ for every $E \in \mathcal{R}$. In view of the space of economies $\mathcal{E}$, we have

**Proposition 2.** $\mathcal{R}$ is open and dense in $\mathcal{E}$ with respect to the “Whitney” topology defined on $\mathcal{E}$.

**Proof.** Since $P$ and $S$ are locally compact, we let $\{K_n\}$ and $\{L_n\}$ be the sequences of compact subsets in $P$ and $S$ respectively, such that $K_n \subseteq K_{n+1}$, $L_n \subseteq L_{n+1}$ and $P = \bigcup K_n$, $S = \bigcup L_n$. For each $u^h \in C^1(P^n \times S, R)$, let $u^h = u^h | K_n^n \times L_n \in C^1(K_n^n \times L_n, R)$. The spaces $C^1(K_n^n \times L_n, R)$ are Banach spaces (see [1], p. 24). Moreover, they are metrizable and separable, hence they are second countable. It is easy to see that the space $C^1(P^n \times S, R)$ is the inverse limit of the sequence $\{C^1(K_n^n \times L_n, R), f\}$. That is,

$$f_\alpha: C^1(K_\alpha^n \times L_\alpha, R) \rightarrow C^1(K_{\alpha-1}^n \times L_{\alpha-1}, R)$$

defined by $f_\alpha(u^h) = u^h | K_\alpha^n \times L_{\alpha-1}$ is clearly continuous. Define $\mathcal{U}_\alpha = \{u^h \in C^1(K_\alpha^n \times L_\alpha, R): u^h(\cdot, \xi, p) \in C^1(K_\alpha, R) \text{ for every } (\xi, p), \text{ and A.1, A.2 are satisfied} \}$, and $\mathcal{E}_\alpha = (\mathcal{U}_\alpha \times P)^n$. Then $\mathcal{U}$ and $\mathcal{E}$ are the inverse limit spaces of the sequences $\{\mathcal{U}_\alpha, f_\alpha\}$ and $\{\mathcal{E}_\alpha, g_\alpha\}$ respectively, where $f_\alpha = f_\alpha | \mathcal{E}_\alpha$, $g_\alpha: \mathcal{E}_\alpha \rightarrow \mathcal{E}_{\alpha-1}$ defined by

$$g_\alpha = (f_\alpha', \ldots, f_\alpha', \text{id})$$

and id, the identity map, from $P^n$ to $P^n$. Clearly, $\mathcal{E}_\alpha$ is a $C^1$ (Banach) manifold and second countable. Define the sequence $\{\mathcal{R}_\alpha, g_\alpha\}$ as $\mathcal{R}_\alpha = \{E_\alpha \in \mathcal{E}_\alpha: \psi_{E_\alpha} \not\in 0\}$, $g_\alpha = g_\alpha | \mathcal{R}_\alpha$ and $\psi_{E_\alpha} = \psi_E | K_\alpha^n \times L_\alpha$. Then $\mathcal{R}$ is the inverse limit of $\{\mathcal{R}_\alpha, g_\alpha\}$.

We now claim that $\mathcal{R}$ is open and dense in $\mathcal{E}_\alpha$ for each $\alpha$. We apply the Transversal Density Theorem 19.1 of [1], p. 48. Conditions (1), (2) and (3) of 19.1 are satisfied. We need to check condition (4) of Theorem 19.1 of [1]. First, let $\psi_{E_\alpha}: \mathcal{E}_\alpha \times K_\alpha^n \times L_\alpha \rightarrow I^n \times R^{n+1}$ defined by $\psi_{E_\alpha}(x, p) = \psi_{E_\alpha}(x, p)$ for each $E_\alpha \in \mathcal{E}_\alpha$ and $(x, p) \in K_\alpha^n \times L_\alpha$ be the evaluation map of $\psi_{E_\alpha}$. It is clear that $\psi_{E_\alpha}$ is

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7 The density and openness of $\mathcal{R}$ in $\mathcal{E}$ implies that any economy can be approximated by a regular economy and any regular economy is still regular under small perturbations of economic data in the model.

8 Let $X_\alpha$ be a topological space and $f_\alpha$ be a continuous map from $X_\alpha$ into $X_{\alpha-1}$, for each $\alpha$. The sequence $\{X_\alpha, f_\alpha\}$ is called an inverse limit sequence. The inverse limit space of the sequence $\{X_\alpha, f_\alpha\}$ is the following subset of $\Pi_\alpha X_\alpha: X = \{x \in \Pi_\alpha X_\alpha: f_\alpha(x_\alpha) = x_{\alpha-1} \text{ for each } \alpha \text{ and } x_\alpha \in X_\alpha, x_{\alpha-1} \in X_{\alpha-1}\}$ (see [23] for its formal definition and properties).
C^1 (for instance, see [1], p. 25). We go on to prove that $\psi_E \neq 0$. By definition, $\psi_a : \mathcal{E}_a \times K_a^n \times L_a \to I^n \times R^{n+1-1}$ is given by

$$
\psi_a(E_a, x, p) = \left(D_h u^a_h(x, p) - D_h u^a_h(x, p)\right) \cdot p, h = 1, ..., n; p = x^h - x^h,
$$

$h = 1, ..., n - 1; \sum_{h=1}^n \bar{x}^h - \sum_{h=1}^n x^h$.

Its derivative

$$D\psi_a(E_a, x, p) : T(E_a, x, p)(\mathcal{E}_a \times K_a^n \times L_a) \to T_{\phi\alpha(E_a, x, p)}(I^n \times R^{n+1-1})$$

at $(E_a, x, p)$ is defined by

$$D\psi_a(E_a, x, p) (E_a, \dot{x}, \dot{p})$$

$$= \left(\frac{\partial^2 u^a_h}{\partial x^h \partial E_a} \dot{E}_a - \sum_{i=1}^l \frac{\partial^2 u^a_h}{\partial x^h \partial p_i} \dot{p}_r E_a\right)$$

$$+ \sum_{k=1}^n \sum_{j=1}^l \frac{\partial^2 u^a_h}{\partial x^h \partial x^k} \dot{x}^j - \sum_{k=1}^n \sum_{j=1}^l \frac{\partial^2 u^a_h}{\partial x^h \partial x^k} \dot{p}_r \dot{x}^j,$n

$$+ \sum_{r=1}^{l} \frac{\partial^2 u^a}{\partial x^h \partial p} \dot{p}_r - \sum_{r=1}^{l} \frac{\partial^2 u^a}{\partial x^h \partial p} \dot{p}_r \dot{p},$$

$$= \sum_{i=1}^l \frac{\partial u^a_h}{\partial x^h} \dot{p}_r, r = 1, ..., l, h = 1, ..., n.$$

$$\dot{p}(\bar{x}^h - x^h) + \dot{p}(\dot{x}^h - x^h), h = 1, ..., n - 1; \sum_{h=1}^n \ddot{x}^h - \sum_{h=1}^n x^h)$$

where

$$(E_a, \dot{x}, \dot{p}) \in T(E_a, x, p)(\mathcal{E}_a \times K_a^n \times L_a)$$

and $\dot{E}_a = (\dot{u}_a, \dot{x})$. Without loss of generality, we take $\dot{E}_a = (0, \dot{x})$ and $\dot{x} = 0$, then

$$D\psi_a(E_a, x, p) ((0, \dot{x}), 0, \dot{p})$$

$$= \left(\sum_{r=1}^{l} \frac{\partial^2 u^a_h}{\partial x^h \partial p_r} \dot{p}_r - \sum_{r=1}^{l} \frac{\partial^2 u^a_h}{\partial x^h \partial p_r} \dot{p}_r \dot{p},$$

$$+ \sum_{r=1}^{l} \frac{\partial u^a}{\partial x^h} \dot{p}_r, r = 1, ..., l, h = 1, ..., n;$$

$$\dot{p}(\bar{x}^h - x^h) + \dot{p}(\dot{x}^h - x^h), h = 1, ..., n - 1; \sum_{h=1}^n \ddot{x}^h - \sum_{h=1}^n x^h)$$

For each $(a, b, c) \in T_{\phi\alpha(E_a, x, p)}(I^n \times R^{n+1-1})$ with $a = (a^1, ..., a^n) \in T_{\phi\alpha(E_a, x, p)}(I^n)$,
Let \( r_\alpha: \mathcal{E} \to \mathcal{E}_\alpha \) for every \( \alpha \) be the canonical restriction maps. To prove that \( \mathcal{R} \) is open and dense in \( \mathcal{E} \) with respect to the “Whitney” topology, we first claim that \( r_\alpha^{-1}(\mathcal{R}) \) is dense in \( \mathcal{E}_\alpha \) with respect to the “Whitney” topology. We note that \( r_\alpha \) is not an open map with respect to the “Whitney” topology on \( \mathcal{E} \). But in fact, we do not need the openness of \( r_\alpha \), and it would suffice if we know that the image of an open set of \( \mathcal{E}_\alpha \) under \( r_\alpha \) contains an open set of \( \mathcal{E}_\alpha \). Let \( N(E) = N^\theta(u) \times N(\bar{x}) \) be a neighborhood of \( E = (u, \bar{x}) \) in \( \mathcal{E} \) with respect to the “Whitney” topology, where \( N(\bar{x}) \) is an usual neighborhood of \( \bar{x} \) in \( P \) and \( N^\theta(u) = \{ u' \in \mathcal{U}_n: \| D^h u^h(x, p) - D^h u^h(x, p) \| < \epsilon^h(x, p) \} \) for all \( (x, p) \in P^n \times S \), \( k = 0, 1 \) and \( h = 1, \ldots, n \) with \( \epsilon^h: P^n \times S \to R \) being a positive continuous function for each \( h \). As we discuss earlier, \( r_\alpha(N(E)) \subset \mathcal{E}_\alpha \) is not an open set in general. However, if we shrink \( N(E) \) to a neighborhood \( N^*(E) = N^\theta(u) \times N(\bar{x}) \) with \( \delta^h \leq \epsilon^h \) and \( \delta^h: P^n \times S \to R \) is a positive continuous function and increasing with respect to \( x^h \in P \) for every \( h \), it can be obvious that for every \( E' = (u'_\alpha, \bar{x}') \in N^*(E) = N^\theta(u_\alpha) \times N(\bar{x}) \), \( u^h \) can be extended to a function \( u^h \in \mathcal{U}_n \) with \( u^h \mid K_\alpha \times L_\alpha \) for every \( h \), where \( N^\theta(u_\alpha) = \{ u' \in \mathcal{U}_n: \| D^h u^h(x, p) - D^h u^h(x, p) \| < \delta^h(x, p) \} \) for every \( (x, p) \in K_\alpha \times L_\alpha \), \( h = 0, 1 \) and \( h = 1, \ldots, n \). Hence \( r_\alpha(N^*(E)) = N^\theta(E_\alpha) \) is open in \( \mathcal{E}_\alpha \), and consequently \( r_\alpha(N(E)) \) contains an open set. Together with the fact that \( \mathcal{R}_\alpha \) is dense in \( \mathcal{E}_\alpha \), we have \( r_\alpha(N(E)) \cap \mathcal{R}_\alpha \neq \phi \). This means that there exists an \( E' \in N(E) \) such that \( r_\alpha(E') \in \mathcal{R}_\alpha \). Hence \( N(E) \cap r_\alpha^{-1}(\mathcal{R}) \neq \phi \), or equivalently \( r_\alpha^{-1}(\mathcal{R}) \) is dense in \( \mathcal{E} \). By definition, \( \mathcal{E} = \bigcap_\alpha r_\alpha^{-1}(\mathcal{R}) \). Therefore, \( \mathcal{E} \) is dense in \( \mathcal{E} \) with respect to the “Whitney” topology since \( \mathcal{E} \) is a Baire space. Moreover, if \( E \in \mathcal{R} \), by definition, \( E_\alpha \in \mathcal{R}_\alpha \) with \( E_\alpha \mid L_\alpha \) for each \( \alpha \). Since \( \mathcal{R}_\alpha \) is open in \( \mathcal{E}_\alpha \), there exists a neighborhood \( N_\alpha(E_\alpha) = N^\theta(u_\alpha) \times N(\bar{x}) \) of \( E_\alpha \in \mathcal{E}_\alpha \) with \( N_\alpha(E_\alpha) \subset \mathcal{R}_\alpha \). In particular, \( N^\theta(u_\alpha) = \{ u' \in \mathcal{U}_n: \| D^h u^h(x, p) - D^h u^h(x, p) \| < \epsilon^h(x, p) \} \) for every \( (x, p) \in K_\alpha \times L_\alpha \), \( h = 0, 1 \) and \( h = 1, \ldots, n \), where \( \epsilon^h: K_\alpha \times L_\alpha \to R \) is a positive continuous function for every \( h \). We now choose a positive continuous function \( \delta^h: P^n \times S \to R \) with \( \delta^h(x, p) \leq \epsilon^h(x, p) \) for every \( (x, p) \in K_\alpha \times L_\alpha \) and all \( \alpha \). Then \( N^*(E) = N^\theta(u) \times N(\bar{x}) \) is a neighborhood of \( E \) in \( \mathcal{E} \) and \( N^*(E) \subset \mathcal{R} \). Hence the openness of \( \mathcal{R} \) in \( \mathcal{E} \) follows with respect to the “Whitney” topology. Q.E.D.
For every regular economy $E \in \mathcal{R}$, we have $\psi_E \not\equiv 0$. By the openness property of $\mathcal{R}$, $\psi_{E'} \not\equiv 0$ for $E' \in \mathcal{R}$ sufficiently near $E$. One might expect that for $E''$ near $E$, $\psi_{E''}(0)$ and $\psi_{E'}(0)$ are close to each other. In other words, we have

**Theorem 1.** The extended equilibrium correspondence $\Phi$ defined by $\Phi(E) = \psi_{E}(O)$ for every $E \in \mathcal{R}$ is continuous, i.e., it is stable for every $E \in \mathcal{R}$, with respect to the “Whitney” topology.

**Proof.** We know that $\psi_\alpha$ is $C^1$. Moreover, for every $E_\alpha \in \mathcal{R}$, $\psi_{E_\alpha}$ is a $C^1$ local diffeomorphism by the inverse function theorem since $D\psi_{E_\alpha}(x, p): T_{(x, p)}(K_{E_\alpha}^n \times L_\alpha) \to T_{\phi_{E_\alpha}(x, p)}(I^n \times R^{n+1-1})$ with $(x, p) \in \psi_{E_\alpha}(0)$ is an isomorphism (see [16]). Hence, the stability property of the map $\Phi_\alpha = \Phi \rightarrow P^n \times S$ follows from an application of the implicit function theorem on the evaluation map $\psi_\alpha$. That is, there exist neighborhoods $N_\alpha(E_\alpha)$ of $E_\alpha \in \mathcal{R}$ and $V$ of $(x, p) \in K_{E_\alpha}^n \times L_\alpha \subset P^n \times S$, and a $C^1$ function $\xi_\alpha: N_\alpha(E_\alpha) \rightarrow V$ such that $\psi_\alpha(E_\alpha', \xi_\alpha(E_\alpha')) = 0$ for every $E_\alpha' \in N_\alpha(E_\alpha)$ and $\xi_\alpha(E_\alpha) = (x, p)$. Since $\Phi_{\alpha-1}(E_\alpha) \subset \Phi(E_\alpha)$ for every $\alpha$, we have the following diagram

\[
\begin{array}{ccc}
N_{\alpha-1}(E_{\alpha-1}) & \xleftarrow{g_{\alpha-1}'} & N_\alpha(E_\alpha) \\
\downarrow \xi_{\alpha-1} & & \downarrow \xi_\alpha \\
V & \xleftarrow{id} & V
\end{array}
\]

which is commutative, i.e., $\xi_{\alpha-1} \circ g_{\alpha-1}'|N_\alpha(E_\alpha) = id \circ \xi_\alpha$ for every $\alpha$. This implies that for each $E \in \mathcal{R}$ there is a continuous function $\xi: N^*(E) \rightarrow V$ such that $\psi(E', \xi(E')) = 0$ for every $E' \in N^*(E)$ and $\xi(E) = (x, p)$, where $N^*(E)$ is a neighborhood as described in the proof of Proposition 2. Hence the extended equilibrium correspondence $\Phi$ is stable for every $E \in \mathcal{R}$ with respect to the “Whitney” topology. Q.E.D.

**Corollary 1.** The classical equilibrium correspondence defined on the space of convex regular economies is continuous. That is, $W(E)$ is stable for every $E \in \mathcal{R}_0$ with respect to the “Whitney” topology.

As an application of Theorem 1 and Corollary 1, we note that the space of exchange economies without externalities described in [20] appears as a subset of $\mathcal{E}_0$, the space of economies with externalities. That is, let $\mathcal{E}_1$ denote the space of economies without external effects, so $\mathcal{E}_1 \subset \mathcal{E}$ since utility functions for every agent $h$, $u^h: P^n \times S \to R$ are constant along $P^{n-1} \times S$. Given $E \in \mathcal{R}_1 \subset \mathcal{E}_1$, a regular economy without external effects, and a family of regular economies with externalities $\{E^q\}$ such that $E^q$ converges to $E$, we have by continuity or stability of $\Phi$ defined on $\mathcal{R}$, $\Phi(E^q)$ converges to $\Phi(E)$ continuously, which is the equilibrium set of an exchange economy without external effects. This asserts the continuity of extended equilibria for economies with vanishing external
effects. By the same argument applied on $W$, one gets the continuity of classical equilibria for convex economies with vanishing external effects (see [8] and compare).

Next, we prove local uniqueness of the equilibria for an open and dense subset $\mathcal{A}$ of the space of all economies $\mathcal{E}$ with respect to the "Whitney" topology.

**Theorem 2.** For every regular economy $E = (u, \bar{x}) \in \mathcal{A}$, the extended equilibrium set $Q(E)$ is a finite set.

**Proof.** Since $Q(E) = \psi^{-1}_E(0)$ is compact for every $E \in \mathcal{E}$ by Proposition 1, and $\psi^{-1}_E(0)$ is a submanifold with zero dimension if $E \in \mathcal{A}$, we have $Q(E)$ is a finite set. Q.E.D.

**Corollary 2.** For every regular economy $E \in \mathcal{A}$, the classical equilibrium set $W(E)$ is also a finite set.

**Remark 1.** As in [20] the local uniqueness and stability of equilibria can be obtained under weaker conditions. In particular, there is no need to assume boundary condition A.1. Proposition 1 now is not true, but still $\Phi(E)$ is closed in $P^n \times S$. Hence, $\Phi(E)$ and $W(E)$ are locally unique for every $E \in \mathcal{A}$. In other words, for every $E \in \mathcal{A}$, $\Phi(E)$ and $W(E)$ are discrete sets in $P^n \times S$. Finiteness is a fairly strong conclusion which follows from a boundary condition imposed on the commodity space for every agent in the economy.

### IV. Existence of Equilibrium

Although the number of extended or classical equilibria for every regular economy $E$ is finite, it is possible that $\Phi(E)$ or $W(E)$ is an empty set. To show $\Phi(E) \neq \emptyset$ and $W(E) \neq \emptyset$, we first prove the following.

**Proposition 3.** There exists a regular convex economy which has unique equilibrium.

**Proof.** We prove this proposition by considering a nonempty subset of $\mathcal{U}_a$ for each agent, which contains additive separable utility functions with respect to $x^1, ..., x^n$ and $p$, denoted by $\mathcal{U}_{S_0} \subseteq \mathcal{U}_a \subseteq \mathcal{U}$. Define $\mathcal{E}_{S_0} = (\mathcal{U}_{S_0} \times P)^n$, then $\mathcal{E}_{S_0} \subseteq \mathcal{E}_0 \subseteq \mathcal{E}$. For an $E = (u, \bar{x}) \in \mathcal{E}_{S_0}$, let $\bar{x}$ be an equilibrium allocation (this is possible if we choose $E = (u, \bar{x})$ with $u^1 = \cdots = u^n, \bar{x}^1 = \cdots = \bar{x}^n$). Then, by the continuous differentiability and monotonicity of $u^h$ for every $h$, there exists a unique $p^* \in S$ such that $\psi_0(\bar{x}, p^*) = 0$. In particular,

$$D_h u^h(\bar{x}, p^*) = |D_h u^h(\bar{x}, p^*)| \cdot p^*$$

for every agent $h$. Since $u^h \in \mathcal{U}_{S_0}$, by a well known result of consumer theory on
convex preferences (for instance, see [17]), $p^*x^h > p^*\bar{x}^h$ for every $h$ with $x^h \neq \bar{x}^h$ and $\psi_E(x, p^*) = 0$. This is a selfcontradiction. Hence $(\bar{x}, p^*)$ is a unique equilibrium for $E$. Furthermore, the derivative matrix of $\psi_E$ has rank $ln + l - 1$ at $(\bar{x}, p^*)$. This follows from the fact that for each agent $h$, $D_hu^h(\bar{x}, p^*) > 0$ and $D_h^2u^h(\bar{x}, p^*)$ as a bilinear symmetric form on the space $\{v \in R^l: v \cdot D_hu^h(\bar{x}, p^*) = 0\}$ is negative definite. Hence $E = (u, \bar{x}) \in \mathcal{P}$.

**Theorem 3.** There exists extended equilibrium for every economy, i.e., $\Phi(E) \neq \emptyset$ for all $E \in \mathcal{E}$.

**Proof.** First, we check $\mathcal{E}$ is arcwise connected. Let $E, E' \in \mathcal{E}$, we construct $E' \cdot tE \cdot (1 - t) E'$ for $t \in [0, 1]$, i.e., $E' = (u', \bar{x}') = (tu + (1 - t) u', t\bar{x} + (1 - t) \bar{x}')$. By the "compact-open" topology given on $\mathcal{E}$, $\psi^{ht} \in C^1(P^n \times S, R)$, $\bar{x}^{ht} \in P$ for every $h$. Moreover, $u^{ht}$ satisfies A.1 and A.2. Thus $E' = (u', \bar{x}') \in \mathcal{E}$.

From Proposition 1, we have the extended equilibrium manifold $\psi_E^{-1}(0)$ is compact in $P^n \times S$. Therefore, the Brouwer degree is defined (see [16]). If $E \in \mathcal{P}$, the degree of the map $\psi_E$ is equal to the algebraic sum of the orientations (see [16]) of the elements of $\psi_E^{-1}(0)$. Let $\deg \psi_E$ denote the degree of map $\psi_E$.

By Proposition 3, there exists $E \in \mathcal{P}_0 \subset \mathcal{E}$, $\deg \psi_E$ is one. Finally, the Brouwer degree is a homotopy invariant, so that $\deg \psi_E$ is one for every $E \in \mathcal{E}$. This implies that $\Phi(E) = \psi_E^{-1}(0) \neq \emptyset$ for every $E \in \mathcal{E}$. Q.E.D.

**Corollary 3.** For every convex economy there is a classical equilibrium, i.e., $W(E) = \psi_E^{-1}(0)$ for all $E \in \mathcal{E}_0$.

**Proof.** It follows directly from $W(E) = \psi_E^{-1}(0)$ for every $E \in \mathcal{E}_0$. Q.E.D.

**Remark 2.** It is obvious from the definition of Brouwer degree, $\Phi(E)$ and $W(E)$ have an odd number of elements for every $E \in \mathcal{P}$ and $E \in \mathcal{E}_0$, respectively. In particular, if the sign of the determinant of the non-singular matrix of the derivatives of $\psi_E$ at $(x, p)$ with rank $ln + l - 1$ were constant for every $(x, p) \in \psi_E^{-1}(0)$, there is only one extended or classical equilibrium for $E$ in $\mathcal{E}$ or $\mathcal{E}_0$.

**References**