Compact differences of composition operators

Jennifer Moorhouse

Department of Mathematics, Colgate University, Hamilton, NY 13346, USA

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Abstract

A characterization of compact difference is given for composition operators acting on the standard weighted Bergman spaces and necessary conditions are given on a larger scale of weighted Dirichlet spaces. Conditions are given under which a composition operator can be written as a finite sum of composition operators modulo the compacts. The additive structure of the space of composition operators modulo the compact operators is investigated further and a sufficient condition is given to insure that two composition operators lie in the same component. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

Let \( \phi \) be an analytic map from the open unit disk \( D \) to itself, then \( \phi \) induces a linear operator \( C_\phi \) via composition; in other words

\[
C_\phi f(z) = f \circ \phi(z).
\]

For the purposes of this paper, we limit our analysis to composition operators acting on the standard weighted Bergman spaces \( A^2_\alpha \) and the standard scale of weighted Dirichlet spaces \( D_\beta \). Definitions, along with some necessary background material, follow in...
Section 2, but first we review some of the history leading up to and motivating the current work.

Much effort has been expended on characterizing those analytic maps \( \varphi \) which induce compact composition operators. Early results of Shapiro and Taylor [22] in 1973 include a necessary condition for the compactness of a composition operator on \( H^2 \), namely that the inducing function not have angular derivative at any point of the boundary of the unit disk. Carl Cowen [5] carried on the investigation, giving essential norm estimates and calculating essential spectra for certain “nice” composition operators on the Hardy space. Finally, MacCluer [13] brought Carleson measure conditions to bear in the study of composition operators on \( H^p(\mathbb{B}^N) \), the Hardy spaces of the unit ball of \( \mathbb{C}^N \). Using these Carleson measure techniques, MacCluer and Shapiro [16] proved the Shapiro–Taylor result in the more general setting of the weighted Dirichlet spaces, \( D^\beta \), \( \beta > 0 \), and showed that, for composition operators acting on \( A^p \), the existence of the angular derivative for the inducing function is equivalent to non-compactness of the composition operator. Then in 1987, with the use of the Nevanlinna counting function, Shapiro [19] gave a characterization of those \( \varphi \) which induce compact composition operators on the Hardy space \( H^2 \), explicitly calculating the essential norm.

Another area of particular interest is the topological structure of the space of composition operators. When \( X \) is a Banach space of analytic functions, we write \( C(X) \) for the space of composition operators on \( X \) under the operator norm topology. In 1981, Berkson [2] focused attention on topological structure with his isolation results on composition operators acting on \( H^p \). In 1989, MacCluer [14] showed that, on \( D^\alpha \) for \( \alpha \geq 1 \), the compact composition operators form an arcwise connected set in \( C(D^\alpha) \) and gave necessary conditions for two composition operators to have compact difference. At about the same time Shapiro and Sundberg [21] gave further results on compact difference and isolation and, among other things, posed the question for \( C(H^2) \), do the composition operators that differ from \( C_\varphi \) by a compact operator form the component of \( C_\varphi \)? An example which answers this question in the negative was recently given by Moorhouse and Toews [18]. Independently Paul Bourdon [3] showed that two linear fractional self-maps of the disk having the same first-order data at a point \( \zeta \) on the boundary of the disk and different second derivatives at \( \zeta \) lie in the same component of \( C(H^2) \), while the induced composition operators do not have compact difference, thus providing a whole class of examples.

Although no characterization of compact difference on the Hardy and Bergman spaces had been found, MacCluer et al. [15] used the pseudo-hyperbolic metric to give equivalent conditions for compactness of composition operators acting on \( H^\infty \). These results were extended to the setting of \( H^\infty(\mathbb{B}^N) \) by Toews [24] and independently by Gorkin et al. [8]. Building on this foundation, this paper answers the question of compact difference for composition operators acting on \( A^2_\alpha \), \( \alpha > -1 \), and gives a partial answer to the component structure of \( C(A^2_\alpha) \).

In Section 3, we show that if the pseudo-hyperbolic distance between the image values \( \varphi(z) \) and \( \psi(z) \) converges to zero as \( z \to \zeta \) for every point \( \zeta \) at which \( \varphi \) and \( \psi \) have finite angular derivative then the difference \( C_\varphi - C_\psi \) yields a compact operator. More precisely, if \( \rho(z) := \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} \) so that \( \rho(z) \) is the pseudo-hyperbolic distance
between \( \varphi(z) \) and \( \psi(z) \), then \( C_\varphi - C_\psi \) is compact if and only if

\[
\lim_{|z| \to 1} \rho(z) \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) = 0.
\]

This result is extended, in Section 4, to give conditions under which a single composition operator can be written as a finite sum of composition operators modulo the compact operators. In Section 5, we investigate the role of the second derivative in determining compact difference, paralleling independent work of Bourdon [3] and Bourdon et al. [4] for those composition operators on the Hardy space induced by “almost linear fractional” maps. We take this idea a step further, giving a method for decomposing a special sub-class of composition operators into sums of linear fractional composition operators modulo the ideal of compact operators. Finally, in Section 6, two composition operators are seen to lie in the same component of \( \mathcal{C}(A^2_\beta) \) under the operator norm topology if the pseudo-hyperbolic distance is uniformly bounded away from 1 on the unit disk.

2. Background and notation

Recall that the Hardy space, denoted \( H^2 \), is the set of functions \( f \) analytic on the unit disk, satisfying the norm condition

\[
\|f\|_{H^2}^2 := \lim_{r \to 1} \int \mathbb{D} |f(r\zeta)|^2 d\sigma(\zeta) < \infty,
\]

where \( \sigma \) is normalized Lebesgue measure on the boundary of the disk. For \( z > -1 \), the standard weighted Bergman space, \( A^2_z \), is the set of analytic functions on the disk with

\[
\|f\|_{A^2_z}^2 := \int \mathbb{D} |f(z)|^2 d\lambda_z(z) < \infty,
\]

where \( d\lambda_z(z) = \frac{(z+1)}{\pi} (1 - |z|^2)^{\frac{3}{2}} dA(z) \) and \( A \) is area measure on the unit disk. These spaces are all reproducing kernel Hilbert spaces: Evaluation at a point \( w \) in the disk is given by inner product with the reproducing kernel function at \( w \), denoted \( K_w \); that is to say \( f(w) = \langle f, K_w \rangle \). In the Hardy space \( H^2 \), we have reproducing kernels \( K_w(z) = \frac{1}{1-w\bar{z}} \), while in the weighted Bergman spaces \( A^2_z \), the reproducing kernels are given by \( K_w(z) = \frac{1}{(1-w\bar{z})^{3/2}} \).

Now for \( \beta > 0 \), we define the weighted Dirichlet space, \( D^\beta \), another reproducing kernel Hilbert space, by its kernels alone, \( K_w(z) = \frac{1}{(1-w\bar{z})^\beta} \). Comparing kernels we see that \( D_1 = H^2 \) while for \( \beta > 1 \) we have \( D^\beta = A^2_{\beta-2} \), so that \( D^\beta \) defines a new space for \( 0 < \beta < 1 \) only. It can be shown (see [6, Section2.1]) that for \( \beta > 1 \), \( D^\beta \) is the set
Theorem 1 (Julia–Carathéodory). For \( \varphi \) an analytic map from the disk to itself the following are equivalent:

1. \( \varphi \) has finite angular derivative at a point \( \zeta \).
2. \( \varphi \) has radial limit of modulus 1 at \( \zeta \) and \( \varphi'(z) \) has a finite non-tangential limit at \( \zeta \).
3. \( d(\zeta) := \lim_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} \) is finite, moreover, \( d(\zeta) = \lim_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} \).

Further, if any of the above conditions is satisfied, we have \( \lim_{z \to \zeta} \varphi'(z) = \varphi'(\zeta) \) and \( d(\zeta) = |\varphi'(\zeta)| \).

With this relationship in mind, we will not differentiate between these conditions, but will use the phrase “\( \varphi \) has finite angular derivative at \( \zeta \)” to mean that \( \varphi \) satisfies any of (1), (2) or (3). If \( \varphi \) and \( \psi \) are two analytic self-maps of the disk with finite angular derivative at \( \zeta \), we will say that \( \varphi \) and \( \psi \) have the same first-order data at \( \zeta \) if \( \varphi(\zeta) = \psi(\zeta) \) and \( \varphi'(\zeta) = \psi'(\zeta) \).

In what follows we make extensive use of Carleson measure techniques, so we give a short introduction to Carleson sets and measures. For a point \( \zeta \) in the boundary of the disk, we define the Carleson set \( S(\zeta, \delta) := \{ z \in D : |\zeta - z| < \delta \} \). Given a positive, finite measure \( \mu \) on the open unit disk, we say that \( \mu \) is an \( \alpha \)–Carleson measure if

\[
\|\mu\|_\alpha := \left[ \sup_{S(\zeta, \delta)} \frac{\mu(S(\zeta, \delta))}{\delta^{2\alpha + 2}} \right]^\frac{1}{2} < \infty, \tag{1}
\]
where the supremum is taken over all $\zeta \in \partial D$ and all $\delta > 0$. If, in addition,

$$\lim_{\delta \to 0} \sup_{\zeta \in \partial D} \frac{\mu(S(\zeta, \delta))}{\delta^{x+2}} = 0$$

then we call $\mu$ a compact $x$-Carleson measure. This notion has applications to the study of composition operators through the following, which was first seen as a Hardy space result due to Carleson and was extended to a variety of spaces by several authors. It is most pertinent to this paper in its incarnation as a theorem on Bergman spaces developed, for the most part, by Hastings [10], Luecking [12], Stegenga [23] and Axler [1] which gives a characterization of those measures on the unit disk, $\mu$, for which the inclusion map from $A^2_\alpha$ into $L^2(\mu)$ is either bounded or compact. For a reference and historical development see [6, Section 2.2].

**Theorem 2.** Fix $\alpha > -1$, and let $\mu$ be a finite positive Borel measure on the open disk, then:

1. $A^2_\alpha \subset L^2(\mu)$ if and only if $\mu$ is an $\alpha$-Carleson measure. In this case the inclusion map $I_\alpha : A^2_\alpha \to L^2(\mu)$ is a bounded linear operator with norm comparable to $\|\mu\|_\alpha$.
2. If $\mu$ is an $\alpha$-Carleson measure, then $I_\alpha$ is compact if and only if $\mu$ is a compact $\alpha$-Carleson measure.

Now, using the measure theoretic change of variables (see [9, Section 39]), one sees that, for $\varphi$ an analytic map of the disk to the disk and $f \in A^2_\alpha$,

$$\|C_\varphi f\|_{A^2_\alpha}^2 = \int_D |f \circ \varphi(z)|^2 d\lambda_\alpha$$

$$= \int_D |f(z)|^2 d\lambda_\alpha \circ \varphi^{-1}$$

$$= \|f\|_{L^2(\lambda_\alpha \circ \varphi^{-1})}^2.$$

Thus, $\|C_\varphi\|_\alpha$, the norm of $C_\varphi$ as an operator from $A^2_\alpha$ to itself, is comparable to $\|\lambda_\alpha \circ \varphi^{-1}\|_\alpha$. Moreover, $C_\varphi$ is compact exactly when $\lambda_\alpha \circ \varphi^{-1}$ is a compact $\alpha$-Carleson measure, and putting these ideas together, MacCluer and Shapiro [16] show:

**Theorem 3.** For $\alpha > -1$, $C_\varphi$ is compact on $A^2_\alpha$ if and only if $\varphi$ has no finite angular derivative at any point $\zeta \in \partial D$.

### 3. Compact difference

As discussed in Section 1, equivalent conditions for compactness of composition operators on $H^\infty$ involving the pseudo-hyperbolic metric are given in [15], and extended to $H^\infty(B_N)$ independently in [8,24]. Thus, we proceed apace with the intuition that
the pseudo-hyperbolic distance is a good measure for characterizing compactness. In this metric, the distance between two points \( u \) and \( v \) in the closed unit disk is given by \( \frac{|u-v|}{1-\bar{u}v} \).

**Theorem 4.** Suppose \( \alpha > -1 \). Let \( \varphi \) and \( \psi \) be analytic maps of the disk to the disk, and define \( \rho(z) = \left| \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)} \psi(z)} \right| \). The following are equivalent:

1. \( \lim_{|z| \to 1} \rho(z) \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) = 0 \).
2. \( C_\varphi - C_\psi \) is compact on the weighted Bergman spaces, \( A_2^\beta \).

In fact, (1) holds when \( C_\varphi - C_\psi \) is compact on any weighted Dirichlet space \( D_\beta \) for \( \beta > 0 \).

The idea for the direction (1) \( \Rightarrow \) (2) will be to break up the disk and use different Carleson measure conditions to analyze “local” behavior near the points of finite angular derivative separately from the behavior far from the angular derivative point.

The following lemma will allow us to examine the behavior of the individual composition operators away from the points of finite angular derivative. The proof is a slight modification of work of Mirzakerimi and Seddighi [17] who prove a similar result for weighted composition operators acting on weighted Dirichlet spaces, using techniques of MacCluer and Shapiro.

**Lemma 1.** Suppose \( \alpha > -1 \). Let \( \varphi \) be an analytic function mapping the disk to the disk and take \( W \) to be a non-negative, bounded, measurable function on \( D \). Define the measure \( W\lambda_\varphi \) by \( W\lambda_\varphi(E) = \int_E W(z) \, d\lambda_\alpha \) on all Borel subsets \( E \), of \( D \). If

\[
\lim_{|z| \to 1} W(z) \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0,
\]

then \( W\lambda_\varphi \circ \varphi^{-1} \) is a compact \( \alpha \)-Carleson measure and hence the inclusion map \( I_\varphi : A_2^\alpha \to L^2(W\lambda_\varphi \circ \varphi^{-1}) \) is compact.

We note here that in the case when \( w \) is an analytic function, bounded on \( D \), letting \( |w|^2 \) play the role of \( W \) and using the ideas outlined in Theorem 3 we have the “if” direction of the following corollary:

**Corollary 1.** Suppose \( \alpha > -1 \). Let \( \varphi \) and \( w \) be analytic functions on the unit disk, with \( w \) bounded and \( \varphi \) mapping the disk to itself. Then the weighted composition operator \( wC_\varphi \) is compact on \( A_2^\alpha \) if and only if

\[
\lim_{|z| \to 1} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.
\]

The “only if” direction is accomplished by evaluating adjoints of weighted composition operators acting on kernel functions which converge weakly to zero.
The computations are a slight modification of those found in the proof of Theorem 4.

**Proof of Lemma 1.** Choose \( \gamma > -1 \) so that \( 0 < \alpha - \gamma \leq 1 \); for ease of notation we write \( \beta = \alpha - \gamma \), and let

\[
\varepsilon_0(\delta) = \sup \left\{ \left( W(z) \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\beta : 1 - |z| < \delta \right\}.
\]

By hypothesis \( \varepsilon_0(\delta) \to 0 \) as \( \delta \to 0 \).

Now for \( \delta > 0 \) let \( S = S(\zeta, \delta) \) be a Carleson set. Using the Schwarz–Pick Theorem (see [6]), one has

\[
\frac{1 - |z|}{1 - |\varphi(z)|} \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} = C < \infty,
\]

so that if \( \varphi(z) \in S \), then

\[
1 - |z| \leq C(1 - |\varphi(z)|) < C\delta.
\]

Thus, for \( z \in \varphi^{-1}(S) \) we have \( 1 - |z| < C\delta \), whence, taking \( M \) to be an upper bound of \( W \),

\[
W(z)(1 - |z|^2)^\beta \leq (1 - |\varphi(z)|^2)^\beta M^{1-\beta} \varepsilon_0(C\delta) \leq \delta^\beta \varepsilon(\delta),
\]

where \( \varepsilon(\delta) = 2^{2\beta} M^{1-\beta} \varepsilon_0(C\delta) \to 0 \) as \( \delta \to 0 \). It then follows that

\[
W\lambda_\alpha \circ \varphi^{-1}(S) = \int_{\varphi^{-1}(S)} W(z) d\lambda_\alpha
\]

\[
= \int_{\varphi^{-1}(S)} W(z)(1 - |z|^2)^\beta (1 - |\varphi(z)|^2)^\gamma dA
\]

\[
\leq \varepsilon(\delta) \delta^\beta \int_{\varphi^{-1}(S)} d\lambda_\gamma
\]

\[
= \varepsilon(\delta) \delta^\beta (\lambda_\gamma \circ \varphi^{-1})(S)
\]

\[
\leq \varepsilon(\delta) \delta^\beta \|\lambda_\gamma \circ \varphi^{-1}\|_\gamma \delta^{\gamma+2}
\]

\[
= \|\lambda_\gamma \circ \varphi^{-1}\|_\gamma \varepsilon(\delta) \delta^{\gamma+2}.
\]

By the statements following Theorem 2 the quantity \( \|\lambda_\gamma \circ \varphi^{-1}\|_\gamma \) is comparable to the norm of \( C_\varphi \) acting on \( A^2_\gamma \), which is known to be bounded. Thus, the measure \( W\lambda_\alpha \circ \varphi^{-1} \)
is a compact \( \alpha \)-Carleson measure and the inclusion map \( I_\alpha : A^2_\alpha \rightarrow L^2(W \lambda_\alpha \circ \varphi^{-1}) \) is compact. \( \square \)

Now, we develop a method for investigating the behavior of \( C_\varphi - C_\psi \) at the points of finite angular derivative.

**Lemma 2.** Suppose that \( \alpha > -1 \). Let \( \varphi \) and \( \psi \) be analytic maps of the disk to the disk and define

\[
\varphi_s(z) = (1 - s) \varphi(z) + s \psi(z) \quad \text{for} \quad 0 \leq s \leq 1,
\]

\[
w(z) = \varphi(z) - \psi(z) \quad \text{and} \quad \rho(z) = \left| \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} \right|.
\]

Let \( E \) be an open set in the disk on which \( \rho(z) \) is bounded away from 1. If \( \chi_E |w|^2 \lambda_\alpha \) is the measure defined on all Borel subsets \( B \) of \( D \) by

\[
\chi_E |w|^2 \lambda_\alpha(B) = \int_B \chi_E(z) |w(z)|^2 \, d\lambda_\alpha,
\]

then \( \chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1} \) is an \( \alpha + 2 \)-Carleson measure for each \( s \), and moreover the inclusion map \( I_{\alpha + 2,s} : A^2_{\alpha + 2} \rightarrow L^2(\chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1}) \) has norm uniformly bounded in \( s \). Further, if

\[
\lim_{|z| \to 1} \chi_E(z) \rho(z) \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right) = 0,
\]

then \( \chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1} \) is a compact \( \alpha + 2 \)-Carleson measure for each \( s \in [0,1] \) and the inclusion map \( I_{\alpha + 2,s} \) is compact.

**Proof.** First, notice that \( \varphi_s(z) \) lies on a straight-line path between \( \varphi(z) \) and \( \psi(z) \), so that

\[
\frac{1}{1 - |\varphi_s(z)|^2} \leq \frac{1}{1 - |\varphi(z)|^2} + \frac{1}{1 - |\psi(z)|^2}.
\]

Thus, the hypotheses imply that

\[
\lim_{|z| \to 1} \chi_E(z) \rho(z) \frac{1 - |z|^2}{1 - |\varphi_s(z)|^2} = 0 \quad (2)
\]

for each \( s \in [0,1] \). Further,

\[
\frac{1 - |\varphi_s(z)|^2}{|1 - \varphi(z)\psi(z)|} = \left| 1 + \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} + \varphi_s(z) \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} \right|
\]

\[
\geq \left| 1 - (1 - s)\rho(z) - s \varphi_s(z) \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} \right|
\]

\[
= 1 - \rho(z).
\]
whence
\[ \chi_E(z)|w(z)| \leq \frac{\chi_E(z)}{1 - \rho(z)} \rho(z)(1 - |\varphi(z)|^2) \]
\[ \leq M \chi_E(z) \rho(z)(1 - |\varphi(z)|^2), \]
where \( M \) is an upper bound for \( \frac{1}{1 - \rho(z)} \) on \( E \).

Now, let \( \zeta \) be any point on the boundary of the disk, \( 0 < \delta \), and define \( S = S(\zeta, \delta) \), a Carleson set, then we have
\[ \chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1}(S) = \int_{\varphi_s^{-1}(S)} \chi_E(z)|w(z)|^2 \, d\lambda_\alpha \]
\[ \leq \int_{\varphi_s^{-1}(S)} M^2 \chi_E(z) \rho^2(z)(1 - |\varphi_s(z)|^2)^2 \, d\lambda_\alpha \]
\[ \leq 4M^2 \delta^2 \int_{\varphi_s^{-1}(S)} \chi_E(z) \rho^2(z) \, d\lambda_\alpha \]
\[ = 4M^2 \delta^2 \chi_E \rho^2 \lambda_\alpha \circ \varphi_s^{-1}(S). \] (3)
The condition in (2) together with Lemma 1 insures that \( \chi_E \rho^2 \lambda_\alpha \circ \varphi_s^{-1} \) is a compact \( \alpha \)-Carleson measure, thus
\[ \frac{\chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1}(S)}{\delta^{\alpha+2}} \leq 4M^2 \chi_E \rho^2 \lambda_\alpha \circ \varphi_s^{-1}(S) \]
\[ \to 0 \] (4)
as \( \delta \to 0 \), so that \( \chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1} \) is a compact \( \alpha + 2 \)-Carleson measure.

It is known that the norm of a composition operator \( C_\varphi \) acting on \( A_\alpha^2 \) is less than some multiple of a power of \( \frac{1}{1 - |\varphi(0)|} \), where the power depends only on \( \alpha \) (see [6, Chapter 3]). Since \( \frac{1}{1 - |\varphi_s(0)|} \leq \frac{1}{1 - |\varphi(0)|} + \frac{1}{1 - |\varphi(0)|} \), the operators \( C_{\varphi_s} \) have uniformly bounded norms on \( A_\alpha^2 \) and, by the statements following Theorem 2, there is a value \( m < \infty \) such that \( \| \lambda_\alpha \circ \varphi_s^{-1} \|_2 \leq m \| C_{\varphi_s} \|_2 \). We can now estimate the quantity in (3) using the fact that \( \rho(z) \leq 1 \) and thus
\[ \int_{\varphi_s^{-1}(S)} \chi_E(z) \rho^2(z) \, d\lambda_\alpha \leq \int_{\varphi_s^{-1}(S)} d\lambda_\alpha = \lambda_\alpha \circ \varphi_s^{-1}(S) \leq m \| C_{\varphi_s} \|_2 \delta^{\alpha+2}. \]
The uniform bound on the norms, \( \| C_{\varphi_s} \|_2 \), together with the inequality in (4) yield the uniform boundedness of the quantities \( \| \chi_E |w|^2 \lambda_\alpha \circ \varphi_s^{-1} \|_{\alpha+2} \) and hence of \( \| I_{\alpha+2} \|_2 \).

Now, using the previous lemmas, we turn our attention back to the question of compact difference and provide a proof of Theorem 4.

**Proof.** Fix \( 0 < r < 1 \) and define the open set \( E := \{ z \in D : \rho(z) < r \} \) and \( E' = D - E \). In this setting we take \( W = \chi_{E'} \) the characteristic function of \( E' \) and
define the measure $\chi_{E'}\lambda_\varphi$ on Borel subsets $B$ of $D$ by $\chi_{E'}\lambda_\varphi(B) = \int_B \chi_{E'}(z) d\lambda_\varphi$ as in Lemma 1.

We note that if $\chi_{E'}(z) \frac{1-|z|^2}{1-|\varphi(z)|^2} > \delta > 0$, then in particular $\chi_{E'}(z) \neq 0$ so that $z$ is in $E'$ and

$$\rho(z) \frac{1-|z|^2}{1-|\varphi(z)|^2} \geq r\delta.$$ 

Thus, the condition in (i) implies that

$$\lim_{|z| \to 1} \frac{\chi_{E'}(z) \frac{1-|z|^2}{1-|\varphi(z)|^2}}{\rho(z) \frac{1-|z|^2}{1-|\varphi(z)|^2}} = 0$$

and similarly

$$\lim_{|z| \to 1} \frac{\chi_{E'}(z) \frac{1-|z|^2}{1-|\psi(z)|^2}}{\rho(z) \frac{1-|z|^2}{1-|\psi(z)|^2}} = 0.$$ 

Therefore, $\chi_{E'}\lambda_\varphi \circ \varphi^{-1}$ and $\chi_{E'}\lambda_\varphi \circ \psi^{-1}$ are each compact $\varphi$-Carleson measures by Lemma 1 and hence the inclusion mappings $I_{\varphi,0} : A_2^2 \to L^2(\chi_{E'}\lambda_\varphi \circ \varphi^{-1})$ and $I_{\psi,1} : A_2^2 \to L^2(\chi_{E'}\lambda_\varphi \circ \psi^{-1})$ are compact.

As in Lemma 2, take $w(z) = \varphi(z) - \psi(z)$, $\varphi_s(z) = s\varphi(z) + (1-s)\psi(z)$ and define the measure $\chi_E|w|^2\lambda_\varphi$ as before. By definition $\rho(z)$ is bounded away from 1 on the set $E$ and thus, by Lemma 2, the inclusion map $I_{\varphi+2,s} : A_2^2 \to L^2(\chi_E|w|^2\lambda_\varphi \circ \varphi^{-1})$ has norm uniformly bounded in $s$ and, moreover, is compact for each $s \in [0, 1]$.

Take $f$ to be an arbitrary function in $A_2^2$ and consider $C_\varphi - C_\psi$ acting on $f$.

$$\| (C_\varphi - C_\psi) f \|^2_{A_2^2}$$

$$= \int_D |f \circ \varphi(z) - f \circ \psi(z)|^2 d\lambda_\varphi$$

$$= \int_{E'} |f \circ \varphi(z) - f \circ \psi(z)|^2 d\lambda_\varphi + \int_E |f \circ \varphi(z) - f \circ \psi(z)|^2 d\lambda_\varphi. \quad (5)$$

We consider the two terms in (5) separately. With an application of Minkowski’s inequality to the first term, we get

$$\int_D |f \circ \varphi(z) - f \circ \psi(z)|^2 \chi_{E'}(z) d\lambda_\varphi(z)$$

$$\leq \left[ \left( \int_D |f \circ \varphi(z)|^2 \chi_{E'}(z) d\lambda_\varphi \right)^{\frac{1}{2}} + \left( \int_D |f \circ \psi(z)|^2 \chi_{E'}(z) d\lambda_\varphi \right)^{\frac{1}{2}} \right]^2$$

$$= (\| f \|_{L^2(\chi_{E'}\lambda_\varphi \circ \varphi^{-1})} + \| f \|_{L^2(\chi_{E'}\lambda_\varphi \circ \psi^{-1})})^2.$$
while the second term in (5) becomes

\[
\int_D |f \circ \varphi(z) - f \circ \psi(z)|^2 \chi_E(z) \, d\lambda_z
\]

\[
= \int_D \left| \int_0^1 w(z) f'(z) \, ds \right|^2 \chi_E(z) \, d\lambda_z
\]

\[
\leq \int_D \int_0^1 |w(z) f'(z)|^2 \, ds \chi_E(z) \, d\lambda_z
\]

\[
= \int_0^1 \int_D |f' \circ \varphi_s(z)|^2 \chi_E(z) |w(z)|^2 \, d\lambda_z \, ds
\]

\[
= \int_0^1 \|f'\|^2_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1}_s)} \, ds.
\]

Thus,

\[
\| (C_\varphi - C_\psi) f \|_{A^2_2}^2 \leq \left( \| f \|_{L^2(\chi_E \lambda_z \circ \varphi^{-1})} + \| f \|_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1})} \right)^2
\]

\[
+ \int_0^1 \|f'\|^2_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1}_s)} \, ds.
\]

Now, let \( \{f_n\} \) be a sequence of unit vectors converging weakly to zero in \( A^2_2 \). Since the inclusion maps \( I_{2,0} \) and \( I_{2,1} \) are compact, we have \( \|f_n\|_{L^2(\chi_E \lambda_z \circ \varphi^{-1})} \to 0 \) and \( \|f_n\|_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1})} \to 0 \). Further, \( \{f'_n\} \) is a bounded sequence in \( A^2_{2,s} \) which also converges weakly to zero, and the uniform boundedness of the norms of the inclusion maps \( I_{2,s+2} \) allows us to apply Dominated Convergence, which together with the compactness of \( I_{2,s+2} \) for individual \( s \) yields:

\[
\lim_{n \to \infty} \int_0^1 \|f'_n\|^2_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1}_s)} \, ds = \int_0^1 \lim_{n \to \infty} \|f'_n\|^2_{L^2(\chi_E |w|^2 \lambda_z \circ \varphi^{-1}_s)} \, ds = 0.
\]

Putting this together, we see that

\[
\| (C_\varphi - C_\psi) f_n \|_{A^2_2} \to 0
\]

and thus \( C_\varphi - C_\psi \) is compact.

We prove the converse, with no extra assumptions, for the weighted Dirichlet spaces \( D_\beta \) with \( \beta > 0 \), which includes both the Hardy space \( H^2 \) and all of the weighted Bergman spaces \( A^2_\alpha \), \( \alpha > -1 \). Fix \( \beta > 0 \) and consider \( C^*_\varphi - C^*_\psi \) acting on reproducing kernel functions \( K_z \). Recall that on the Dirichlet space, \( D_\beta \), \( K_z(w) = \frac{1}{(1-\bar{w}z)^\beta} \). Note that \( \frac{K_z}{\|K_z\|} \to 0 \) weakly as \( |z| \to 1 \). Thus, if there is some sequence \( \{z_n\} \subset D \), with \( |z_n| \to 1 \), along which \( \frac{\|C^*_\varphi - C^*_\psi\| K_z}{\|K_z\|} \to 0 \) then \( C^*_\varphi - C^*_\psi \) and thus \( C_\varphi - C_\psi \) is not compact.
We use the following well-known equality, for which we refer the reader to [7]:

$$
\frac{(1 - |\varphi(z)|^2)(1 - |\psi(z)|^2)}{|1 - \varphi(z)\overline{\psi(z)}|^2} = 1 - \rho^2(z),
$$

(6)

from which it follows that

$$
|\left\langle K_{\varphi(z)}, K_{\psi(z)} \right\rangle| \leq (1 - \rho^2(z))^{\beta} \frac{\|K_{\varphi(z)}\|^2}{\|K_z\|^2} \frac{\|K_{\psi(z)}\|^2}{\|K_z\|^2}.
$$

Thus, taking $u(z) = (1 - \rho^2(z))^{\beta/2}$ we have

$$
\frac{\|(C_{\varphi'} - C_{\psi'})K_z\|^2}{\|K_z\|^2} = \frac{\|K_{\varphi(z)}\|^2}{\|K_z\|^2} - 2\Re\left(\frac{K_{\varphi(z)}}{\|K_z\|}, \frac{K_{\psi(z)}}{\|K_z\|}\right) + \frac{\|K_{\psi(z)}\|^2}{\|K_z\|^2}
\geq \frac{\|K_{\varphi(z)}\|^2}{\|K_z\|^2} - 2u(z) \frac{\|K_{\varphi(z)}\|}{\|K_z\|} \frac{\|K_{\psi(z)}\|}{\|K_z\|} + \frac{\|K_{\psi(z)}\|^2}{\|K_z\|^2}
= \left(\frac{\|K_{\varphi(z)}\| - \|K_{\psi(z)}\|}{\|K_z\|}\right)^2 + 2(1 - u(z)) \frac{\|K_{\varphi(z)}\|}{\|K_z\|} \frac{\|K_{\psi(z)}\|}{\|K_z\|}.
$$

We assume that the condition in (1) does not hold, thus there exists a sequence $\{z_n\} \subset D$ with $|z_n| \to 1$ along which either

$$
a_n = \rho(z_n) \frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2} = \rho(z_n) \frac{\|K_{\varphi(z_n)}\|^2}{\|K_{z_n}\|^2}
$$

or

$$
b_n = \rho(z_n) \frac{1 - |z_n|^2}{1 - |\psi(z_n)|^2} = \rho(z_n) \frac{\|K_{\psi(z_n)}\|^2}{\|K_{z_n}\|^2},
$$

does not converge to zero. By passing to a subsequence, we may assume that $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$ exist and that one is non-zero, by symmetry we may further assume that $a \neq 0$. If $a \neq b$ then

$$
\lim_{n \to \infty} \left(\frac{\|K_{\varphi(z_n)}\| - \|K_{\psi(z_n)}\|}{\|K_{z_n}\|^2}\right)^2 \neq 0.
$$

If $a = b \neq 0$ then, in particular, we may choose $N < \infty$ and $\delta > 0$ such that $\rho^2(z_n) > \delta$ for all $n > N$, and thus, $u(z_n) < (1 - \delta)^{\beta} < 1$. Hence, $(1 - u(z_n))\frac{\|K_{\varphi(z_n)}\|}{\|K_{z_n}\|} \frac{\|K_{\psi(z_n)}\|}{\|K_{z_n}\|}$ is bounded away from zero.
Thus, in either case, we see that

$$\lim_{n \to \infty} \frac{\| (C_{\varphi}^* - C_{\psi}^*) K_{z_n} \|}{\| K_{z_n} \|} \neq 0$$

so that $C_{\varphi} - C_{\psi}$ is not compact. □

4. The sum theorem

So far we have concentrated on the question of compactness of the difference of two composition operators, $C_{\varphi} - C_{\psi}$. Another way to approach this question, is to ask when $C_{\varphi}$ can be written as a sum, $C_{\psi} + K$, where $K$ is a compact operator, in other words, when $C_{\varphi}$ is equivalent to $C_{\psi}$ modulo the compact operators. In this idiom the next natural step is to express $C_{\varphi}$ as a finite sum of composition operators modulo the compacts. The following theorem gives conditions under which this form can be realized.

**Theorem 5.** Suppose $\alpha > -1$. Let $\varphi, \varphi_1, \ldots, \varphi_N$ be analytic maps from the open unit disk $D$ to itself. For each $i = 1, \ldots, N$ let $F_i$ be the set of points on the boundary of the disk at which $\varphi_i$ has finite angular derivative and let $F$ be the angular derivative set for $\varphi$. Suppose that $F_i \cap F_j = \emptyset$ whenever $i \neq j$, and $\bigcup_{i=1}^{N} F_i = F$. Define $\rho_i(z) = \left| \frac{\varphi(z) - \varphi_i(z)}{1 - \varphi(z) \varphi_i(z)} \right|$. If

$$\lim_{z \to \zeta} \rho_i(z) \frac{1 - |z|^2}{1 - |\varphi_i(z)|^2} = 0 \quad (7)$$

and

$$\lim_{z \to \zeta} \rho_i(z) \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0 \quad (8)$$

for every $\zeta \in F_i$, $i = 1, \ldots, N$, then there exists a compact operator $K$ on $A^2_\alpha$ such that

$$C_{\varphi} = C_{\varphi_1} + \ldots + C_{\varphi_N} + K.$$

**Proof.** The proof will be quite similar to the proof of Theorem 4, with a few added complications since we must further subdivide the disk into regions near to those points at which each $\varphi_i$ has angular derivative and regions far from the points of finite angular derivative for each $\varphi_i$. 
Define $D_i := \{z \in D : \frac{1-|z|^2}{1-|\varphi_i(z)|^2} \geq \frac{1-|z|^2}{1-|\varphi_j(z)|^2}, \text{ for all } j \neq i \}$ for $i = 1, \ldots, N$ (notice that this is, in fact, the set on which $|\varphi_j(z)| \geq |\varphi_j(z)|$). Fix $0 < r < 1$ and define $E_i := \{z \in D_i : \rho_i(z) < r\}$ and $E'_i := D_i - E_i$.

For $i = 1, \ldots, N$ we will show the following:

$$\lim_{|z| \to 1} \chi_{E_i}(z) \frac{1-|z|^2}{1-|\varphi_j(z)|^2} = 0 \quad \text{(9)}$$

when $j = 0, 1, \ldots, N$ (for ease of notation we write $\varphi_0 := \varphi$) and

$$\lim_{|z| \to 1} \chi_{E_i}(z) \frac{1-|z|^2}{1-|\varphi_j(z)|^2} = 0 \quad \text{(10)}$$

for $j \neq 0$ and $j \neq i$. Thus, if the measures $\chi_{E_i} \lambda_\alpha$ and $\chi_{E_i} \lambda_\alpha$ are defined as in Lemma 1, then, for $j = 0, 1, \ldots, N$, $\chi_{E_i} \lambda_\alpha \circ \varphi_j^{-1}$ is a compact $\alpha$-Carleson measure and $\chi_{E_i} \lambda_\alpha \circ \varphi_j^{-1}$ is a compact $\alpha$-Carleson measure for $j \neq 0$ and $j \neq i$, and the corresponding inclusion maps are each compact.

To prove our claim, in the case when $j \neq 0$, $j \neq i$ note that for $z \in D_i$ we have

$$\frac{1-|z|^2}{1-|\varphi_j(z)|^2} \geq \frac{1-|z|^2}{1-|\varphi_j(z)|^2}.$$

Thus, if either limit in Eqs. (9) and (10) is non-zero then $\varphi_i$ and $\varphi_j$ each have finite angular derivative at some point $\zeta \in \partial \mathbb{D}$ contradicting the hypothesis that $F_i \cap F_j = \emptyset$. Thus, (9) and (10) hold for $j \neq 0$ and $j \neq i$.

Now, for $z \in E'_i$ we have $\rho_i(z) \geq r$, and therefore, assuming that the limit in Eq. (9) is non-zero contradicts the condition given in (7) if $j = i$ and that of (8) for $j = 0$. Thus, (9) holds for $j = 0$ and $j = i$.

Next, we consider the regions of the disk near to the finite angular derivative set of each $\varphi_i$. By definition $\rho_i(z) < 1$ on the set $E_i$ for each $i$ and the hypothesis in Eq. (7) insures that

$$\lim_{|z| \to 1} \chi_{E_i}(z) \rho_i(z) \frac{1-|z|^2}{1-|\varphi_i(z)|^2} = 0. \quad \text{(11)}$$

We assert further that

$$\lim_{|z| \to 1} \chi_{E_i}(z) \rho_i(z) \frac{1-|z|^2}{1-|\varphi_i(z)|^2} = 0. \quad \text{(12)}$$

Again, we argue by contradiction. If the limit in (12) is non-zero, then there exists an $\varepsilon > 0$ and a path $\gamma \subset E_i$ converging to a point $\zeta \in \partial \mathbb{D}$ on which $\rho_i(z) > \varepsilon$ and

$$\frac{1-|\varphi_i(z)|^2}{1-|\varphi_i(z)|^2} > \varepsilon. \text{ But,}$$

$$\left| \frac{1-\varphi_i(z)}{\varphi_i(z)} \right| = \frac{1-|\varphi_i(z)|^2}{|\varphi_i(z)|^2} + |\varphi_i(z)| \geq \frac{1-|\varphi_i(z)|^2}{|\varphi(z)|^2 - |\varphi_i(z)|^2} - 1$$
and on $\gamma$, $\frac{1}{\rho_i(z)} < \frac{1}{\varepsilon}$ so that

$$\frac{1 - |\phi_i(z)|^2}{|\phi(z) - \phi_i(z)|} < \frac{1}{\varepsilon} + 1 < \infty.$$ 

Thus, since $\frac{1 - |z|^2}{1 - |\phi_i(z)|^2} \to 0$ as $z \to \zeta$, we have

$$\frac{1 - |z|^2}{|\phi(z) - \phi_i(z)|} = \frac{1 - |\phi_i(z)|^2}{|\phi(z) - \phi_i(z)|} \frac{1 - |z|^2}{1 - |\phi_i(z)|^2} \to 0.$$ 

But we can also write $\frac{1}{\rho_i(z)}$ as below

$$\left| \frac{1 - \overline{\phi(z)} \phi_i(z)}{\phi(z) - \phi_i(z)} \right| = \left| \frac{1 - |\phi(z)|^2}{1 - |z|^2} \frac{1 - |z|^2}{\phi(z) - \phi_i(z)} + \frac{\phi(z)}{\phi_i(z)} \right|$$

and $\frac{1 - |\phi(z)|^2}{1 - |z|^2}$ is bounded on $\gamma$, by our assumption, so that $\frac{1}{\rho_i(z)} \to |\phi(\zeta)| = 1$ as $z \to \zeta$ along $\gamma$. But this is not possible since $\gamma \subset E_i$ and $\rho_i(z) < r < 1$ on the set $E_i$. Thus, we have reached a contradiction.

If we now take $w_i(z) = \phi(z) - \phi_i(z)$ and $\phi_{i,s}(z) = (1 - s)\phi(z) + s\phi_i(z)$, and define the measure $\lambda_{E_i} |w_i|^2 \lambda_2$ as in Lemma 2 then the hypotheses of the lemma are satisfied and hence the inclusion map $L_2(\lambda_{E_i} |w_i|^2 \lambda_2) \to L^2(\lambda_{E_i} |w_i|^2 \lambda_2)$ has norm uniformly bounded in $s$ and is compact for each $s \in [0, 1]$. If we let $f$ be an arbitrary function in $A^2_{\lambda_2}$, then

$$\| (C_\phi - C_{\phi_1} - \ldots - C_{\phi_N}) f \|_{A^2_{\lambda_2}}^2$$

$$= \int_D \left| f \circ \phi(z) - f \circ \phi_1(z) - \ldots - f \circ \phi_N(z) \right|^2 d\lambda_2$$

$$\leq \sum_{i=1}^N \int_{E_i} \left| f \circ \phi(z) - f \circ \phi_1(z) - \ldots - f \circ \phi_N(z) \right|^2 d\lambda_2$$

$$+ \sum_{i=1}^N \int_{E'_i} \left| f \circ \phi(z) - f \circ \phi_1(z) - \ldots - f \circ \phi_N(z) \right|^2 d\lambda_2.$$ 

We consider each piece separately.

$$\left( \int_{E'_i} \left| f \circ \phi(z) - f \circ \phi_1(z) - \ldots - f \circ \phi_N(z) \right|^2 d\lambda_2 \right)^{1/2}$$
\[
\begin{align*}
\leq & \left( \int_D |f \circ \varphi(z)|^2 \chi_{E_1}(z) \, d\lambda_x \right)^{\frac{1}{2}} + \sum_{k=1}^{N} \left( \int_D |f \circ \varphi_k(z)|^2 \chi_{E_1'}(z) \, d\lambda_x \right)^{\frac{1}{2}} \\
& = \|f\|_{L^2(\chi_{E_1'}, \lambda_x \circ \varphi^{-1})} + \sum_{k=1}^{N} \|f\|_{L^2(\chi_{E_1'}, \lambda_x \circ \varphi_k^{-1})}
\end{align*}
\] (13)

and

\[
\begin{align*}
\left( \int_{E_i} |f \circ \varphi(z) - f \circ \varphi_1(z) - \ldots - f \circ \varphi_N(z)|^2 \, d\lambda_x \right)^{\frac{1}{2}} \\
& \leq \left( \int_D |f \circ \varphi(z) - f \circ \varphi_i(z)|^2 \chi_{E_i}(z) \, d\lambda_x \right)^{\frac{1}{2}} \\
& \quad + \sum_{0<j \neq i} \left( \int_D |f \circ \varphi_j(z)|^2 \chi_{E_i}(z) \, d\lambda_x \right)^{\frac{1}{2}} \\
& \leq \left( \int_0^1 \int_D |f' \circ \varphi_{i,s}(z)|^2 |w_i(z)|^2 \chi_{E_i}(z) \, d\lambda_x \, ds \right)^{\frac{1}{2}} \\
& \quad + \sum_{0<j \neq i} \left( \int_D |f \circ \varphi_j(z)|^2 \chi_{E_i}(z) \, d\lambda_x \right)^{\frac{1}{2}} \\
& = \left( \int_0^1 \|f'-\|_{L^2(|w_i|^2 \chi_{E_1'}, \lambda_x \circ \varphi^{-1}_{i,s})}^2 \, ds \right)^{\frac{1}{2}} + \sum_{0<j \neq i} \|f\|_{L^2(\chi_{E_1'}, \lambda_x \circ \varphi_j)}.
\end{align*}
\] (14)

We have already established the compactness of each of the necessary inclusion maps into the respective $L^2$ spaces of lines (13) and (14) as well as the uniform boundedness of $I_{x+2,i,s}$ for each $i$ and we use this to see that if $f_k$ is a sequence of unit vectors converging weakly to zero in $A^2_x$ then

\[
\|(C_\varphi - C_{\varphi_1} - \ldots - C_{\varphi_n}) f_k\|_{A^2_x} \to 0.
\]

5. The role of second-order data and the pseudo-hyperbolic metric

We have seen that the pseudo-hyperbolic distance plays a key role in questions of compactness and, in the last section, we will show that the pseudo-hyperbolic metric is also a good measure of connectedness, so it would behoove us to gain a better understanding of convergence in this metric. We begin by looking at a sub-class of maps
for which computations are possible, namely those which are twice differentiable at the boundary points in question. In [3] and [4], slightly different methods are employed to obtain a similar result for a class of “almost linear fractional” composition operators acting on the Hardy space.

Suppose that \( \varphi \) is a map from the disk to the disk, analytic on the open disk, having finite angular derivative at a point, \( \zeta \in \partial D \), and further assume (giving up complete generality for the sake of computability) that \( \varphi \) is twice differentiable at \( \zeta \), by which we mean that, if \( \varphi \) is considered as a function of \( D \cup \{ \zeta \} \), it is twice continuously differentiable and has the expansion

\[
\varphi(z) = \varphi(\zeta) + \varphi'(\zeta)(z - \zeta) + \frac{\varphi''(\zeta)}{2}(z - \zeta)^2 + f(z),
\]

where \( f(z) = o(|z - \zeta|^2) \), as \( z \to \zeta \) inside the disk. Now, if \( \psi \) is another such map, and we have \( \varphi(\zeta) = \psi(\zeta) \), \( \varphi'(\zeta) = \psi'(\zeta) \) and \( \varphi''(\zeta) = \psi''(\zeta) \) then we say that \( \varphi \) and \( \psi \) have the same second-order data at \( \zeta \).

We can see that simply having the same second-order data is not sufficient to guarantee compactness of the difference by considering the class of maps \( \varphi_t(z) = z - t(z - 1)^3 \), where \( t \) is chosen small enough so that \( \varphi_t \) gives a map of the unit disk to itself (see [11] for details). For any such \( t, s > 0 \), \( \varphi_t \) and \( \varphi_s \) have the same second-order data at 1 but for \( t \neq s \) the induced composition operators have non-compact difference. We see this by letting \( z \rightarrow 1 \) along the path \( \gamma = \{z : 1 - |z|^2 = |1 - z|^3 \} \) upon which both \( \rho(z) \) and \( \frac{1 - |z|^2}{1 - |\varphi_t(z)|^2} \) are bounded below and invoking Theorem 4. Thus, we can see that a further restriction is needed, and this will require a new definition: For \( \varphi \) an analytic self-map of the disk, we say that \( \varphi \) has order of contact at most \( k \) at a point \( \zeta \in \partial D \) if there exists a circle \( C \), centered at \( \zeta \) such that for \( \varphi(z) \in C \) the quantity \( \frac{1 - |\varphi(z)|^2}{|\varphi(z) - \varphi(z)|^k} \) is bounded below.

**Proposition 1.** Let \( \varphi \) and \( \psi \) be analytic maps of the unit disk to itself with second-order data at a point \( \zeta \in \partial D \). Define \( \rho(z) = \left| \frac{\varphi(z) - \psi(z)}{1 - \varphi(z)\psi(z)} \right| \).

1. If \( \varphi \) and \( \psi \) have the same second-order data at \( \zeta \) and, moreover, \( \varphi \) has order of contact at most 2 at the point \( \zeta \) then \( \rho(z) \rightarrow 0 \) as \( z \rightarrow \zeta \).
2. If \( \varphi \) and \( \psi \) have the same first-order data and different second derivatives, then there exists a path, \( \gamma \in D \) with \( \gamma \rightarrow \zeta \) such that \( \rho(z) \rightarrow 0 \) along \( \gamma \). Moreover, along this path \( \gamma \), \( \frac{1 - |\varphi(z)|^2}{1 - |\varphi(z)|^k} \rightarrow 0 \).

**Proof.** In what follows we will use the equality

\[
\frac{1 - \overline{\varphi(z)}\psi(z)}{\varphi(z) - \psi(z)} = \frac{1 - |\varphi(z)|^2}{\varphi(z) - \psi(z)} + \frac{\overline{\varphi(z)}}{\varphi(z)},
\]

so that \( \rho(z) \rightarrow 0 \) if and only if \( \frac{\varphi(z) - \psi(z)}{1 - |\varphi(z)|^2} \rightarrow 0 \).
Since \( \varphi \) and \( \psi \) have second-order data at \( \zeta \), we have

\[
\varphi(z) = \varphi(\zeta) + \varphi'(\zeta)(z - \zeta) + \frac{\varphi''(\zeta)}{2}(z - \zeta)^2 + f(z)
\]

and

\[
\psi(z) = \psi(\zeta) + \psi'(\zeta)(z - \zeta) + \frac{\psi''(\zeta)}{2}(z - \zeta)^2 + g(z)
\]

where \( f(z) \) and \( g(z) \) are \( o(|z - \zeta|^2) \), so that

\[
\frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|} = |\varphi'(\zeta) + \varphi''(\zeta)(z - \zeta) + \frac{f(z)}{z - \zeta}| \to |\varphi'(\zeta)| > 0
\]

as \( z \to \zeta \) along any path in the interior of the disk. Thus, the quantity \( \frac{|\varphi(z) - \varphi(\zeta)|}{|z - \zeta|} \) is bounded both above and below for all \( z \) in some neighborhood of \( \zeta \).

For \( \varphi \) and \( \psi \) with the same second-order data we can write

\[
\psi(z) = \varphi(z) + h(z)
\]

where \( h(z) \) is \( o(|z - \zeta|^2) \) and we have

\[
\frac{|\varphi(z) - \psi(z)|}{1 - |\varphi(z)|^2} = \frac{|h(z)|}{|\varphi(\zeta) - \varphi(z)|^2} \frac{|\varphi(\zeta) - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq C \frac{|h(z)|}{|\zeta - z|^2} \frac{|\varphi(\zeta) - \varphi(z)|^2}{1 - |\varphi(z)|^2}
\]

for all \( z \) near \( \zeta \). Since \( \varphi \) has order of contact at most 2, the quantity \( \frac{|\varphi(\zeta) - \varphi(z)|^2}{1 - |\varphi(z)|^2} \) is bounded for \( z \) near \( \zeta \), and since \( h(z) \) is \( o(|z - \zeta|^2) \) the quantity in (15) is seen to converge to 0 as \( z \to \zeta \) so that \( \rho(z) \to 0 \).

Next, suppose that \( \varphi \) and \( \psi \) have the same first-order data but different second derivatives at \( \zeta \), then we write

\[
\psi(z) = \varphi(z) + a_2(z - \zeta)^2 + h(z),
\]

where \( a_2 = \frac{\varphi''(\zeta) - \psi''(\zeta)}{2} \neq 0 \) and \( h(z) \) is \( o(|z - \zeta|^2) \). Observe that \( \frac{|\varphi(z) - \psi(z)|}{|z - \zeta|^2} = |a_2 + \frac{h(z)}{(z - \zeta)^2}| \to |a_2| > 0 \) so that \( \frac{|\zeta - z|^2}{|\varphi(z) - \psi(z)|} \) is bounded near \( \zeta \), and

\[
\frac{1 - |\varphi(z)|^2}{|\varphi(z) - \psi(z)|} = \frac{1 - |\varphi(z)|^2}{|z - \zeta|^2} \frac{|\zeta - z|^2}{|\varphi(z) - \psi(z)|},
\]
so that we need only concentrate on the quantity \( \frac{1 - |\phi(z)|^2}{|z - \zeta|^2} \). For ease of computation, we will assume that \( \zeta = 1 \) and \( \phi(1) = 1 \) and note that \( \phi(\zeta)\phi(\bar{\zeta}z) \) has fixed point 1 so that this assumption is no loss of generality. Let \( \gamma = \{z : 1 - |z|^2 = |1 - z|^2 \} \). Notice that \( \frac{1 - |\phi(z)|^2}{|1 - z|^2} \) being bounded along the path \( \gamma \) is equivalent to \( \frac{1 - |\phi(z)|^2}{1 - |z|^2} \) bounded on \( \gamma \). For \( z \in \gamma \),

\[
\frac{1 - |\phi(z)|^2}{|1 - z|^2} = \frac{2\phi'(1)\Re(1 - z) - (\phi'(1))^2|1 - z|^2 - \Re(\phi''(1)(1 - z)^2) + o(|1 - z|^2)}{|1 - z|^2}
\]

and which is bounded along \( \gamma \) since for \( z \in \gamma \) we have \( \Re(1 - z) = |1 - z|^2 \). Thus, \( \frac{1 - |\phi(z)|^2}{|\phi(z)|^2 - \psi(z)} \) is bounded so that \( \rho(z) \) is bounded away from 0 on \( \gamma \) as is \( \frac{1 - |z|^2}{1 - |\phi(z)|^2} \). \( \square \)

We sum up what we know in the following, combining the results above with Theorem 4.

**Theorem 6.** Let \( z > -1 \). Suppose \( \phi \) and \( \psi \) are analytic self-maps of the disk, and let \( F(\phi) \) be the set of points at which \( \phi \) has finite angular derivative, and similarly define \( F(\psi) \). If \( F(\phi) = F(\psi) := F \) and, for each \( \zeta \in F \), \( \phi \) and \( \psi \) have second-order data, then the following hold:

1. \( C_\phi - C_\psi \) is compact on \( A^2_\mathbb{R} \) implies that \( \phi \) and \( \psi \) have the same second-order data at each point \( \zeta \) in \( F \).
2. If at each \( \zeta \in F \), \( \phi \) has at most order 2 contact and \( \phi \) and \( \psi \) have the same second-order data at \( \zeta \) then \( C_\phi - C_\psi \) is compact on \( A^2_\mathbb{R} \).

For one particular class of maps second-order data at a boundary point is definitive. Consider, every linear fractional map of the disk to itself which has contact with the boundary of the disk at a point \( \zeta \) has second-order data at that point and, moreover, is completely determined by its second-order data. Thus, we can read off an immediate result, found independently by Bourdon [3] on the Hardy space:

**Corollary 2.** No two distinct linear fractional maps of the disk to itself, having contact with the boundary of the disk, induce composition operators with compact difference.

But what is more interesting to us is the fact that for any map \( \phi \) with second-order data at a boundary point \( \zeta \), there exists a corresponding linear fractional map with the same second-order data. This is not completely obvious, but when one considers that the second-order data of \( \phi \) completely determines the curvature as well as the normal vector for the image curve \( \phi(\bar{\cdot}D) \) at the point \( \zeta \), it becomes apparent. We state our observations as a lemma without proof.

**Lemma 3.** If \( \phi \) is an analytic self-map of the disk to itself with second-order data at a point \( \zeta \), then there exists a linear fractional map \( \psi \) with the same second-order data at \( \zeta \).
These results supply us with a means for analyzing the additive structure of a subset of the composition operators on $A^2_\infty$, $\omega > -1$, modulo the compact operators. Suppose $\varphi$ has finite angular derivative on a finite set of distinct points $\{z_k\}_{k=1}^N$ and is twice differentiable at each $z_k$, with curvature greater than 1. By Lemma 3, we see that for each $z_k$ there is a linear fractional map, $\varphi_k$, such that $\varphi$ and $\varphi_k$ have the same second-order data at $z_k$ and hence $\rho_k(z) = \frac{|\varphi(z) - \varphi_k(z)|}{1 - \varphi(z)\varphi_k(z)} \to 0$ as $z \to z_k$. Now, for each linear fractional map $\varphi_k$, the set of finite angular derivatives, $F_k$, consists of a single point $z_k$, and thus $F_i \cap F_j = \emptyset$ when $i \neq j$. Therefore, the hypotheses of Theorem 5 have been satisfied and we can write $C_\varphi = C_{\varphi_1} + \ldots + C_{\varphi_N} + K$, where $K$ is a compact operator in $A^2_\infty$. We summarize these results in our final compactness theorem.

**Theorem 7.** Let $\varphi$ be an analytic self-map of the disk, with finite angular derivative on a finite set of points $F$. If $\varphi$ has second-order data and order of contact at most 2 at each point $z \in F$, then there exist unique linear fractional maps of the disk, $\varphi_1, \varphi_2, \ldots, \varphi_N$ such that $C_\varphi$ is equivalent to the sum $C_{\varphi_1} + \ldots + C_{\varphi_N}$ modulo the compact operators.

### 6. Component structure

We have seen that the pseudo-hyperbolic distance plays a key role in questions of compactness. We now apply the methods already detailed, to partially answer the question of when two composition operators lie in the same component; the result will follow almost immediately from Lemma 2, as seen in [18].

**Theorem 8.** Let $\varphi$ and $\psi$ be analytic self-maps of the unit disk, and, as usual, $\rho(z) = \frac{|\varphi(z) - \psi(z)|}{1 - \varphi(z)\psi(z)}$. If there exists an $r < 1$ such that $\rho(z) < r$ for every $z \in D$ then $C_\varphi$ and $C_\psi$ are arc connected in $C(A^2_\infty)$.

**Proof.** Let $w(z) = \varphi(z) - \psi(z)$ and $\varphi_s(z) = s\varphi(z) + (1 - s)\psi(z)$. By Lemma 2, $|w|^2\lambda_2 \circ \varphi_s^{-1}$ is an $\omega + 2$-Carleson measure, moreover $wC_{\varphi_s}$, as an operator from $A^2_{\omega+2}$ to $A^2_\infty$, has norm uniformly bounded in $s$.

Thus, as in Theorem 4,

$$\| (C_{\varphi_s} - C_{\varphi_t}) f \|_{A^2_\omega} = \int_D |f \circ \varphi_s(z) - f \circ \varphi_t(z)|^2 d\lambda_{\omega} \leq \int_s^t \int_D |w(z)C_{\varphi_s} f'(z)|^2 d\lambda_{\omega} dr = \int_s^t \| wC_{\varphi_s} f' \|_{A^2_\omega} dr.$$

By the uniform boundedness of the norms of $wC_{\varphi_s}$ we see that the last quantity can be made as small as desired by choosing $s$ and $t$ close. \(\square\)
This condition is certainly not necessary for arc-connectedness, which we see by considering two maps $\phi$ and $\psi$ having no finite angular derivative, so that the induced composition operators are both compact and therefore lie in the same component: Suppose, that the image of $\psi$ is contained in a disk $rD$ but that there exists a point $\zeta \in \partial D$ with $|\phi(\zeta)| = 1$ then it is easy to see that $|\rho(\zeta)| = 1$, so that $\phi$ and $\psi$ do not satisfy the conditions of Theorem 8. However, for the moment we restrict our attention to points of finite angular derivative only.

**Proposition 2.** Let $\phi$ and $\psi$ be analytic maps of the unit disk to itself with second-order data at a point $\zeta \in \partial D$ and order of contact at most 2. Define $\rho(z) = \frac{|\phi(z) - \psi(z)|}{|1 - \phi(z)\psi(z)|}$. If $\phi$ and $\psi$ have the same first-order data at $\zeta$ then $\rho(z)$ is bounded away from 1 near $\zeta$.

**Proof.** As before, $\phi$ and $\psi$ have second-order data at $\zeta$, so that we can write

$$\phi(z) = \phi(\zeta) + \phi'(\zeta)(z - \zeta) + \frac{\phi''(\zeta)}{2}(z - \zeta)^2 + f(z)$$

and

$$\psi(z) = \psi(\zeta) + \psi'(\zeta)(z - \zeta) + \frac{\psi''(\zeta)}{2}(z - \zeta)^2 + g(z),$$

where $f$ and $g$ are $o(|\zeta - z|^2)$. Assuming only that $\phi$ and $\psi$ have the same first-order data at $\zeta$ we have

$$\phi(z) - \psi(z) = a_2(z - \zeta)^2 + h(z)$$

where $a_2 = \frac{\phi''(\zeta) - \psi''(\zeta)}{2}$ and $h(z)$ is $o(|\zeta - z|^2)$. As in Proposition 1, $\frac{|\phi(z) - \psi(z)|}{|\zeta - z|}$ is bounded both above and below in some neighborhood of $\zeta$, and since $\phi$ has order of contact 2 or less,

$$\frac{|\phi(z) - \psi(z)|}{1 - |\phi(z)|^2} \leq C \frac{|\phi(z) - \psi(z)|}{|\zeta - z|^2} \frac{|\phi(\zeta) - \phi(z)|^2}{1 - |\phi(z)|^2}$$

$$\leq m \left| a_2 + \frac{h(z)}{(1 - z)^2} \right|$$

is uniformly bounded in a neighborhood of $\zeta$. Thus,

$$\frac{|1 - \overline{\phi(z)}\psi(z)|}{1 - |\phi(z)|^2} \leq 1 + \frac{|\phi(z) - \psi(z)|}{1 - |\phi(z)|^2} \leq K_1 < \infty$$

and similarly $\frac{|1 - \overline{\phi(z)}\psi(z)|}{1 - |\psi(z)|^2} \leq K_2 < \infty$, for $z$ near $\zeta$. 
Now, we use equality (6) to see that
\[
1 - \rho^2(z) = \frac{(1 - |\varphi(z)|^2)}{|1 - \varphi(z)\psi(z)|} \frac{(1 - |\psi(z)|^2)}{|1 - \varphi(z)\psi(z)|} \geq \frac{1}{K_1 K_2} > 0
\]
and thus \( \rho \) is bounded away from 1 near \( \zeta \). \( \square \)

Combining these two results gives the following.

**Corollary 3.** Let \( \varphi \) and \( \psi \) be analytic maps of the unit disk to itself having the same angular derivative set, \( F \). Assume that the ranges of both \( \varphi \) and \( \psi \) are bounded away from the unit circle on the complement of every open set containing \( F \) and have second-order data and order of contact at most two at each point of \( F \). If \( \varphi \) and \( \psi \) have the same first-order data at every point in \( F \), then \( C_\varphi \) and \( C_\psi \) lie in the same component.

We note that for those maps which satisfy the conditions of the previous corollary but have different second-order data, the induced composition operators have non-compact difference and yet lie in the same component, thus supplying us with a large class of examples which answer the Shapiro–Sundberg question in the negative.

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**References**