Fractional and Wiener–Hopf factorizations

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Abstract

A connection between Wiener–Hopf factorizations of an analytic matrix function \( a(t) \) and fractional factorizations of the rational part of \( a^{-1}(t) \) is obtained. The result is applied to an explicit construction of Wiener–Hopf factorizations of \( a(t) \). © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The problem of an explicit construction of Wiener–Hopf factorizations for matrix functions has fundamental significance in the theory of systems of singular integral equations and systems of Wiener–Hopf equations. It also plays an important role in the theory of differential equations and in the integration of nonlinear equations in mathematical physics.

The first step in solving the problem was done in 1978 by Gohberg et al. [1], who obtained explicit formulas for the left partial indices of matrix polynomials. Full proofs appeared in [2], where, in addition, factorization factors for matrix polynomials were also computed. Moreover, in that work the authors showed that these results can be extended to the case of analytic matrix functions.

In [3,4] a method of an explicit construction of Wiener–Hopf factorizations of meromorphic matrix functions was suggested. The method is based on notions of...
essential polynomials. It allows us to obtain the left and right factorizations simultaneously. Besides, as shown in these works, to determine the factorizations of an analytic matrix function \( a(t) \) it suffices to know \( 2\kappa + 1 \) moments of \( a^{-1}(t) \), where \( \kappa = \text{ind}_t \det a(t) \). The number of moments appearing in formulas from [2] is a priori unknown.

It turns out that there is the relation between the Wiener–Hopf factorizations and linear system theory. In particular, in [5] an attempt to clarify a connection between the Wiener–Hopf factorizations of rational matrix functions and coprime factorizations has been done. Formulas for the Wiener–Hopf factorization of a rational matrix function (or an analytic operator function) \( W(\lambda) \) can be found in terms of a realization of the transfer matrix \( W(\lambda) \) [6]. However, in the analytic case these formulas cannot be regarded as explicit.

All the same there is a simple connection between the Wiener–Hopf factorizations of an analytic matrix function \( a(t) \) and the coprime factorizations of the rational matrix function \( r(t) \); here \( r(t) \) is the rational part of \( a^{-1}(t) \). Applying this connection, we will find an explicit solution of the Wiener–Hopf factorization problem in the analytic case. The main goal of the work is to obtain this solution.

2. Fractional and Wiener–Hopf factorizations: basic definitions

In this section, we recall some definitions [7,8].

Let \( r(t) \) be a strictly proper rational \( p \times q \) matrix function. The following matrix fractional factorization

\[
    r(t) = D_l^{-1}(t)N_l(t)
\]

plays an important role in the system theory. Here \( D_l(t) \), \( N_l(t) \) are left coprime matrix polynomials in \( t \) and \( D_l(t) \) is nonsingular, i.e., \( \det D_l(t) \neq 0 \). We note that \( D_l(t) \) and \( N_l(t) \) are left coprime iff the following Bezout equation

\[
    D_l(t)U_l(t) + N_l(t)V_l(t) = I_p
\]

is solvable. Here \( U_l(t) \) and \( V_l(t) \) are matrix polynomials in \( t \). As is well known, an arbitrary strictly proper rational matrix function \( r(t) \) admits the representation (2.1) that is called a left coprime fractional factorization of \( r(t) \).

For any nonsingular \( p \times p \) matrix polynomial \( \bar{L}(t) \) there exists a unimodular matrix polynomial \( S(t) \) such that \( L(t) = S(t)\bar{L}(t) \) is row proper (see, e.g., [7]). This means that the constant matrix \( L_\text{row} \) consisting of the coefficients of the highest degrees in each row of \( L(t) \) is invertible. Now we can always assume that the denominator in (2.1) is row proper. In the system theory it is shown that the row degrees \( \lambda_1, \ldots, \lambda_p \) of the row proper denominator are uniquely determinated by \( r(t) \). They are called observability indices of the pair \( (A, C) \), where \( (A, B, C) \) is any minimal realization of \( r(t) \). The sum \( \delta \) of the indices coincides with degree of \( \det D_l(t) \) and is called the McMillan degree of the system. We can suppose that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \).

In an analogous manner we define a right coprime fractional factorization of \( r(t) \):
Here \( N_r(t), D_r(t) \) are right coprime matrix polynomials. This is equivalent to solvability of the Bezout equation
\[
U_r(t)D_r(t) + V_r(t)N_r(T) = I_q,
\]
where \( U_r(t) \) and \( V_r(t) \) are matrix polynomials.

Moreover, we can suppose that \( D_r(t) \) is a nonsingular column proper \( p \times p \) matrix polynomial, and the column degrees \( \rho_j \) (reachability indices of \( (A, B) \)) are in increasing order. By \( D_r^{\text{col}} \) we denote the invertible matrix consisting of the coefficients of the highest degrees in each column of \( D_r(t) \).

It is known (see, e.g., [9]) that there exist solutions of the Bezout equations such that
\[
\deg[V_l(t)]_j < \rho_j, \quad j = 1, \ldots, q; \quad \deg[V_r(t)]^j < \lambda_j, \quad j = 1, \ldots, p.
\]
Here by \( [A]_j \) (\( [A]^j \)) we denote the \( j \)-th row (column) of the matrix \( A \).

We will always suppose that solutions of the Bezout equations satisfy the above conditions.

Let \( \Gamma \) be a simple closed contour in the complex plane \( \mathbb{C} \) bounding a domain \( D_+ \). The complement of \( D_+ \cup \Gamma \) in the extended complex plane \( \mathbb{C} \cup \{\infty\} \) will be denoted by \( D_- \). Without loss of generality we may assume that \( 0 \in D_+ \), \( \infty \in D_- \). Let \( a(t) \) be a continuous invertible \( p \times p \) matrix function on the contour \( \Gamma \).

A right Wiener–Hopf factorization of the matrix function \( a(t) \) is defined to be a representation of it in the form
\[
a(t) = r_-(t)d_r(t)r_+(t), \quad t \in \Gamma,
\]
where \( r_{\pm}(t) \) are continuous matrix functions on \( \Gamma \) that can be extended analytically to \( D_\pm \) and are invertible there, \( d_r(t) = \text{diag}[t^{\rho_1}, \ldots, t^{\rho_p}] \), the numbers \( \rho_1, \ldots, \rho_p \) being integers uniquely determined by \( a(t) \). These integers are called the right partial indices of \( a(t) \). We may assume that the right partial indices are in increasing order: \( \rho_1 \leq \cdots \leq \rho_p \).

A left factorization is defined similarly: \( a(t) = l_+(t)d_l(t)l_-(t) \), \( d_l(t) = \text{diag}[t^{\lambda_1}, \ldots, t^{\lambda_p}] \), \( \lambda_1 \geq \cdots \geq \lambda_p \) are left partial indices of \( a(t) \). It is easily seen that \( \sum_{j=1}^p \rho_j = \sum_{j=1}^p \lambda_j = \text{ind}_r \det a(t) \). Here \( \text{ind}_r \det a(t) \) is the Cauchy index of the determinant of \( a(t) \).

3. Connection between fractional and Wiener–Hopf factorizations

In this section, we will reduce the Wiener–Hopf factorization problem for an analytic matrix function to the coprime fractional factorization of a rational matrix function. Applying this connection, we will obtain an explicit solution of the Wiener–Hopf factorization problem for analytic matrix functions.

Let \( a(t) \) be a \( p \times q \) matrix function which is meromorphic in a domain \( D_+ \) and continuous on \( D_+ \cup \Gamma \). Then the matrix function \( a(t) \) can be represented in the form
\[
a(t) = r(t) + a_+(t).
\]
Here the matrix function $a_+(t)$ is continuous on $D_+ \cup \Gamma$ and analytic in $D_+$. The latter means that $r(t)$ is the sum of the principal parts of the Laurent series in neighbourhoods of the poles. Obviously, $r(t)$ is analytic in $D_-$ and continuous on $D_- \cup \Gamma$.

We now extend the definition of the fractional factorization to the case of meromorphic matrix functions.

**Definition 1.** Let $a(t)$ be a matrix function which is continuous on $D_+ \cup \Gamma$ and meromorphic in $D_+$. We say that $a(t)$ admits a left (right) coprime fractional factorization if

$$a(t) = D_l^{-1}(t)N_l^+(t), \quad (a(t) = N_r^+(t)D_r^{-1}(t)). \quad (3.1)$$

Here $D_l(t)$ ($D_r(t)$) is a nonsingular $p \times p$ ($q \times q$) matrix polynomial and the zeros of $\det D_l(t)$ ($\det D_l(t)$) lie in $D_+$. The matrix function $N_l^+(t)$ ($N_r^+(t)$) is continuous on $D_+ \cup \Gamma$ and analytic in $D_+$. Moreover, $D_l(t)$ and $N_l^+(t)$ ($D_r(t)$ and $N_r^+(t)$) are left (right) coprime. This means that any common left (right) polynomial divisor of $D_l(t)$ and $N_l^+(t)$ ($D_r(t)$ and $N_r^+(t)$) is a unimodular matrix polynomial.

Let us establish a connection between the fractional factorizations of $a(t)$ and $r(t)$.

**Theorem 2.** Let $a(t) = r(t) + a_+(t)$ and let

$$r(t) = D_l^{-1}(t)N_l(t), \quad r(t) = N_r(t)D_r^{-1}(t)$$

be coprime fractional factorizations of $r(t)$. Then

$$a(t) = D_l^{-1}(t)N_l^+(t), \quad a(t) = N_r^+(t)D_r^{-1}(t)$$

are coprime fractional factorizations of $a(t)$. Here

$$N_l^+(t) = N_l(t) + D_l(t)a_+(t), \quad N_r^+(t) = N_r(t) + a_+(t)D_r(t).$$

Conversely, if we know the fractional factorizations (3.1) of $a(t)$, then putting $N_l(t) = N_l^+(t) - D_l(t)a_+(t)$, $N_r(t) = N_r^+(t) - a_+(t)D_r(t)$ we receive the fractional factorizations of $r(t)$.

**Proof.** Let $r(t) = D_l^{-1}(t)N_l(t)$ be a left coprime fractional factorization of $r(t)$. Since all poles of $r(t)$ lie in $D_+$, $\det D_l(t) \neq 0$ in $D_-$. Moreover,

$$a(t) = D_l^{-1}(t)N_l(t) + a_+(t)$$

$$= D_l^{-1}(t) \left[ N_l(t) + D_l(t)a_+(t) \right]$$

$$= D_l^{-1}(t)N_l^+(t).$$

It is obvious that $N_l^+(t)$ is continuous on $D_+ \cup \Gamma$ and analytic in $D_+$. Let $G(t)$ be a common left polynomial divisor of $D_l(t)$ and $N_l^+(t)$:
\[ D_l(t) = G(t) \tilde{D}_l(t), \quad N_l^+(t) = G(t) \tilde{N}_l^+(t). \]

Then \( N_l(t) = G(t)[\tilde{N}_l^+(t) - \tilde{D}_l(t)a_+(t)] \). Since \( N_l(t), G(t) \) are matrix polynomials and \( \tilde{N}_l^+(t) - \tilde{D}_l(t)a_+(t) \) is analytic in \( D_+ \), we see that \( \tilde{N}_l^+(t) - \tilde{D}_l(t)a_+(t) \) is also a matrix polynomial. Thus \( G(t) \) is a common left polynomial divisor of \( N_l(t) \) and \( D_l(t) \). Hence \( G(t) \) is unimodular. Therefore the left coprime fractional factorization of \( a(t) \) is constructed.

The remaining statements can be proved in a similar manner. \( \square \)

**Remark 3.** Now it is not difficult to prove that the polynomials \( D_l(t) \) and \( N_l^+(t) \) are left coprime iff the following Bezout equation

\[ D_l(t)U_l^+(t) + N_l^+(t)V_l^+(t) = I_p. \]

is solvable. Here \( U_l^+(t) \) is analytic in \( D_+ \) and continuous on \( D_+ \cup \Gamma \), and \( V_l^+(t) \) is a matrix polynomial in \( t \).

If \( a(t) \) is a meromorphic function and \( a(t) \neq 0 \) in \( D_+ \), i.e., \( a^{-1}(t) \) is an analytic matrix function, then the fractional factorization \( a(t) \) is its Wiener–Hopf factorization. This statement holds also in the matrix case.

**Theorem 4.** Let \( a(t) \) be a \( p \times p \) matrix function that is continuous and invertible on \( \Gamma \) and extends analytically to the domain \( D_+ \). If

\[ a(t) = l_+(t)d_l(t)l_-(t), \quad a(t) = r_-(t)d_r(t)r_+(t) \]

are its Wiener–Hopf factorizations, then

\[ a^{-1}(t) = D_l^{-1}(t)l_+^{-1}(t), \quad a^{-1}(t) = r_+^{-1}(t)D_r^{-1}(t) \]

are the coprime fractional factorizations of \( a^{-1}(t) \). Here the denominators

\[ D_l(t) = d_l(t)l_-(t), \quad D_r(t) = r_-(t)d_r(t), \]

have the row and column proper form, respectively.

Conversely, if

\[ a^{-1}(t) = D_l^{-1}(t)N_l^+(t), \quad a^{-1}(t) = N_r^+(t)D_r^{-1}(t) \]

are arbitrary coprime fractional factorizations of \( a^{-1}(t) \) with row/column proper denominators, then the Wiener–Hopf factorizations of \( a(t) \) can be constructed by the formulas:

\[ l_+(t) = (N_l^+(t))^{-1}, \quad d_l(t) = \text{diag}[t^{\lambda_1}, \ldots, t^{\lambda_p}], \quad l_-(t) = d_l^{-1}(t)D_l(t); \]

\[ r_+(t) = (N_r^+(t))^{-1}, \quad d_r(t) = \text{diag}[t^{\rho_1}, \ldots, t^{\rho_p}], \quad r_-(t) = D_r(t)d_r^{-1}(t). \]

Here \( \lambda_1, \ldots, \lambda_p \) are the row degrees of \( D_l(t) \), and \( \rho_1, \ldots, \rho_p \) are the column degrees of \( D_r(t) \).
Proof. Let \(a(t) = l_+(t)d_+(t)l_-(t)\) be the left Wiener–Hopf factorization of \(a(t)\). It is easily seen that for the analytic matrix function \(a(t)\) the left factorization indices satisfy the inequality \(\lambda_j \geq 0\) and the factor \(l_-(t)\) is a matrix polynomial in \(t^{-1}\). Moreover, for the row degrees of \([l_-(t)]_j\) we have the inequality \(\deg[l_-(t)]_j \leq \lambda_j\). Hence \(D_l(t) = d_l(t)l_-(t)\) is a matrix polynomial in \(t\) and all zeros of \(\det D_l(t)\) lie in \(D_+\). If \(G(t)\) is a common left polynomial divisor of \(D_l(t)\) and \(N_i^+(t) = l_i^-(t)\), then \(\det G(t)\) has no zeros in \(D_-\). In addition, since \(l_+(t)\) is analytic in \(D_+\) matrix function, \(\det G(t)\) has no zeros in \(D_+\) also. Thus \(G(t)\) is a unimodular matrix polynomial and \(a^{-1}(t) = D_l^{-1}(t)N_i^+(t)\) is the left coprime fractional factorization of \(a^{-1}(t)\).

Conversely, let \(a^{-1}(t) = D_l^{-1}(t)N_i^+(t)\) be a left coprime fractional factorization of \(a^{-1}(t)\), and the denominator \(D_l(t)\) is row proper. Put \(l_-(t) = d_l^{-1}(t)D_l(t)\). Obviously, \(l_-(t)\) is a matrix polynomial in \(t^{-1}\). Moreover, all zeros of \(\det l_-(t)\) lie in \(D_+\). This means that the matrix function \(l_-(t)\) has the inverse which is analytic in \(D_-\).

By Definition 1, the matrix function \(N_i^+(t)\) is analytic in \(D_+\), continuous on \(D_+ \cup \Gamma\) and invertible on \(\Gamma\). It remains to prove that \((N_i^+(t))^{-1}\) is also analytic in \(D_+\). From solvability of the Bezout equation \(D_l(t)U_i^+(t) + N_i^+(t)V_i^+(t) = I_p\) it follows that \((N_i^+(t))^{-1} = V_i^+(t) + a(t)U_i^+(t)\) is analytic in \(D_+\). Hence \(a(t) = (N_i^+(t))^{-1}d_l(t)l_-(t)\) is the left Wiener–Hopf factorization of \(a(t)\).

The second part of the proposition can be proved similarly. \(\square\)

4. The explicit construction of coprime factorizations for rational matrix functions

As we have seen in the previous section, an explicit construction of the Wiener–Hopf factorization for analytic matrix functions is reduced to an explicit construction of the fractional factorization for rational matrix functions. By an explicit solution of the factorization problems we understand here a reduction of them to the investigation of finitely many systems of linear algebraic equations with the coefficient matrices written out in closed form (i.e., with the help of quadratures).

There exist few algorithms for a construction of the fractional factorizations of rational matrix functions (see, e.g., [7]). For the purpose of this paper it will be convenient to obtain another method of computation of these factorizations. The method is based on notions of indices and essential polynomials. These concepts were first introduced in the scalar case in [10]; the summarizing work is [11]. The same notions were independently introduced by Heinig et al. (see, e.g., [12]).

Let \(r(t)\) be a strictly proper rational \(p \times q\) matrix function. Then for \(t \in \mathbb{C}\) such that \(|t|\) is sufficiently large we have

\[ r(t) = \sum_{j=-\infty}^{-1} r_j t^j. \]
In this section, we will construct the coprime factorizations of \( r(t) \) in terms of indices and essential polynomials of the sequence \( r_{-m}, \ldots, r_{-1} \) consisting of the Laurent coefficients of \( r(t) \) for \( m \geq 2\delta > 0 \). We will use definitions, notations, and results from the work [11]. For more information we refer to this paper.

First we find the defects of the sequence \( r_{-m}, \ldots, r_{-1} \) in terms of the fractional factorizations of \( a(t) \).

**Theorem 5.** Let \( \delta \) be the McMillan degree of the rational matrix function \( r(t) \). Then for any \( m \geq \delta \) the left (right) defect of the sequence \( r_{-m}, \ldots, r_{-1} \) coincides with the number of constant columns (rows) of the denominator \( D_r(t) \) (\( D_l(t) \)) of \( r(t) \).

**Proof.** Let \( \rho_1 = \cdots = \rho_d = 0 < \rho_{d+1} \), i.e., \([D_r(t)]^1, \ldots, [D_r(t)]^d\) are constant vectors from \( \mathbb{C}^{q \times 1} \). From the factorization \( r(t)D_r(t) = N_r(t) \) it follows that \( r_{-k}[D_r(t)]^j = 0 \) for \( k \geq 1 \) and \( j = 1, \ldots, d \). Hence the columns \([D_r(t)]^1, \ldots, [D_r(t)]^d\) belong to \( \mathcal{N}_{-m}^R \) for any sequence \( r_{-m}, \ldots, r_{-1} \). Obviously, these columns are linearly independent by virtue of nonsingularity of \( D_r(t) \). Therefore \( \alpha = \dim \mathcal{N}_{-m}^R \geq d \).

Now we prove that the inverse inequality is also fulfilled if \( m \geq \delta \). It is easily seen that

\[
\ker \begin{pmatrix} r_{-m} \\ \vdots \\ r_{-1} \end{pmatrix} \supseteq \ker \begin{pmatrix} r_{-m-1} \\ \vdots \\ r_{-1} \end{pmatrix}.
\] (4.1)

Hence there exists an integer \( m_0 \) such that the sequence of these spaces will stabilize if \( m \geq m_0 \). Let us estimate \( m_0 \). To do this, we consider the left coprime fractional factorization \( r(t) = D_l^{-1}(t)N_l(t) \), where the denominator \( D_l(t) \) is row proper.

Let \( [D_l(t)]_j = d_0^j + d_1^jt + \cdots + d_{\lambda_j}^j t^{\lambda_j}, j = 1, \ldots, p \). It follows from the equality \([D_l(t)]_jr(t) = [N_l(t)]_j\) that

\[
d_0^j r_{-1} + d_1^j r_{-2} + \cdots + d_{\lambda_j}^j r_{-\lambda_j} = 0, \\
d_0^j r_{-2} + d_1^j r_{-3} + \cdots + d_{\lambda_j}^j r_{-\lambda_j} = 0, \\
\vdots
\]

Let \( r_{-1}x = \cdots = r_{-\lambda_1}x = 0 \). From the above equations it is easily seen that \( d_{\lambda_j}^j r_{-\lambda_j-1}x = 0, j = 1, \ldots, p \). Since the matrix \( D_l^{\text{row}} \) consisting of the leading coefficients \( d_{\lambda_j}^j \) of the rows of \( D_l(t) \) is invertible, we have \( r_{-\lambda_1-1}x = 0 \). By induction, we can prove that \( r_{-k}x = 0 \) for \( k \geq \lambda_1 \).

Thus, taking into account that \( \lambda_1 \leq \delta \), we obtain \( r_{-\delta-j}x = 0 \) for any \( j \geq 1 \) if the equations \( r_{-1}x = \cdots = r_{-\delta}x = 0 \) are fulfilled. This means that the sequence (4.1) is stabilized if \( m \geq \delta \).
Let $R_1, \ldots, R_\alpha$ be a basis of the $\alpha$-dimensional space $\mathcal{N}_{-m}^R$, $m \geq \delta$. Then $r_{-k}R_j = 0$ for all $k \geq 1$ and $j = 1, \ldots, \alpha$.

We complete the columns $R_1, \ldots, R_\alpha$ to a basis of the space $\mathbb{C}^{p \times 1}$ and form the invertible matrix $C$ from its elements. By the definition of $R_1, \ldots, R_\alpha$, we have $r(t)C = (0 \ r_1(t))$, where $r_1(t)$ is a rational $p \times (q - \alpha)$ matrix function. Let $r_1(t) = N_1(t)D_1^{-1}(t)$ be its right coprime factorization with $D_1(t)$ in the column proper form. We form the matrix polynomials

$$
D_r(t) = C \begin{pmatrix} I_\alpha & 0 \\ 0 & D_1(t) \end{pmatrix}, \quad N_r(t) = \begin{pmatrix} 0 & N_1(t) \end{pmatrix}.
$$

It is easily seen that $r(t) = N_r(t)D_r^{-1}(t)$ and $D_r(t)$ is column proper. Moreover, the first $\alpha$ columns of $D_r(t)$ coincide with $R_1, \ldots, R_\alpha$. Let us prove that $N_r(t), D_r(t)$ are right coprime matrix polynomials. Denote by $(U_1(t), V_1(t))$ a solution of the Bezout equation

$$
U_1(t)D_1(t) + V_1(t)N_1(t) = I_{q-\alpha}
$$

and form the matrix polynomials

$$
V_r(t) = \begin{pmatrix} 0 \\ V_1(t) \end{pmatrix}, \quad U_r(t) = \begin{pmatrix} I_\alpha & 0 \\ 0 & U_1(t) \end{pmatrix} C^{-1}.
$$

It is not difficult to verify that $(U_r(t), V_r(t))$ is a solution of the Bezout equation

$$
U_r(t)D_r(t) + V_r(t)N_r(t) = I_q.
$$

This means that $N_r(t), D_r(t)$ are right coprime, $r(t) = N_r(t)D_r^{-1}(t)$ is the right coprime fractional factorization of $r(t)$, and $D_r(t)$ is column proper. The first $\alpha$ columns of $D_r(t)$ in this factorization are constant. Hence the number of constant columns $d$ is greater than or equal to $\alpha$. Thus $d = \alpha$. The second part of the theorem is proved in a similar manner. □

The relation between fractional factorizations of $r(t)$ and indices and essential polynomials of the sequence $r_{-m}, \ldots, r_{-1}$ is contained in the following theorem.

**Theorem 6.** Let

$$
r(t) = N_r(t)D_r^{-1}(t), \quad r(t) = D_l^{-1}(t)N_l(t)
$$

be coprime fractional factorizations of $r(t)$ such that $D_r(t)$ ($D_l(t)$) is column (row) proper. Denote by $\rho_1 \leq \cdots \leq \rho_q$ ($\lambda_1 \geq \cdots \geq \lambda_p$) the column (row) degrees of $D_r(t)$ ($D_l(t)$). Let $(U_r(t), V_r(t)), (U_l(t), V_l(t))$ be solutions of the Bezout equations (2.4) and (2.2), respectively.

Then the integers

$$
\rho_1 - m - 1, \ldots, \rho_q - m - 1, -\lambda_1, \ldots, -\lambda_p
$$

are indices, the vector polynomials
are right essential polynomials, and
\[ t^{ρ_j - m - 1}[V_r(t)]_1, \ldots, t^{ρ_j - m - 1}[V_r(t)]_q, \quad r^{-λ_1}[D_r(t)]_1, \ldots, r^{-λ_p}[D_r(t)]_p \]
(4.4) are left essential polynomials of the sequence \( r_{-m}, \ldots, r_{-1} \) for any \( m \geq 2δ > 0 \).

**Proof.** Let us verify that all conditions of the essentialness criterion (Theorem 4.1 in [11]) are fulfilled for integers (4.2) and polynomials (4.3) and (4.4).

From the inequality \( m \geq 2δ \) it follows at once that integers (4.2) are in increasing order. Moreover, their sum coincides with \(-(m + 1)q\).

Let \( α (ω) \) be the number of constant columns (rows) of the denominator \( D_r(t) \) \( (D_l(t)) \). By Theorem 5, \( α (ω) \) is the left (right) defect of the sequence \( r_{-m}, \ldots, r_{-1} \) for \( m \geq δ \). Moreover, in this theorem we proved that the first \( α \) columns of \( D_r(t) \) belong to \( \mathcal{A}^{R}_{ρ_j - m} \). Now we show that \([D_r(t)]_j^j \in \mathcal{A}^{R}_{ρ_j - m} \) for \( j = α + 1, \ldots, q \) if \( m \geq 2δ \).

Since \( D_r(t) \) is column proper, \( \text{deg}[D_r(t)]_j^j = ρ_j \). Recall that \( r(t) = \sum_{i=-∞}^{-1} r_it^i \) and let \( r^{(m)}(t) = \sum_{i=-m}^{-1} r_it^i \). Define \( b^{(m)}(t) = \sum_{i=-m}^{-1} r_it^i \). Then from factorization (2.3) we have
\[ r^{(m)}(t)[D_r(t)]_j^j = [N_r(t)]_j^j - b^{(m)}(t)[D_r(t)]_j^j. \]
The Laurent coefficients of \( r(t) \) are found by the formula
\[ r_j = -\frac{1}{2\pi i} \int_{Γ} t^{-j-1} r(t) \, dt, \quad j = -1, -2, \ldots, \]
where \( Γ \) is a simple closed contour such that \( z = 0 \) lies inside \( Γ \). Hence
\[ σ_R[R(t)] = -\frac{1}{2\pi i} \int_{Γ} t^{-1} r^{(m)}(t)R(t) \, dt. \]
Thus for \( k = ρ_j - m, ρ_j - m + 1, \ldots, -1 \) we obtain
\[ σ_R\{t^{-k}[D_r(t)]_j^j\} = -\frac{1}{2\pi i} \int_{Γ} t^{-k-1} ([N_r(t)]_j^j - b^{(m)}(t)[D_r(t)]_j^j) \, dt = 0, \]
because \([N_r(t)]_j^j\) is a polynomial in \( t \), and the rational matrix function \( b^{(m)}(t)[D_r(t)]_j^j \) has the vanishing Laurent coefficients for \( k = ρ_j - m, ρ_j - m + 1, \ldots, -1 \).

Therefore, \([D_r(t)]_j^j \in \mathcal{A}^{R}_{ρ_j - m} \) for \( j = 1, \ldots, q \).

Let us now prove that \([V_l(t)]_j^j \in \mathcal{A}^{R}_{λ_j + 1} \) for \( j = 1, \ldots, q-ω \). Recall that \( \text{deg}[V_l(t)]_j^j < ρ_q \) for the appropriate solution of the Bezout equation (see Eq. (2.5)). Hence \( \text{deg}[V_l(t)]_j^j < m - λ_j + 1 \) if \( m \geq 2δ \). From the factorization (2.1) and the Bezout equation (2.2) we have
\[ r^{(m)}(t)[V_l(t)]_j^j = [D_l^{-1}(t)]_j^j - [U_l(t)]_j^j - b^{(m)}(t)[V_l(t)]_j^j \]
\[ = t^{-λ_j}[D_l^{-1}(t)]_j^j - [U_l(t)]_j^j - b^{(m)}(t)[V_l(t)]_j^j. \]
Here we define the matrix polynomial \( D_\nu(t) \) in \( t^{-1} \) by the equality

\[
D_l(t) = d_l(t)D_\nu(t), \quad d_l(t) = \text{diag}[t^{\lambda_1}, \ldots, t^{\lambda_p}].
\]

Since \( D_l(t) \) is row proper, \( D_\nu(\infty) = D_l^{\text{row}} \) is invertible and \( D_l^{-1}(t) \) is analytic at infinity.

Then for \( k = -\lambda_j + 1, -\lambda_j + 2, \ldots, -1 \) we obtain

\[
\sigma_R\{t^{-k}[V_l(t)]^j\} = -\frac{1}{2\pi i} \int \left( t^{-k-\lambda_j-1}[D_l^{-1}(t)]^j - t^{-k-1}[U_l(t)]^j - t^{-k-1}b^{(m)}(t)[V_l(t)]^j \right) dt = 0,
\]

because \([U_l(t)]^j\) is a polynomial in \( t \), \([D_l^{-1}(t)]^j\) has the vanishing Laurent coefficients for \( k = 1, 2, \ldots, \lambda_j - 1 \), and \([b^{(m)}(t)[V_l(t)]^j\) has the vanishing Laurent coefficients for \( k = -\lambda_j + 1, -\lambda_j + 2, \ldots, -1 \).

Thus \([V_l(t)]^j \in \mathcal{N}_Rk + \lambda_j + 1 \) for \( j = 1, \ldots, p - \omega \).

Now we form the test matrix \( A_R \) for the polynomials (4.3). We will need \( \tilde{\sigma}_R\{t^{k_j}[V_l(t)]^j\}, j = 1, \ldots, p \). It is easily seen that

\[
\tilde{\sigma}_R\{t^{k_j}[V_l(t)]^j\} = -\frac{1}{2\pi i} \int t^{k-1}[D_l^{-1}(t)]^j \ dt = -[(D_l^{\text{row}})^{-1}]^j.
\]

First let \( \omega = 0 \). Taking into account that the leading coefficients of the last \( p \) polynomials (4.3) are equal to zero, we have the following test matrix:

\[
A_R = \begin{pmatrix}
* & -(D_l^{\text{row}})^{-1} \\
D_l^{\text{col}} & 0
\end{pmatrix}.
\]

Thus in this case \( A_R \) is invertible. If \( \omega \neq 0 \), then the test matrix is obtained from the above matrix by deleting the last \( \omega \) columns. Hence it has the full rank. By the essentialness criterion we obtained that the integers \( \rho_1 - m - 1, \ldots, \rho_q - m - 1, -\lambda_1, \ldots, -\lambda_p \) are indices, and the polynomials \([D_r(t)]^1, \ldots, [D_r(t)]^q, [V_l(t)]^1, \ldots, [V_l(t)]^{p-\omega} \) are the right essential polynomials of the sequence.

Moreover, if \( \omega \neq 0 \), then the system of the right essential polynomials can be completed by the polynomials \([V_l(t)]^{p-\omega+1}, \ldots, [V_l(t)]^p \) because this completion leads to the invertible matrix \( A_R \). The second part of the theorem is proved similarly.

In Theorem 6 we construct the essential polynomials having the following additional properties:

1. The leading coefficients of the polynomials \( R_{q+1}(t), \ldots, R_{p+q}(t) \) are equal to zero. Hence the leading coefficients of \( R_1(t), \ldots, R_q(t) \) form an invertible matrix.
2. The constant terms of the polynomials \( L_1(t), \ldots, L_q(t) \) are equal to zero. Therefore the constant terms of \( L_{q+1}(t), \ldots, L_{p+q}(t) \) form an invertible matrix.
The essential polynomials \( R_1(t), \ldots, R_{p+q}(t); L_1(t), \ldots, L_{p+q}(t) \) having these properties will be called factorization polynomials. In the following lemma we will use conforming essential polynomials. For the definition see in [11, Definition 5.3].

**Lemma 7.** Let \( R_1(t), \ldots, R_{p+q}(t), L_1(t), \ldots, L_{p+q}(t) \) be arbitrary conforming essential polynomials of the sequence.

If \( R_1(t), \ldots, R_{p+q}(t), (L_1(t), \ldots, L_{p+q}(t)) \) have property 1 (property 2), then \( L_1(t), \ldots, L_{p+q}(t) \ (R_1(t), \ldots, R_{p+q}(t)) \) have property 2 (property 1). In other words, the sequence always has conforming factorization polynomials.

**Proof.** Suppose that the right essential polynomials \( R_1(t), \ldots, R_{p+q}(t) \) satisfy condition 1. Let \( R_1(t), \ldots, R_{p+q}(t), L_1(t), \ldots, L_{p+q}(t) \) be the conforming essential polynomials, that is, the equality

\[
(t^{-m-1}R(t)d_{-1}(t)) \begin{pmatrix} \alpha_{-1}(t) & \beta_{-1}(t) & \mathcal{L}(t) \end{pmatrix} = I_{p+q}
\]

is fulfilled (see [11, Section 5]). It follows from this that

\[
t^{-m-1}R_1(t)d_{-1}(t)\mathcal{L}_1(t) = -t^{-m-1}R_2(t)d_{-1}(t)\mathcal{L}_2(t).
\]

Here

\[
\mathcal{R}_1(t) = \begin{pmatrix} R_1(t) & \cdots & R_q(t) \end{pmatrix}, \quad \mathcal{R}_2(t) = \begin{pmatrix} R_{q+1}(t) & \cdots & R_{p+q}(t) \end{pmatrix};
\]

\[
\mathcal{L}_1(t) = \begin{pmatrix} L_1(t) \\ \vdots \\ L_q(t) \end{pmatrix}, \quad \mathcal{L}_2(t) = \begin{pmatrix} L_{q+1}(t) \\ \vdots \\ L_{p+q}(t) \end{pmatrix};
\]

\[
d_1(t) = \text{diag}[t^{\mu_1}, \ldots, t^{\mu_q}], \quad d_2(t) = \text{diag}[t^{\mu_{q+1}}, \ldots, t^{\mu_{p+q}}].
\]

By \( \mathcal{R}_1^{\text{col}}, \mathcal{R}_2^{\text{col}} \) we denote the matrices consisting of the coefficients of highest degrees in each column of the matrix polynomials \( \mathcal{R}_1(t), \mathcal{R}_2(t) \), respectively. Let \( \mathcal{L}_{1,0} \) and \( \mathcal{L}_{2,0} \) be the constant terms of \( \mathcal{L}_1(t) \) and \( \mathcal{L}_2(t) \), respectively. Then from (4.5) we get

\[
\mathcal{R}_1^{\text{col}} \mathcal{L}_{1,0} = -\mathcal{R}_2^{\text{col}} \mathcal{L}_{2,0}.
\]

Since \( \mathcal{R}_1^{\text{col}} \) is invertible, and \( \mathcal{R}_2^{\text{col}} = 0 \), we have \( \mathcal{L}_{1,0} = 0 \). Then by the essentialness criterion (Theorem 4.1 in [11]) \( \mathcal{L}_{2,0} \) is invertible. This completes the proof of the first part of the lemma. In a similar manner we verify the second part. 

Now we can obtain the main result of this section.

**Theorem 8.** Let \( r(t) \) be a strictly proper rational \( p \times q \) matrix function, and let \( \delta \) be its McMillan degree. Let \( r_{-m}, \ldots, r_{-1} \) be the sequence formed from the Laurent coefficients of \( r(t) \) for \( m \geq 2\delta > 0 \).
Then the sequence has conforming factorization polynomials. Let \( \mu_1, \ldots, \mu_{p+q} \) be the indices, and let \( R_1(t), \ldots, R_{p+q}(t); L_1(t), \ldots, L_{p+q}(t) \) be arbitrary essential polynomials of the sequence. Denote
\[
D_r(t) = \begin{pmatrix} R_1(t) & \cdots & R_q(t) \end{pmatrix}, \quad D_l(t) = \begin{pmatrix} t^{-\mu_{q+1}} L_{q+1}(t) \\ \vdots \\ t^{-\mu_{p+q}} L_{p+q}(t) \end{pmatrix}
\]
and
\[
N_r(t) = r(t) D_r(t), \quad N_l(t) = D_l(t) r(t).
\]
Then \( \det D_r(t) \not\equiv 0 \), \( \det D_l(t) \not\equiv 0 \), and
\[
r(t) = N_r(t) D_r^{-1}(t), \quad r(t) = D_l^{-1}(t) N_l(t)
\]
are the right and left coprime fractional factorizations of \( r(t) \). If the essential polynomials are factorization polynomials, then \( D_r(t) \) is column proper and \( D_l(t) \) is row proper, the integers \( m + \mu_1 + 1, \ldots, m + \mu_q + 1 \) are the column degrees of \( D_r(t) \), and \( -\mu_{q+1}, \ldots, -\mu_{p+q} \) are the row degrees of \( D_l(t) \).

Let, in addition, the essential polynomials \( R_1(t), \ldots, R_{p+q}(t), L_1(t), \ldots, L_{p+q}(t) \) be conforming factorization polynomials. Denote
\[
V_l(t) = \begin{pmatrix} R_{q+1}(t) & \cdots & R_{p+q}(t) \end{pmatrix}, \quad V_r(t) = -\begin{pmatrix} t^{-\mu_1} L_1(t) \\ \vdots \\ t^{-\mu_q} L_q(t) \end{pmatrix}
\]
and
\[
U_l(t) = D_l^{-1}(t) (I_p - N_l(t) V_l(t)), \quad U_r(t) = (I_q - V_r(t) N_r(t)) D_r^{-1}(t).
\]
Then
\[
(U_l(t), V_l(t)), \quad (U_r(t), V_r(t))
\]
are solutions of the Bezout equations (2.2) and (2.4), respectively.

**Proof.** The formula for the column/row degrees and the existence of factorization polynomials are proved in Theorem 6. The existence of conforming factorization polynomials follows from Lemma 7.

Let us now prove that we can construct the fractional factorizations in terms of arbitrary essential polynomials. Let \( r(t) = \hat{N}_r(t) \hat{D}_r^{-1}(t) \) be a right coprime factorization of \( r(t) \). By Theorem 6, \( \hat{R}_1(t) = [\hat{D}_r(t)]^1, \ldots, \hat{R}_q(t) = [\hat{D}_r(t)]^q \) are the first \( q \) right essential polynomials of the sequence. Since \( m \geq 2\delta \), we have \( \mu_q < \mu_{q+1} \). Hence it follows from the kernel structure of block Toeplitz matrices [11, Theorem 3.1], that
\[ \hat{R}_j(t) = \sum_{i=1}^{q} s_{ij}(t) R_i(t), \quad j = 1, \ldots, q, \]

where \( s_{ij}(t) \equiv 0 \) if \( \mu_j - \mu_i < 0 \), and \( s_{ij}(t) \) is a scalar polynomial in \( t \) of degree \( \leq \mu_j - \mu_i \) if \( \mu_j - \mu_i > 0 \). These equalities can be rewritten in the matrix form

\[ \hat{D}_r(t) = D_r(t) S(t). \]

It is easily seen that \( S(t) = \|s_{ij}\|_{i,j=1}^q \) has the constant determinant. Since \( \det \hat{D}_r(t) \neq 0 \), \( S(t) \) is a unimodular matrix polynomial. Hence \( D_r(t) \) is the denominator of a certain right coprime factorization. If, in addition, the right essential polynomials are factorization ones, then the matrix \( D_r^{\text{col}} \) is invertible, i.e., \( D_r(t) \) is column proper.

In a similar way we can construct a left coprime factorization in terms of the left essential polynomials.

To conclude the proof, it remains to show that solutions of the Bezout equations can be obtained in terms of conforming factorization polynomials.

Define matrix functions (4.7) and (4.8) and verify that (4.8) is a matrix polynomial in \( t \).

Since the essential polynomials are conforming ones, we have

\[ \begin{pmatrix} \alpha_1^+(t) & d_1^{-1}(t) L_1(t) \\ \alpha_2^+(t) & d_2^{-1}(t) L_2(t) \end{pmatrix} \begin{pmatrix} R_1(t) & R_2(t) \\ \beta_1^+(t) & \beta_2^+(t) \end{pmatrix} = I_{p+q}. \]  \hspace{1cm} (4.9)

We already proved that

\[ R_1(t) = D_r(t), \quad d_2^{-1}(t) L_2(t) = D_l(t). \]

Put

\[ V_l(t) = R_2(t), \quad V_r(t) = -d_1^{-1}(t) L_1(t). \]

Let us find \( \beta_1^+(t) \) and \( \alpha_2^+(t) \).

The matrix polynomial \( \beta_1^+(t) \) is uniquely determined by the decomposition

\[ r^{(m)}(t) R_1(t) = \alpha_1^-(t) d_1(t) - \beta_1^+(t). \]

Since \( r(t) = r^{(m)}(t) + b^{(m)}(t) \) and \( R_1(t) = D_r(t) \), we obtain

\[ r^{(m)}(t) R_1(t) = r(t) D_r(t) - b^{(m)} D_r(t) \]

\[ = N_r(t) - t^{m+1} b^{(m)}(t) D_- (t) d_1(t). \]

Here \( D_- (t) = D_r(t) \text{diag}[t^{-\rho_1}, \ldots, t^{-\rho_q}] \) is a matrix polynomial in \( t^{-1} \) because \( D_r(t) \) is column proper. It is easy to verify that \( t^{m+1} b^{(m)}(t) D_- (t) \) is also a matrix polynomial in \( t^{-1} \). This means that we obtain the required decomposition and \( \beta_1^+(t) = -N_r(t) \).

In the same way, \( \alpha_2^+(t) = N_l(t) \). From (4.9) we get

\[ \alpha_1^+(t) D_r(t) + V_r(t) N_r(t) = I_q, \quad N_l(t) V_l(t) + D_l(t) \beta_2^+(t) = I_p. \]
Put \( U_r(t) = \alpha_1^+(t) \) and \( U_l(t) = \beta_2^+(t) \). We see that \( U_r(t), U_l(t) \) are matrix polynomials in \( t \) and \((U_r(t), V_r(t)), (U_l(t), V_l(t))\) are solutions of the Bezout equations. 

5. The explicit construction of Wiener–Hopf factorizations for analytic matrix functions

We can now explicitly construct the Wiener–Hopf factorization of an analytic matrix function \( a(t) \) in terms of indices and essential polynomials of a finite sequence formed from moments of \( a^{-1}(t) \) with respect to \( \Gamma \).

**Theorem 9.** Let \( a(t) \) be a \( p \times p \) matrix function that is analytic in \( \mathbb{D}_+ \), continuous on \( \mathbb{D}_+ \cup \Gamma \), and invertible on \( \Gamma \). Form the sequence \( c_{-m}, \ldots, c_{-1} \) from the moments

\[
c_j = \frac{1}{2\pi i} \int \limits_{\Gamma} t^{-j-1} a^{-1}(t) \, dt, \quad j \in \mathbb{Z},
\]

of \( a^{-1}(t) \) for \( m \geq 2\kappa \). Here \( \kappa \) is the Cauchy index of \( \det a(t) \) with respect to \( \Gamma \).

Then the sequence has factorization essential polynomials. If \( \mu_1, \ldots, \mu_{2p} \) are the indices and \( R_1(t), \ldots, R_{2p}(t); L_1(t), \ldots, L_{2p}(t) \) are any factorization essential polynomials of the sequence, then

\[
\rho_j = m + \mu_j + 1, \quad \lambda_j = -\mu_{p+j}, \quad j = 1, \ldots, p,
\]

are the right and left factorization indices of \( a(t) \), respectively, and the polynomial factors \( l_-(t), r_-(t) \) from the Wiener–Hopf factorizations are constructed by the formulas:

\[
l_-(t) = L_2(t), \quad r_-(t) = R_1(t)d_r^{-1}.
\]

Here

\[
L_2(t) = \begin{pmatrix} L_{p+1}(t) \\ \vdots \\ L_{2p}(t) \end{pmatrix}, \quad R_1(t) = \left( R_1(t) \cdots R_{2p}(t) \right).
\]

**Proof.** Let us represent the meromorphic matrix function \( a^{-1}(t) \) in the form

\[
a^{-1}(t) = r(t) + c_+(t),
\]

where \( r(t) \) is a rational matrix function with poles lying in \( \mathbb{D}_+ \). By Theorems 2 and 4, the construction of the Wiener–Hopf factorizations of \( a(t) \) is reduced to the construction of the coprime fractional factorizations of \( r(t) \). It follows from Theorem 4 that the McMillan degree of \( r(t) \) coincides with the Cauchy index \( \kappa \).

Furthermore, the Laurent coefficients of \( r(t) \) at infinity differ only by sign from the corresponding moments \( c_j \) of the matrix function \( a^{-1}(t) \). The statements of the theorem follow at once from Theorem 8. 

\[\square\]
References


