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Sublines and subplanes of $PG(2, q^3)$ in the Bruck–Bose representation in $PG(6, q)$

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ABSTRACT

In this article we look at the Bruck–Bose representation of $PG(2, q^3)$ in $PG(6, q)$. We look at sublines and subplanes of order q in $PG(2, q^3)$ and describe their representation in $PG(6, q)$. We then show how these results can be generalized to the Bruck–Bose representation of $PG(2, q^n)$ in $PG(2n, q)$.

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1. Introduction

The Bruck–Bose representation of $PG(2, q^2)$ in $PG(4, q)$ has been studied in great detail. Many authors have investigated the representation of Baer sublines, subplanes and unitals of $PG(2, q^2)$ in $PG(4, q)$ (see [2] for a survey and proofs of many of these results). In this article, we investigate a cubic extension, namely the Bruck–Bose representation of $PG(2, q^3)$ in $PG(6, q)$. We study sublines and secant and tangent subplanes of $PG(2, q^3)$ of order q and determine their representation in $PG(6, q)$. We then generalize these results to the Bruck–Bose representation of $PG(2, q^n)$ in $PG(2n, q)$, in that we completely determine the representation in $PG(2n, q)$ of sublines and secant and tangent subplanes of $PG(2, q^n)$ of order q. The results in this paper form a foundation for further work by the authors investigating the representation in $PG(6, q)$ of unitals and Baer subplanes of $PG(2, q^3)$ when q is square.

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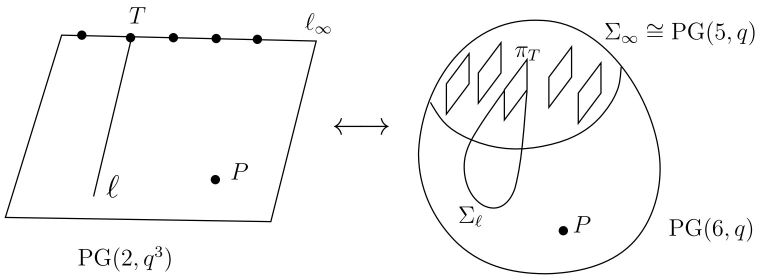

Fig. 1. The Bruck–Bose construction of PG(2, q^3) in PG(6, q).

2. The Bruck–Bose representation of PG(2, q^3) in PG(6, q)

2.1. The 2-spreads and 2-reguli of PG(5, q)

A 2-*spread* of PG(5, q) is a set of $q^3 + 1$ planes that partition PG(5, q). A 2-*regulus* of PG(5, q) is the system of maximal 2-spaces of a Segre variety $\mathcal{S}_{1;2}$ (see [7, Section 25.5] for full details on Segre varieties). That is, a 2-regulus $\mathcal{R}$ is a set of $q + 1$ mutually disjoint planes $\pi_1, \dots, \pi_{q+1}$ with the property that if a line meets three of the planes, then it meets all $q + 1$ of them. Thus there are $q^2 + q + 1$ mutually disjoint lines associated with $\mathcal{R}$ (these are the maximal 1-spaces of $\mathcal{S}_{1;2}$). Three mutually disjoint planes in PG(5, q) lie on a unique 2-regulus. A 2-spread $\mathcal{S}$ is *regular* if for any three planes in $\mathcal{S}$, the 2-regulus containing them is contained in $\mathcal{S}$. In a regular 2-spread, any $q + 1$ spread elements meeting a line form a 2-regulus.

The following construction of a regular 2-spread of PG(5, q) will also be useful. Embed PG(5, q) in PG(5, q^3) and let g be a line of PG(5, q^3) disjoint from PG(5, q). Let g^q, g^{q^2} be the conjugate lines of g, both of these are disjoint from PG(5, q). Let P_i be a point on g; then the plane $\langle P_i, P_i^q, P_i^{q^2}\rangle$ meets PG(5, q) in a plane. As P_i ranges over all the points of g, we get $q^3 + 1$ planes of PG(5, q) that partition the space. These planes form a regular spread $\mathcal{S}$ of PG(5, q). The lines g, g^q, g^{q^2} are called the (conjugate skew) *transversal lines* of the spread $\mathcal{S}$. Conversely, given a regular 2-spread in PG(5, q), there is a unique set of three (conjugate skew) transversal lines in PG(5, q^3) that generate $\mathcal{S}$ in this way. See [7, Section 25.6] for more information on 2-reguli and 2-spreads.

2.2. The Bruck–Bose representation

In this section we introduce the linear representation of a finite translation plane $\mathcal{P}$ of dimension at most three over its kernel, an idea which was developed independently by André [1] and Bruck and Bose [4,5]. We will use the vector space construction as developed by Bruck and Bose.

Let Σ_∞ be a hyperplane of PG(6, q) and let $\mathcal{S}$ be a 2-spread of Σ_∞. We use the phrase *a subspace of* PG(6, q)$\backslash\Sigma_\infty$ to mean a subspace of PG(6, q) that is not contained in Σ_∞. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG(6, q)$\backslash\Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of PG(6, q)$\backslash\Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG(6, q). Fig. 1 illustrates this construction. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^3. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the 2-spread $\mathcal{S}$.

The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane PG(2, q^3) if and only if $\mathcal{S}$ is a regular 2-spread of $\Sigma_\infty \cong$ PG(5, q) [3].

In the case $\mathcal{P}(\mathcal{S}) \cong$ PG(2, q^3), we can relate the coordinates of PG(2, q^3) and PG(6, q) as follows. Let τ be a primitive element in GF(q^3) with primitive polynomial

$$x^3 - t_2x^2 - t_1x - t_0.$$

Then every element $\alpha \in \mathrm{GF}(q^3)$ can be uniquely written as $\alpha = a_0 + a_1\tau + a_2\tau^2$ with $a_0, a_1, a_2 \in \mathrm{GF}(q)$. Points in $\mathrm{PG}(2, q^3)$ have homogeneous coordinates (x, y, z) with $x, y, z \in \mathrm{GF}(q^3)$. Let the line at infinity ℓ_∞ have equation $z = 0$; so the affine points of $\mathrm{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\mathrm{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in \mathrm{GF}(q)$. Let Σ_∞ have equation $z = 0$. Let $P = (\alpha, \beta, 1)$ be a point of $\mathrm{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1\tau + a_2\tau^2$ and $\beta = b_0 + b_1\tau + b_2\tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathrm{GF}(q)$. Then the map $\phi : \mathrm{PG}(2, q^3)\backslash\ell_\infty \to \mathrm{PG}(6, q)\backslash\Sigma_\infty$ such that $\phi(\alpha, \beta, 1) = (a_0, a_1, a_2, b_0, b_1, b_2, 1)$ is the Bruck–Bose map.

To complete this to a projective map, we generalize the construction of Bruck–Bose coordinates of $\mathrm{PG}(2, q^2)$ in $\mathrm{PG}(4, q)$ (see [2, Section 3.4.4] for full details of this 2-dimensional case). First define $\sigma(\alpha, \beta, 0) = (a_0, a_1, a_2, b_0, b_1, b_2, 0)$. Let $(1, \delta, 0)$ be a point of ℓ_∞ in $\mathrm{PG}(2, q^3)$; then we can write $\delta = f_0 + f_1\tau + f_2\tau^2$ for unique $f_0, f_1, f_2 \in \mathrm{GF}(q)$. Then the spread element of Σ_∞ in $\mathrm{PG}(6, q)$ corresponding to $(1, \delta, 0)$ is the plane spanned by the three points D_0, D_1, D_2 given by

$$D_0 = \sigma(1, \delta, 0) = (1, 0, 0, f_0, f_1, f_2, 0),$$

$$D_1 = \sigma(\tau, \delta\tau, 0) = (0, 1, 0, f_2t_0, f_0 + f_2t_1, f_1 + f_2t_2, 0),$$

$$D_2 = \sigma\big(\tau^2, \delta\tau^2, 0\big) = \big(0, 0, 1, f_1t_0 + f_2t_0t_2, f_2t_0 + f_2t_1t_2 + f_1t_1, f_0 + f_2t_1 + f_2t_2^2 + f_1t_2, 0\big).$$

Note also that the point $(0, 1, 0)$ of ℓ_∞ in $\mathrm{PG}(2, q^3)$ corresponds to the spread element that is spanned by the three points $(0, 0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 0, 1, 0)$.

Later we will need the equation of the three (conjugate skew) transversals of $\mathcal{S}$ in the cubic extension $\overline{\Sigma} = \mathrm{PG}(6, q^3)$; so we calculate their equations here.

Lemma 2.1. *Let g be the line of $\mathrm{PG}(6, q^3)$ through the points $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0)$ and $A_2 = (0, 0, 0, p_0, p_1, p_2, 0)$ where $p_0 = t_1 + t_2\tau - \tau^2$, $p_1 = t_2 - \tau$, $p_2 = -1$. Then g is one of the three (conjugate skew) transversals of the regular 2-spread $\mathcal{S}$. (The remaining transversals are g^q, g^{q^2}.)*

Proof. Label the planes of $\mathcal{S}$ by π_δ, $\delta \in \mathrm{GF}(q^3) \cup \{\infty\}$, such that π_δ corresponds to the point $(1, \delta, 0)$ of ℓ_∞ in $\mathrm{PG}(2, q^3)$. Note that g is not a line of $\Sigma_\infty \cong \mathrm{PG}(5, q)$, but lies in the cubic extension. We need to show that the line g meets each of the planes π_δ (considering π_δ as a plane in the cubic extension). Now π_∞ corresponds to the point $(0, 1, 0)$ and so is the plane spanned by the points $(0, 0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 0, 1, 0)$. Clearly π_∞ meets the line g in the point A_2. The plane π_δ, $\delta \in \mathrm{GF}(q^3)$, is spanned by the three points D_0, D_1, D_2 calculated above. The point $P = A_1 + \delta A_2$ is on the line g. Moreover, $P = p_0D_0 + p_1D_1 + p_2D_2$ and so P is on the spread element π_δ. Hence g meets every spread element, and so g and its conjugates g^q, g^{q^2} are the unique transversals of the spread $\mathcal{S}$. □

2.3. Sublines and subplanes of $\mathrm{PG}(2, q^3)$

In any plane $\mathcal{P}$ of order q^3, a natural subplane B to consider is an *order-q-subplane*, that is, a set of $q^2 + q + 1$ points where every line of $\mathcal{P}$ meets B in 0, 1 or $q + 1$ points. Hence every line of the subplane B has $q + 1$ points and we call these *order-q-sublines*. In particular, if $\mathcal{P} = \mathrm{PG}(2, q^3)$, then every order-$q$-subplane is isomorphic to $\mathrm{PG}(2, q)$.

We consider the representation of order-q-sublines and order-q-subplanes in the Bruck–Bose representation of $\mathrm{PG}(2, q^3)$ in $\mathrm{PG}(6, q)$. There are six cases to consider: Theorem 2.4 looks at order-q-sublines contained in ℓ_∞; Theorem 2.3 looks at order-q-sublines tangent to ℓ_∞; Theorem 2.5 looks at order-q-sublines disjoint from ℓ_∞, Theorem 2.2 looks at order-q-subplanes secant to ℓ_∞ and Theorem 2.7 looks at order-q-subplanes tangent to ℓ_∞. Note that the case of order-q-subplanes exterior to ℓ_∞ is not covered in this paper and a succinct description of its representation in $\mathrm{PG}(6, q)$ remains an open problem.

Theorem 2.2. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ defined from a 2-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(6,q)$.

(a) *A plane of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *that meets $q+1$ elements of $\mathcal{S}$ represents an order-q-subplane of $\mathcal{P}(\mathcal{S})$ secant to ℓ_∞.*
(b) *If $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^3)$, then every order-q-subplane of $\mathcal{P}(\mathcal{S})$ that is secant to ℓ_∞ is represented by a plane of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *that meets $q+1$ elements of $\mathcal{S}$.*

Proof. Let π be a plane of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$, so π meets Σ_∞ in a line. We are interested in the case when this line is not contained in a spread element, so it meets $q+1$ spread elements, each in a point. (Note that if $\mathcal{S}$ were a regular spread then these $q+1$ spread elements would form a 2-regulus of $\mathcal{S}$.) Then, in $\mathcal{P}(\mathcal{S})$, π corresponds to a set A of q^2 affine points, and $q+1$ points on ℓ_∞ (corresponding to the $q+1$ spread elements that π meets). In $\mathrm{PG}(6,q)$, a 3-space about a spread element meets π in 0, 1 or $q+1$ points; hence, in $\mathcal{P}(\mathcal{S})$, lines meet A in 0, 1 or $q+1$ points. Thus A is a subplane of order q and (a) holds.

To prove (b), we count the number of order-q-subplanes secant to ℓ_∞ in $\mathrm{PG}(2,q^3)$. Since a quadrangle (A,B,C,D) with $A,B\in\ell_\infty$ and $C,D\notin\ell_\infty$ uniquely determines an order-q-subplane, the number of order-q-subplanes secant to ℓ_∞ is

$$\frac{(q^3+1)(q^3)(q^6)(q^6-2q^3+1)}{(q+1)(q)(q^2)(q^2-2q+1)}=\left(q^2-q+1\right)q^6\left(q^2+q+1\right)^2. \tag{1}$$

We now count the number of planes of $\mathrm{PG}(6,q)$ that meet Σ_∞ in a line which does not lie in a spread element. We first count the number of lines of $\mathrm{PG}(5,q)$; it is

$$\frac{\frac{q^6-1}{q-1}\left(\frac{q^6-1}{q-1}-1\right)}{(q-1)q}=\left(q^4+q^2+1\right)\left(\frac{q^5-1}{q-1}\right).$$

The number of lines in one spread element is q^2+q+1; so we subtract $(q^3+1)(q^2+q+1)$ to find the number of lines not in any spread element. Given that any such line determines $q^6/q^2=q^4$ planes of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ it is easy to show that the required number (1) follows. Hence, when $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^3)$, the correspondence of part (a) is exact. □

As an immediate corollary, we have the representation of order-q-sublines that are tangent to ℓ_∞.

Theorem 2.3. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ defined from a 2-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(6,q)$.

(a) *A line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *represents an order-q-subline of $\mathcal{P}(\mathcal{S})$ tangent to ℓ_∞.*
(b) *If $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^3)$, then every order-q-subline of $\mathcal{P}(\mathcal{S})$ tangent to ℓ_∞ is represented by a line of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$.

Theorem 2.4. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ defined from a 2-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(6,q)$.

(a) *A 2-regulus $\mathcal{R}$ of $\mathcal{S}$ represents an order-q-subline of ℓ_∞ in $\mathcal{P}(\mathcal{S})$.*
(b) *If $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^3)$, then every order-q-subline of ℓ_∞ in $\mathcal{P}(\mathcal{S})$ is represented by a 2-regulus of $\mathcal{S}$.*

Proof. Part (a) is an immediate corollary of Theorem 2.2. To prove (b) we first note that three points on ℓ_∞ in $\mathrm{PG}(2,q^3)$ are contained in a unique subline of order q. Hence the number of sublines of ℓ_∞ of order q is $\binom{q^3+1}{3}/\binom{q+1}{3}=q^2(q^4+q^2+1)$. Alternatively, suppose $\mathcal{S}$ is a regular 2-spread in $\Sigma_\infty\cong\mathrm{PG}(5,q)$. Then the number of 2-reguli in $\mathcal{S}$ is $q^2(q^4+q^2+1)$ by [7, Theorem 25.6.6]. As these two numbers are equal, if $\mathcal{P}(\mathcal{S})\cong\mathrm{PG}(2,q^3)$, then the correspondence in (a) is exact. □

Theorem 2.5.

(a) *Let ℓ be a line of* $\mathrm{PG}(2,q^3)$ *and let b be an order-q-subline of ℓ that is disjoint from ℓ_∞. Then in* $\mathrm{PG}(6,q)$*, b corresponds to a normal rational curve in the* 3*-space Σ corresponding to ℓ.*
(b) *Let Σ be a* 3*-space of* $\mathrm{PG}(6,q)\backslash\Sigma_\infty$ *about a spread element and let $\mathcal{N}$ be a normal rational curve in Σ that is disjoint from Σ_∞. Then $\mathcal{N}$ corresponds to an order-q-subline of* $\mathrm{PG}(2,q^3)$ *disjoint from ℓ_∞ if and only if in the cubic extension* $\mathrm{PG}(6,q^3)$*, the cubic extension $\overline{\mathcal{N}}$ of $\mathcal{N}$ meets the conjugate transversal lines g, g^q, g^{q^2} of the spread $\mathcal{S}$.*

Proof. We use coordinates to prove (a), generalizing the argument in [2, Theorem 3.17]. Let ℓ be the line of equation $x=y$ in $\mathrm{PG}(2,q^3)$ and let $\omega\in \mathrm{GF}(q^3)\backslash \mathrm{GF}(q)$. We find an order-$q$-subline of ℓ disjoint from ℓ_∞ as follows. The set $\{(1,d,0)\mid d\in \mathrm{GF}(q)\cup\{\infty\}\}$ is clearly an order-q-subline of ℓ_∞. The homography with matrix

$$A=\begin{pmatrix}0&1&0\\0&1&1\\\omega&1&0\end{pmatrix}$$

maps this to the set $\ell_\omega=\{P_d=(d,d,d+\omega)\mid d\in \mathrm{GF}(q)\cup\{\infty\}\}$. So ℓ_ω is an order-q-subline of ℓ which is clearly disjoint from ℓ_∞.

As we are using homogeneous coordinates, for all $d\in\mathrm{GF}(q)$ we can write the points P_d of ℓ_ω as

$$P_d=(\alpha,\alpha,c)=\bigl(d\bigl(d+\omega^q\bigr)\bigl(d+\omega^{q^2}\bigr),d\bigl(d+\omega^q\bigr)\bigl(d+\omega^{q^2}\bigr),(d+\omega)\bigl(d+\omega^q\bigr)\bigl(d+\omega^{q^2}\bigr)\bigr).\qquad(2)$$

We claim that $c=c(d)=(d+\omega)(d+\omega^q)(d+\omega^{q^2})$ is a polynomial in d with $c:\mathrm{GF}(q)\to\mathrm{GF}(q)$. We have $c(d)\in\mathrm{GF}(q)$ as $c(d)^q=c(d)$ for all $d\in\mathrm{GF}(q)$. So we can write $c(d)=c_0+c_1d+c_2d^2+d^3$ for some constants $c_0,c_1,c_2\in\mathrm{GF}(q)$. Write $\omega=w_0+w_1\tau+w_2\tau^2$ for constants $w_0,w_1,w_2\in\mathrm{GF}(q)$. Now recall that τ is a solution to $x^3-t_2x^2-t_1x-t_0$ and so the other two roots are τ^q, τ^{q^2}. Hence $t_2=\tau+\tau^q+\tau^{q^2}$, $-t_1=\tau\tau^q+\tau\tau^{q^2}+\tau^q\tau^{q^2}$ and $t_0=\tau\tau^q\tau^{q^2}$. Many relationships follow, for example $t_2^2+2t_1=\tau^2+\tau^{2q}+\tau^{2q^2}$ and $t_1^2-2t_0t_2=\tau^{2q}\tau^{2q^2}+\tau^2\tau^{2q^2}+\tau^2\tau^{2q}$. Using these and other relationships we can show that

$$\begin{aligned}c_0&=w_0^3+w_0^2w_1t_2+w_0^2w_2\bigl(t_2^2+2t_1\bigr)-w_0w_1^2t_1-w_0w_1w_2(3t_0+t_1t_2)\\&\quad+w_0w_2^2\bigl(t_1^2-2t_0t_2\bigr)+w_1^3t_0+w_1^2w_2t_0t_2-w_1w_2^2t_0t_1+w_2^3t_0^2,\\c_1&=3w_0^2+2w_0w_1t_2+w_0w_2\bigl(2t_2^2+4t_1\bigr)-w_1^2t_1-w_1w_2(3t_0+t_1t_2)+w_2^2\bigl(t_1^2-2t_0t_2\bigr),\\c_2&=3w_0+w_1t_2+w_2\bigl(t_2^2+2t_1\bigr).\end{aligned}$$

However, the element $\alpha=d(d+\omega^q)(d+\omega^{q^2})$ is in $\mathrm{GF}(q^3)$ and not necessarily $\mathrm{GF}(q)$, so we will write it as $\alpha=a_0+a_1\tau+a_2\tau^2$ for unique functions $a_0,a_1,a_2:\mathrm{GF}(q)\to\mathrm{GF}(q)$ of d. Using similar methods as for the c_i we calculate

$$\begin{aligned}a_0(d)&=d^3+d^2\bigl(2w_0+w_1t_2+w_2\bigl(t_2^2+2t_1\bigr)\bigr)\\&\quad+d\bigl(w_0^2+w_0w_1t_2+w_0w_2\bigl(t_2^2+2t_1\bigr)-w_1^2t_1-w_1w_2(t_0+t_1t_2)+w_2^2\bigl(t_1^2-t_0t_2\bigr)\bigr),\\a_1(d)&=-d^2w_1+d\bigl(-w_0w_1-w_1^2t_2-w_1w_2t_2^2+w_2^2(t_0+t_1t_2)\bigr),\\a_2(d)&=-d^2w_2+d\bigl(-w_0w_2+w_1^2+w_1w_2t_2-w_2^2t_1\bigr).\end{aligned}$$

In particular, we have $a_0(d)$ is a cubic in d, and $a_1(d)$, $a_2(d)$ are quadratics in d, that is,

$$a_0(d)=a_{00}+a_{01}d+a_{02}d^2+d^3,\qquad a_1(d)=a_{10}+a_{11}d+a_{12}d^2,\qquad a_2(d)=a_{20}+a_{21}d+a_{22}d^2,$$

for constants $a_{ij}\in \mathrm{GF}(q)$, and where $a_{00}=a_{10}=a_{20}=0$.

Hence the point $P_d=(\alpha,\alpha,c)$ in $\mathrm{PG}(2,q^3)$ corresponds to the point

$$\boldsymbol{P}_d=\big(a_0(d),a_1(d),a_2(d),a_0(d),a_1(d),a_2(d),c(d)\big) \tag{3}$$

in $\mathrm{PG}(6,q)$.

To show that $\mathcal{N}=\{\boldsymbol{P}_d \mid d\in \mathrm{GF}(q)\cup\{\infty\}\}$ is a normal rational curve in Σ, it is sufficient to exhibit a homography that maps the point $(d^3,d^2,d,1)$ to the point $(a_0(d),a_1(d),a_2(d),c(d))$. Consider the matrix

$$B=\begin{pmatrix}1 & a_{02} & a_{01} & a_{00}\\ 0 & a_{12} & a_{11} & a_{10}\\ 0 & a_{22} & a_{21} & a_{20}\\ 1 & c_2 & c_1 & c_0\end{pmatrix},\quad \text{where}\ \begin{pmatrix}a_0(d)\\ a_1(d)\\ a_2(d)\\ c(d)\end{pmatrix}=B\begin{pmatrix}d^3\\ d^2\\ d\\ 1\end{pmatrix}. \tag{4}$$

To show that B corresponds to a homography, we need to show that the determinant of B is non-zero. It is sufficient to show that there are four distinct values of d for which the points $\boldsymbol{P}_d$ are linearly independent, since this would mean that B has full rank.

We will do this in the following way. First, we show that for $d\in \mathrm{GF}(q)$ we can obtain the coordinates of $\boldsymbol{P}_d$ by solving a set of simultaneous linear equations. Then we can extend this definition to $d\in \mathrm{GF}(q^3)$, so the extended curve $\overline{\mathcal{N}}=\{\boldsymbol{P}_d \mid d\in \mathrm{GF}(q^3)\cup\{\infty\}\}$ contains $\mathcal{N}$. It is sufficient to find four values of $d\in \mathrm{GF}(q^3)$ with points $\boldsymbol{P}_d$ being linearly independent. The points we will consider correspond to $d=0,-\omega,-\omega^q,-\omega^{q^2}$.

Recall the definition of P_d ($d\in \mathrm{GF}(q)$) from (2). So we have

$$d\big(d+\omega^q\big)\big(d+\omega^{q^2}\big)=a_0(d)+a_1(d)\tau+a_2(d)\tau^2 \tag{5}$$

where the $a_i:\mathrm{GF}(q)\to \mathrm{GF}(q)$. How can we determine, without explicit calculation, the values of the $a_i(d)$ for a given value of d? As above, for $n=3$ we can explicitly calculate the a_i and hence $a_i(d)$ (for $d\in \mathrm{GF}(q)$) but for general n this will be a difficult problem.

Eq. (5) determines uniquely the values of $a_i(d)$ for $d\in \mathrm{GF}(q)$. However, when we extend to $d\in \mathrm{GF}(q^3)$, then one equation will not uniquely determine the values of $a_i(d)$ for a given value of d. As there are three unknowns $a_0(d)$, $a_1(d)$, $a_2(d)$, we will need three linearly independent equations to uniquely determine the values of $a_i(d)$. Applying the field automorphism to (5), and then again to the resultant equation, we obtain for all $d\in \mathrm{GF}(q)$,

$$d\big(d+\omega^{q^2}\big)(d+\omega)=a_0(d)+a_1(d)\tau^q+a_2(d)\tau^{2q}, \tag{6}$$

$$d(d+\omega)\big(d+\omega^q\big)=a_0(d)+a_1(d)\tau^{q^2}+a_2(d)\tau^{2q^2}. \tag{7}$$

Adding our last equation from the definition of $\boldsymbol{P}_d$ we obtain the system $D=TR_d$, where D, T, R_d are defined below

$$D=\begin{pmatrix}d(d+\omega^q)(d+\omega^{q^2})\\ d(d+\omega^{q^2})(d+\omega)\\ d(d+\omega)(d+\omega^q)\\ (d+\omega)(d+\omega^q)(d+\omega^{q^2})\end{pmatrix},\qquad T=\begin{pmatrix}1 & \tau & \tau^2 & 0\\ 1 & \tau^q & \tau^{2q} & 0\\ 1 & \tau^{q^2} & \tau^{2q^2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\qquad R_d=\begin{pmatrix}a_0(d)\\ a_1(d)\\ a_2(d)\\ c(d)\end{pmatrix}.$$

Note that the determinant of T is 1 times the determinant of a Vandermonde matrix, and so $|T| \neq 0$. Hence we have obtained expressions for $a_0(d)$, $a_1(d)$, $a_2(d)$ and $c(d)$. The expression for $\boldsymbol{P}_d$ given in (3) is defined for $d \in \mathrm{GF}(q)$, but extends in the natural way for $d \in \mathrm{GF}(q^3)$ using our expression $R_d = T^{-1}D$. We will now consider the value $d = -\omega$.

We work in $\mathrm{PG}(6, q^3)$ and show that

$$\boldsymbol{P}_{-\omega} = \alpha_\omega\big(\tau^2 - t_2\tau - t_1, \tau - t_2, 1, \tau^2 - t_2\tau - t_1, \tau - t_2, 1, 0\big),$$

where $\alpha_\omega \in \mathrm{GF}(q^3)$ (defined below) is a constant depending on ω. For $d = -\omega$, the last coordinate $c(d)$ of $\boldsymbol{P}_d$ is zero, so we only need to show that our choice of $\boldsymbol{P}_{-\omega}$ satisfies (5), (6), (7). First note that

$$-\omega\big(\omega^q - \omega\big)\big(\omega^{q^2} - \omega\big) = \big(\tau^q - \tau\big)\big(\tau^{q^2} - \tau\big)\alpha_\omega, \tag{8}$$

where $\alpha_\omega = -\omega(w_1 + w_2(\tau^q + \tau))(w_1 + w_2(\tau^{q^2} + \tau)) \neq 0$ since the LHS of (8) is non-zero. So substituting $d = -\omega$ into (5), (6) and (7) and letting $A_i = a_i/\alpha_\omega : \mathrm{GF}(q^3) \to \mathrm{GF}(q^3)$ $(1 \leqslant i \leqslant 3)$, we obtain

$$-\omega\big(-\omega + \omega^q\big)\big(-\omega + \omega^{q^2}\big) = \alpha_\omega\big(A_0 + A_1\tau + A_2\tau^2\big), \tag{9}$$

$$0 = \alpha_\omega\big(A_0 + A_1\tau^q + A_2\tau^{2q}\big), \tag{10}$$

$$0 = \alpha_\omega\big(A_0 + A_1\tau^{q^2} + A_2\tau^{2q^2}\big); \tag{11}$$

that is

$$\big(-\tau + \tau^q\big)\big(-\tau + \tau^{q^2}\big) = A_0 + A_1\tau + A_2\tau^2, \tag{12}$$

$$0 = A_0 + A_1\tau^q + A_2\tau^{2q}, \tag{13}$$

$$0 = A_0 + A_1\tau^{q^2} + A_2\tau^{2q^2}. \tag{14}$$

Now we check that RHS (12) $= (\tau^2 - t_2\tau - t_1) + (\tau - t_2)\tau + \tau^2 = 3\tau^2 - 2t_2\tau - t_1$. Also, LHS (12) $= \tau^q\tau^{q^2} - \tau\tau^{q^2} - \tau\tau^q + \tau^2 = (-t_1 - \tau\tau^q - \tau\tau^{q^2}) - \tau\tau^{q^2} - \tau\tau^q + \tau^2 = -t_1 - 2\tau(\tau^q + \tau^{q^2}) + \tau^2 = -t_1 - 2\tau(t_2 - \tau) + \tau^2 = 3\tau^2 - 2t_2\tau - t_1 =$ RHS (12). Further, RHS (13) $= (\tau^2 - t_2\tau - t_1) + (\tau - t_2)\tau^q + \tau^{2q} = (\tau^2 - t_2\tau - t_1) + (-\tau^q - \tau^{q^2})\tau^q + \tau^{2q} = \tau^2 - t_2\tau - t_1 - \tau^q\tau^{q^2} = \frac{1}{\tau}(\tau^3 - t_2\tau^2 - t_1\tau - \tau\tau^q\tau^{q^2}) = \frac{1}{\tau}(t_0 - t_0) = 0$ as required. Similarly for (14). This proves our equation for $\boldsymbol{P}_{-\omega}$ is correct.

Comparing the coordinates of $\boldsymbol{P}_{-\omega} = \alpha_\omega(\tau^2 - t_2\tau - t_1, \tau - t_2, 1, \tau^2 - t_2\tau - t_1, \tau - t_2, 1, 0)$ to the definition of the transversal g calculated in Lemma 2.1, we see that $\boldsymbol{P}_{-\omega} \in g$. Note that $c(-\omega) = 0$, and as the coefficients of the polynomial $a_i(d)$ are in $\mathrm{GF}(q)$, it follows that $a_i(d)^q = a_i(d^q)$. Hence

$$\begin{aligned}(\boldsymbol{P}_{-\omega})^q &= \big(a_0(-\omega)^q, a_1(-\omega)^q, a_2(-\omega)^q, a_0(-\omega)^q, a_1(-\omega)^q, a_2(-\omega)^q, 0\big)\\ &= \big(a_0\big(-\omega^q\big), a_1\big(-\omega^q\big), a_2\big(-\omega^q\big), a_0\big(-\omega^q\big), a_1\big(-\omega^q\big), a_2\big(-\omega^q\big), 0\big)\\ &= \boldsymbol{P}_{-\omega^q}.\end{aligned}$$

So $(\boldsymbol{P}_{-\omega})^q = \boldsymbol{P}_{-\omega^q}$ lies on the transversal g^q, and $(\boldsymbol{P}_{-\omega})^{q^2} = \boldsymbol{P}_{-\omega^{q^2}}$ lies on the transversal g^{q^2}. As the lines g, g^q and g^{q^2} are independent and lie in the cubic extension $\overline{\Sigma}_\infty = \mathrm{PG}(5, q^3)$ of $\Sigma_\infty = \mathrm{PG}(5, q)$, and $\boldsymbol{P}_0 \in \Sigma = \mathrm{PG}(6, q)\backslash\Sigma_\infty$, it follows that the points $\boldsymbol{P}_{-\omega}$, $\boldsymbol{P}_{-\omega^q}$, $\boldsymbol{P}_{-\omega^{q^2}}$, $\boldsymbol{P}_0$ are independent points in the 3-space $\overline{\Sigma}$ defined by the $\boldsymbol{P}_d$. This is the result we need to show that $\overline{\mathcal{N}} = \{\boldsymbol{P}_d \mid d \in \mathrm{GF}(q^3) \cup \{\infty\}\}$ is a normal rational curve in Σ, and hence $\mathcal{N} = \{\boldsymbol{P}_d \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$ is a normal rational curve in 3-space Σ corresponding to ℓ, completing the proof of part (a).

Note that we have shown above that $\mathcal{N}$ meets the plane $\Sigma \cap \Sigma_\infty$ when $c = 0$, and in the cubic extension $\mathrm{PG}(6, q^3)$, $\overline{\mathcal{N}}$ meets the plane $\overline{\Sigma} \cap \overline{\Sigma}_\infty$ in the three points $\boldsymbol{P}_{-\omega}$, $\boldsymbol{P}_{-\omega^q}$, $\boldsymbol{P}_{-\omega^{q^2}}$.

We now prove part (b). Let ℓ be a line of $\mathrm{PG}(2, q^3)$ and let Σ be the 3-space of $\mathrm{PG}(6, q)$ corresponding to ℓ. We now count to show that the number of normal rational curves in Σ that when extended to $\mathrm{PG}(6, q^3)$ meet the transversals of $\mathcal{S}$, is equal to the number of order-q-sublines of ℓ that are disjoint from ℓ_∞. In $\mathrm{PG}(6, q^3)$, let $P_1 = g \cap \overline{\Sigma}$, $P_2 = g^q \cap \overline{\Sigma}$, and $P_3 = g^{q^2} \cap \overline{\Sigma}$. Now $\langle P_1, P_2, P_3\rangle$ is a plane that meets $\mathrm{PG}(6, q)$ in the spread element $\pi = \Sigma \cap \Sigma_\infty$. Further, if the extension of a plane of $\mathrm{PG}(6, q)$ contains one of the P_i, then it contains them all. Hence if Q_1, Q_2, Q_3 are three non-collinear points of $\Sigma\backslash\pi$, then in $\mathrm{PG}(6, q^3)$, $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ is a set of six points, no four coplanar. Hence, by [6, Theorem 21.1.1], they lie in a unique normal rational curve $\overline{\mathcal{N}}$ of $\overline{\Sigma} \subset \mathrm{PG}(6, q^3)$. That is, Q_1, Q_2, Q_3 lie in a unique normal rational curve of Σ which contains the points P_1, P_2, P_3 in the cubic extension $\mathrm{PG}(6, q^3)$.

Let Q_1, Q_2, Q_3 be three points of $\ell\backslash\ell_\infty$. They lie in a unique order-q-subline of ℓ. This subline meets ℓ_∞ if and only if, in $\mathrm{PG}(6, q)$, the points Q_1, Q_2, Q_3 are collinear. If Q_1, Q_2, Q_3 are not collinear in $\mathrm{PG}(6, q)$, then, by the above argument, they lie in a unique normal rational curve of Σ that contains P_1, P_2, P_3 in the cubic extension $\mathrm{PG}(6, q^3)$. Hence the number of order-q-sublines of ℓ that are disjoint from ℓ_∞ is equal to the number of normal rational curves of Σ that contain P_1, P_2, P_3 in the cubic extension $\mathrm{PG}(6, q^3)$, proving part (b). □

We next look at the representation of subplanes that are tangent to ℓ_∞. We will need to use coordinates; so we first present a lemma that calculates results about the coordinates of a representative order-q-subplane that is tangent to ℓ_∞ in $\mathrm{PG}(2, q^3)$. Note that $\mathrm{PG}(2, q) = \{(x, y, z) \mid x, y, z \in \mathrm{GF}(q)$, not all zero$\}$ is an order-q-subplane that is secant to ℓ_∞. We find a homography that maps this to a tangent order-q-subplane. Note also that the point $(x, y, 1)$ lies in $\mathrm{PG}(2, q)$ if and only if $x^q = x$ and $y^q = y$.

Lemma 2.6. *In* $\mathrm{PG}(2, q^3)$, *let B be the unique order-q-subplane containing the quadrangle* $(1, 0, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ *and* $(1 + \omega, 1, 1 + \omega)$ *for some fixed* $\omega \in \mathrm{GF}(q^3)\backslash\mathrm{GF}(q)$. *Then B is tangent to ℓ_∞, and B contains the points* $(1, 0, 1)$, $(1, 1, 1 + \omega)$, *and the order-q-subline* $\ell_\omega = \{P_d = (d, d, d + \omega) \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$. *Further, B is the image of the order-q-subplane* $\mathrm{PG}(2, q)$ *under the homography σ_1 with matrix A_1; and also under the homography σ_2 with matrix A_2 where*

$$A_1 = \begin{pmatrix} -\omega & 1+\omega & 0 \\ 0 & 1 & 0 \\ 0 & 1+\omega & -\omega \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \omega & 1 & -\omega \\ 0 & 1 & 0 \\ \omega & 1 & 0 \end{pmatrix}$$

such that $\sigma_i : \mathrm{PG}(2, q) \to B$ *is given by* $\sigma_i(X) = A_i X$, *writing each point X as a column vector* $(1 \leqslant i \leqslant 2)$.

Proof. There is a unique homography σ_1 satisfying $\sigma_1(1, 0, 0) = (1, 0, 0)$, $\sigma_1(0, 1, 0) = (1 + \omega, 1, 1 + \omega)$, $\sigma_1(0, 0, 1) = (0, 0, 1)$, and $\sigma_1(1, 1, 1) = (1, 1, 1)$, namely the homography with matrix A_1 given above. So B is the image of $\mathrm{PG}(2, q)$ under the homography σ_1. Note that $\sigma_1^{-1} : B \to \mathrm{PG}(2, q)$ such that $\sigma_1^{-1}(X) = A_1' X$ has matrix given by

$$A_1' = \begin{pmatrix} -1 & 1+\omega & 0 \\ 0 & \omega & 0 \\ 0 & 1+\omega & -1 \end{pmatrix}.$$

To show that B is tangent to ℓ_∞ (of equation $z = 0$), we show that $(1, 0, 0)$ is the only point of B which is on ℓ_∞. Now, a point $(x, y, z) \in \mathrm{PG}(2, q)$ maps to the point $\sigma_1(x, y, z) = (-x\omega + y(1 + \omega), y, y(1 + \omega) - z\omega)$ of B, which is on ℓ_∞ if and only if $y(1 + \omega) - z\omega = 0$ for some $y, z \in \mathrm{GF}(q)$, if and only if $y = z = 0$, giving the point $(1, 0, 0)$. Hence B meets ℓ_∞ in the point $(1, 0, 0)$. It is easy to check that $\sigma_1^{-1}(1, 0, 1)$, $\sigma_1^{-1}(1, 1, 1 + \omega)$, and $\sigma_1^{-1}(d, d, d + \omega)$ for $d \in \mathrm{GF}(q) \cup \{\infty\}$ all lie in $\mathrm{PG}(2, q)$. Hence $(1, 0, 1)$, $(1, 1, 1 + \omega)$ and ℓ_ω all lie in B.

Alternatively, we can map $\mathrm{PG}(2,q)$ to an order-q-subplane B' via the homography σ_2 uniquely determined by $\sigma_2(1,0,0)=(1,0,1)$, $\sigma_2(0,1,0)=(1,1,1)$, $\sigma_2(0,0,1)=(1,0,0)$, and $\sigma_2(1,1,1)=(1,1,1+\omega)$. We calculate $\sigma_2(X)=A_2X$ where A_2 is given in the statement of the lemma. As B and B' both contain the quadrangle $(1,0,1)$, $(1,1,1)$, $(1,0,0)$, and $(1,1,1+\omega)$, we have $B=B'$. Note also that σ_2^{-1} has matrix A_2' given by

$$A_2'=\begin{pmatrix}0 & -1 & 1\\ 0 & \omega & 0\\ -1 & 0 & 1\end{pmatrix}. \qquad \square$$

Theorem 2.7. *Let B be an order-q-subplane of $\mathrm{PG}(2,q^3)$ that is tangent to ℓ_∞ in the point T. Let π_T be the spread element corresponding to T. Then B determines a set $\boldsymbol{B}$ of points in $\mathrm{PG}(6,q)$ (where the affine points of B correspond to the affine points of $\boldsymbol{B}$) such that:*

(a) *$\boldsymbol{B}$ is a ruled surface with conic directrix $\mathcal{C}$ contained in the plane $\pi_T\in\mathcal{S}$, and normal rational curve directrix $\mathcal{N}$ contained in a 3-space Σ that meets Σ_∞ in a spread element (distinct from π_T). The points of $\boldsymbol{B}$ lie on $q+1$ pairwise disjoint generator lines joining $\mathcal{C}$ to $\mathcal{N}$.*
(b) *The $q+1$ generator lines of $\boldsymbol{B}$ joining $\mathcal{C}$ to $\mathcal{N}$ are determined by a projectivity from $\mathcal{C}$ to $\mathcal{N}$.*
(c) *When we extend $\boldsymbol{B}$ to $\mathrm{PG}(6,q^3)$, it contains the conjugate transversal lines g, g^q, g^{q^2} of the spread $\mathcal{S}$.*
(d) *$\boldsymbol{B}$ is the intersection of nine quadrics in $\mathrm{PG}(6,q)$.*

Proof. Let B be an order-q-subplane of $\mathrm{PG}(2,q^3)$ that meets ℓ_∞ in the point T. Consider the $q+1$ order-q-sublines of B through T. By Theorem 2.3(b), these correspond to $q+1$ lines $m_1,\dots,m_{q+1}$ of $\mathrm{PG}(6,q)\backslash\Sigma_\infty$. Let $\boldsymbol{B}$ be the set of points that lie on these lines, so the affine part of $\boldsymbol{B}$ corresponds to the affine part of the order-q-subplane B. Let $m_i\cap\pi_T=M_i$, $i=1,\dots,q+1$.

First note that $M_1,\dots,M_{q+1}$ are distinct. Since if $M_i=M_j$, then $\langle m_i,m_j\rangle$ is a plane and either meets π_T in a point and consequently corresponds to an order-q-subplane secant to ℓ_∞ (Theorem 2.2), or, meets π_T in a line. In the first case we have two order-q-subplanes in $\mathrm{PG}(2,q^3)$ with a common quadrangle, a contradiction, and in the second case the 3-subspace spanned by m_i and π_T corresponds to a line of $\mathrm{PG}(2,q^3)$ containing at least $2(q+1)$ points of B, another contradiction. If ℓ is a line of $\mathrm{PG}(2,q^3)$ that does not contain T and meets B in an order-q-subline (necessarily disjoint from ℓ_∞) then in $\mathrm{PG}(6,q)$, ℓ corresponds to a 3-space Σ that meets $\boldsymbol{B}$ in a normal rational curve $\mathcal{N}$ (by Theorem 2.5(a)).

We will show that the points $\{M_1,\dots,M_{q+1}\}$ form a conic. Hence we can conclude that $\boldsymbol{B}$ consists of $q+1$ mutually disjoint lines $m_1,\dots,m_{q+1}$ joining the conic $\mathcal{C}=\{M_1,\dots,M_{q+1}\}$ in π_T, and the normal rational curve $\mathcal{N}$. That is, $\boldsymbol{B}$ is a ruled surface with a conic directrix $\mathcal{C}$ and a normal rational curve directrix $\mathcal{N}$. We use coordinates to show that the points $\{M_1,\dots,M_{q+1}\}$ form a conic. As in the proof of Theorem 2.5, fix $\omega\in\mathrm{GF}(q^3)\backslash\mathrm{GF}(q)$ and consider the point set of $\mathrm{PG}(2,q^3)$ given by $\ell_\omega=\{P_d=(d,d,d+\omega)\mid d\in\mathrm{GF}(q)\cup\{\infty\}\}$. Then ℓ_ω is an order-q-subline of the line $x=y$ which is disjoint from ℓ_∞. Let B be the unique order-q-subplane containing ℓ_ω, $T=(1,0,0)$ and $A=(1,0,1)$ (as calculated in Lemma 2.6). Note that B is tangent to ℓ_∞ at T. The line $x=z$ has two points in B, and so meets B in an order-q-subline. Let R_d, $d\in\mathrm{GF}(q)\cup\{\infty\}$ be the point of intersection of the line $x=z$ and the line TP_d, so $R_d=(d+\omega,d,d+\omega)$, see Fig. 2.

Using the notation from Theorem 2.5, the coordinates of $\boldsymbol{P}_d$, $\boldsymbol{R}_d$ in $\mathrm{PG}(6,q)$, are $\boldsymbol{P}_d=(a_0(d),a_1(d),a_2(d),a_0(d),a_1(d),a_2(d),c(d))$ and $\boldsymbol{R}_d=(c(d),0,0,a_0(d),a_1(d),a_2(d),c(d))$. If $d\in\mathrm{GF}(q)$, the line joining $\boldsymbol{P}_d$ and $\boldsymbol{R}_d$ meets Σ_∞ of equation $z=0$ in the point $\boldsymbol{Q}_d=(a_0(d)-c(d),a_1(d),a_2(d),0,0,0,0)$. Now consider the case $d=\infty$. In $\mathrm{PG}(2,q^3)$, $R_\infty=P_\infty=(1,1,1)$, so to find the final line through T we need to find another point of B on the line through $T=(1,0,0)$ and $P_\infty=(1,1,1)$, that is, the line of equation $y=z$. We note that $P_1=(1,1,1+\omega)$ and $(1,0,1)$ are points of B, so the line joining them is a line of B and hence meets $y=z$ in a point of B; namely the point with coordinates $F=(1-\omega,1,1)$. The two points P_∞ and F on the line $y=z$ are represented in $\mathrm{PG}(6,q)$ by the points $\boldsymbol{P}_\infty=(1,0,0,1,0,0,1)$ and $\boldsymbol{F}=(1-w_0,-w_1,-w_2,1,0,0,1)$. The line joining $\boldsymbol{P}_\infty$ and $\boldsymbol{F}$ meets Σ_∞ in the point $(-w_0,-w_1,-w_2,0,0,0,0)$, which is the point $\boldsymbol{Q}_\infty$. Hence we

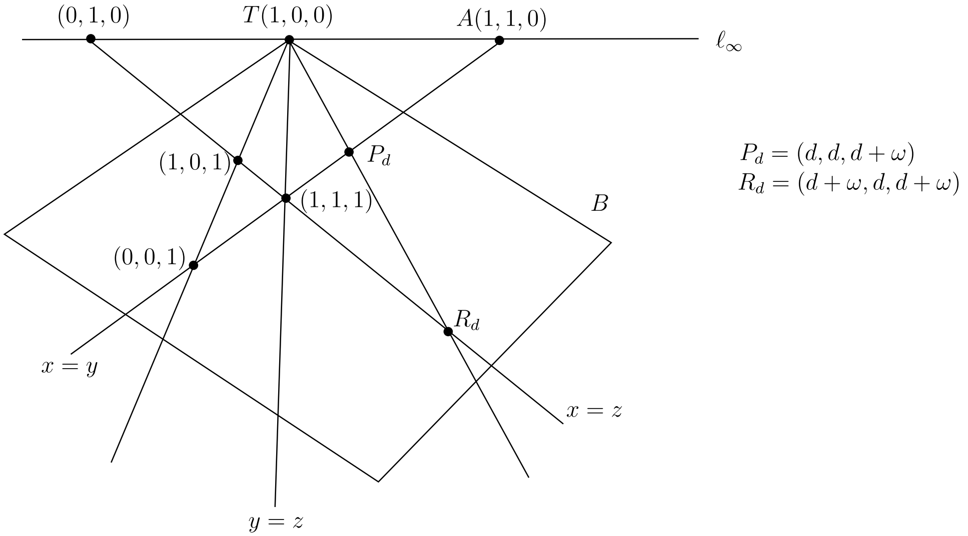

Fig. 2. A tangent order-q-subplane of $\mathrm{PG}(2, q^3)$.

have $\{M_1, \ldots, M_{q+1}\} = \{\boldsymbol{Q}_d \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$. Now $a_0(d) - c(d)$, $a_1(d)$, $a_2(d)$ are quadratics in d over $\mathrm{GF}(q)$ with "leading terms" $-d^2 w_0$, $-d^2 w_1$, $-d^2 w_2$ respectively, so at least one is nonlinear. Hence we need to find a homography that maps the set $\{\boldsymbol{Q}_d \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$ to the set of points $\{(d^2, d, 1) \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$. In the proof of Theorem 2.5, the matrix B was defined as follows, and now suppose C is the matrix with

$$\begin{pmatrix} a_0(d) \\ a_1(d) \\ a_2(d) \\ c(d) \end{pmatrix} = B \begin{pmatrix} d^3 \\ d^2 \\ d \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} a_0(d) - c(d) \\ a_1(d) \\ a_2(d) \end{pmatrix} = C \begin{pmatrix} d^2 \\ d \\ 1 \end{pmatrix}.$$

In Theorem 2.5 we showed that the four points P_d with $d = 0, -\omega, -\omega^q, -\omega^{q^2}$ were independent. Note that for $d = -\omega, -\omega^q, -\omega^{q^2}$, $c(d) = 0$, hence it follows that Q_d for $d = -\omega, -\omega^q, -\omega^{q^2}$ are three independent points. Thus C represents a homography, and hence the points $\{M_1, \ldots, M_{q+1}\}$ form a conic in π_T.

Hence we have shown that $\boldsymbol{B}$ is a ruled surface with conic directrix $\mathcal{C} = \{\boldsymbol{Q}_d \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$ in π_T and normal rational curve directrix $\mathcal{N} = \{\boldsymbol{P}_d \mid d \in \mathrm{GF}(q) \cup \{\infty\}\}$ in Σ, with $q+1$ generator lines $\boldsymbol{Q}_d\boldsymbol{P}_d$, $d \in \mathrm{GF}(q) \cup \{\infty\}$, proving (a). From the coordinates, we have a natural projectivity from the points $\boldsymbol{Q}_d$ of $\mathcal{C}$ to the points $\boldsymbol{P}_d$ of $\mathcal{N}$, proving (b).

Now consider the cubic extension $\mathrm{PG}(6, q^3)$. We can naturally extend the conic $\mathcal{C}$ and normal rational curve $\mathcal{N}$ in $\mathrm{PG}(6, q)$ to $\bar{\mathcal{C}}$ and $\overline{\mathcal{N}}$ in $\mathrm{PG}(6, q^3)$. Hence the projectivity $\mathcal{C} \to \mathcal{N}$ is extended to $\bar{\mathcal{C}} \to \overline{\mathcal{N}}$, and so $\boldsymbol{B}$ is naturally extended to $\overline{\boldsymbol{B}}$ in $\mathrm{PG}(6, q^3)$. We have from Theorem 2.5 that $\overline{\mathcal{N}}$ contains the points $g \cap \overline{\Sigma} = \boldsymbol{P}_{-\omega}$, $g^q \cap \overline{\Sigma} = \boldsymbol{P}_{-\omega^q}$, $g^{q^2} \cap \overline{\Sigma} = \boldsymbol{P}_{-\omega^{q^2}}$. Similarly, $\bar{\mathcal{C}}$ contains the point $\boldsymbol{Q}_{-\omega}$ which we calculate to be $\boldsymbol{Q}_{-\omega} = (\tau^2 - t_2\tau - t_1, \tau - t_2, 1, 0, 0, 0, 0)$, which lies on g (using Lemma 2.1). It follows that $\bar{\mathcal{C}}$ contains the three points $g \cap \overline{\pi}_T = \boldsymbol{Q}_{-\omega}$, $g^q \cap \overline{\pi}_T = \boldsymbol{Q}_{-\omega^q}$, $g^{q^2} \cap \overline{\pi}_T = \boldsymbol{Q}_{-\omega^{q^2}}$. Thus $\overline{\boldsymbol{B}}$ contains the three transversal lines $g = \boldsymbol{P}_{-\omega}\boldsymbol{Q}_{-\omega}$, $g^q = \boldsymbol{P}_{-\omega^q}\boldsymbol{Q}_{-\omega^q}$, $g^{q^2} = \boldsymbol{P}_{-\omega^{q^2}}\boldsymbol{Q}_{-\omega^{q^2}}$, proving (c).

Finally we show that $\boldsymbol{B}$ is an algebraic variety by showing that it is the intersection of nine quadrics. We continue to work with the order-q-subplane B with coordinates given in Lemma 2.6. So the affine points of B are the points $(x, y, 1)$ satisfying $\sigma_1^{-1}(x, y, 1) \in \mathrm{PG}(2, q)$ (where σ_1 is specified in Lemma 2.6). That is, we require $P = ((1+\omega)y - x, \omega y, (1+\omega)y - 1) \in \mathrm{PG}(2, q)$.

We first consider the case $(1+\omega)y-1\neq 0$. So $P\equiv(((1+\omega)y-x)/((1+\omega)y-1),\omega y/((1+\omega)y-1),1)$, and so $P\in \mathrm{PG}(2,q)$ if and only if

$$\left(\frac{(1+\omega)y-x}{(1+\omega)y-1}\right)^q=\frac{(1+\omega)y-x}{(1+\omega)y-1}\quad\text{and}\quad\left(\frac{\omega y}{(1+\omega)y-1}\right)^q=\frac{\omega y}{(1+\omega)y-1}.$$

The second equation is

$$\big((1+\omega)y-1\big)(\omega y)^q-\big((1+\omega)y-1\big)^q\omega y=0. \tag{15}$$

We simplify the first equation as follows:

$$\left(\frac{(1+\omega)y-1+1-x}{(1+\omega)y-1}\right)^q=\frac{(1+\omega)y-1+1-x}{(1+\omega)y-1},$$
$$(1-x)^q\big((1+\omega)y-1\big)=(1-x)\big((1+\omega)y-1\big)^q. \tag{16}$$

We use Eq. (15) in (16) to obtain

$$(1-x)(\omega y)^q-(1-x)^q\omega y=0, \tag{17}$$

which we will need later.

Consider the two equations (15) and (16) in $\mathrm{PG}(2,q^3)$. If we write $x=x_0+x_1\tau+x_2\tau^2$, $y=y_0+y_1\tau+y_2\tau^2$, and $\omega=w_0+w_1\tau+w_2\tau^2$ for $x_i,y_i,w_i\in \mathrm{GF}(q)$, then simplify and equate coefficients of powers of τ, we will obtain the equations of six affine quadrics in $\mathrm{PG}(6,q)$. We can homogenize these, and so the points of B in $\mathrm{PG}(2,q^3)$ correspond to points in $\mathrm{PG}(6,q)$ that lie on all six quadrics. The intersection is not exact though, as we need to consider the case $(1+\omega)y-1=0$, that is, $y=1/(1+\omega)$. The points of $\mathrm{PG}(2,q^3)$ that satisfy $y=1/(1+\omega)$ lie on a line ℓ_1 that corresponds to a 3-space Σ_1 of $\mathrm{PG}(6,q)$. The points on ℓ_1 satisfy Eqs. (15) and (16), and so the points of Σ_1 will lie on all six quadrics, that is, all six quadrics will contain Σ_1.

We can show that $\boldsymbol{B}$ is the precise intersection of nine quadrics by considering a second representation of B in $\mathrm{PG}(2,q^3)$. That is, we repeat the above argument using σ_2 from Lemma 2.6. So we have the affine points of B are the points $(x,y,1)$ satisfying $\sigma_2^{-1}(x,y,1)\in \mathrm{PG}(2,q)$. That is, we require $P=(1-y,\omega y,1-x)\in \mathrm{PG}(2,q)$.

Consider the case $x\neq 1$. So $P\equiv((1-y)/(1-x),\omega y/(1-x),1)$, and so $P\in \mathrm{PG}(2,q)$ if and only if

$$\left(\frac{1-y}{1-x}\right)^q=\frac{1-y}{1-x}\quad\text{and}\quad\left(\frac{\omega y}{1-x}\right)^q=\frac{\omega y}{1-x}.$$

Rearranging these two equations in $\mathrm{PG}(2,q^3)$ yields

$$(1-y)^q(1-x)-(1-y)(1-x)^q=0, \tag{18}$$
$$(1-x)(\omega y)^q-(1-x)^q\omega y=0. \tag{19}$$

As before, writing $x=x_0+x_1\tau+x_2\tau^2$, $y=y_0+y_1\tau+y_2\tau^2$, and $\omega=w_0+w_1\tau+w_2\tau^2$, simplifying, and equating coefficients of powers of τ yields the equations of six affine quadrics in $\mathrm{PG}(6,q)$. These six quadrics all contain the 3-space Σ_2 corresponding to the line ℓ_2 of $\mathrm{PG}(2,q^3)$ with affine equation $x=1$, so $\boldsymbol{B}$ is the residual intersection of these six quadrics.

Putting these two sets of six quadrics together, we have $\boldsymbol{B}$ contained in twelve quadrics. As the 3-space Σ_1 meets the 3-space Σ_2 in an affine point corresponding to the point $(1,\frac{1}{1+\omega},1)$ of

PG$(2, q^3)$, we have that $\boldsymbol{B}$ is the exact intersection of all twelve quadrics. However, as Eq. (19) is the same as Eq. (17), $\boldsymbol{B}$ is the exact intersection of nine quadrics in PG$(6, q)$. $\square$

Note that in proving part (d) of the above theorem, we have shown that, in PG$(6, q)$, $\boldsymbol{B}$ is the (residual) intersection of six quadrics that each contain a common 3-space. This generalizes the 2-dimensional result that a tangent Baer subplane of PG$(2, q^2)$ corresponds in PG$(4, q)$ to the (residual) intersection of two quadrics that contain a common plane (see Vincenti [9], and Quinn and Casse [8]). The argument in the proof of part (d) above can be generalized to the case of the Bruck–Bose representation of PG$(2, q^2)$ in PG$(4, q)$. In this case, Eqs. (15) and (16) yield four quadrics in PG$(4, q)$. However, on closer inspection, in this special case the four quadrics collapse into the two quadrics given by Vincenti. Hence the proof of part (d) generalizes to this case to show that a tangent Baer subplane corresponds to the residual intersection of two quadrics with a common plane; and to the *exact* intersection of three quadrics. Moreover, the proof by Quinn and Casse in [8, Lemma 2.6] gives a geometric construction of these two quadrics. We note that it is straightforward to generalize the proof of Quinn and Casse to obtain a geometric construction of the six quadrics of $\boldsymbol{B}$ in PG$(6, q)$.

3. The Bruck–Bose representation of PG$(2, q^n)$ in PG$(2n, q)$

We can generalize all the results for the case $n = 3$ to the Bruck–Bose representation of PG$(2, q^n)$ in PG$(2n, q)$ for general $n \geqslant 4$. That is, we fully determine the representation of order-q-sublines and secant and tangent order-q-subplanes of PG$(2, q^n)$ in PG$(2n, q)$ for $n \geqslant 4$.

3.1. The t-spreads and t-reguli of PG$(2t + 1, q)$

A *t-spread* of PG$(2t + 1, q)$ is a set of $q^t + 1$ t-spaces that partition PG$(2t + 1, q)$. A *t-regulus* $\mathcal{R}$ of PG$(2t + 1, q)$ is a set of $q + 1$ mutually disjoint t-spaces with the property that if a line meets three of the t-spaces in $\mathcal{R}$, then it meets all $q + 1$ of them. Three mutually disjoint t-spaces in PG$(2t + 1, q)$ lie on a unique t-regulus. A t-spread $\mathcal{S}$ is *regular* if for any three t-spaces in $\mathcal{S}$, the t-regulus containing them is contained in $\mathcal{S}$. In a regular t-spread, any $q + 1$ spread elements meeting a line form a t-regulus.

A regular t-spread of PG$(2t + 1, q)$ has a set of t transversal lines that lie in PG$(2t + 1, q^t)\backslash$PG$(2t + 1, q)$. Embed PG$(2t + 1, q)$ in PG$(2t + 1, q^t)$ and let g be a line of PG$(2t + 1, q^t)$ disjoint from PG$(2t + 1, q)$. Let $g^q, \ldots, g^{q^{t-1}}$ be the conjugate lines of g. Let $\boldsymbol{P}_i$ be a point on g, then the t-space $\langle \boldsymbol{P}_i, \boldsymbol{P}_i^q, \ldots, \boldsymbol{P}_i^{q^{t-1}} \rangle$ meets PG$(2t + 1, q)$ in a t-space. As $\boldsymbol{P}_i$ ranges over all the points of g, we get $q^t + 1$ t-spaces of PG$(2t + 1, q)$ that partition the space. These t-spaces form a regular spread $\mathcal{S}$ of PG$(2t + 1, q)$. The lines $g, g^q, \ldots, g^{q^{t-1}}$ are called the (conjugate skew) *transversal lines* of the spread $\mathcal{S}$. Given a regular t-spread in PG$(2t + 1, q)$, there is a unique set of t conjugate transversal lines in PG$(2t + 1, q^t)$ that generate $\mathcal{S}$ in this way. See [7, Section 25.6] for more information on t-reguli and t-spreads.

3.2. The Bruck–Bose representation

Let Σ_∞ be a hyperplane of PG$(2n, q)$ and let $\mathcal{S}$ be an $(n - 1)$-spread of Σ_∞. Consider the following incidence structure: the *points* of $\mathcal{A}(\mathcal{S})$ are the points of PG$(2n, q)\backslash\Sigma_\infty$; the *lines* of $\mathcal{A}(\mathcal{S})$ are the n-spaces of PG$(2n, q)\backslash\Sigma_\infty$ that contain an element of $\mathcal{S}$; and *incidence* in $\mathcal{A}(\mathcal{S})$ is induced by incidence in PG$(2n, q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order q^n. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$ where the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the $(n - 1)$-spread $\mathcal{S}$. Further, $\mathcal{P}(\mathcal{S})$ is Desarguesian if and only if $\mathcal{S}$ is regular.

It is straightforward to generalize the coordinates from Section 2.2 to PG$(2n, q)$. In this case, τ is a primitive element of GF(q^n) with primitive polynomial

$$x^n - t_{n-1}x^{n-1} - \cdots - t_1 x - t_0.$$

We can calculate the transversal lines of the regular $(n-1)$-spread $\mathcal{S}$.

Lemma 3.1. *Let g be the line of* $\mathrm{PG}(2n, q^n)$ *through the points* $(p_0, \ldots, p_{n-1}, 0, \ldots, 0, 0)$, *and* $(0, \ldots, 0, p_0, \ldots, p_{n-1}, 0)$, *where* $p_i = t_{i+1} + t_{i+2}\tau + \cdots + t_{n-1}\tau^{n-2-i} - \tau^{n-1-i}$, $i = 0, \ldots, n-2$, *and* $p_{n-1} = -1$. *Then g is one of the conjugate skew transversals of the regular $(n-1)$-spread $\mathcal{S}$.*

We omit the proof which involves induction to generalize the proof of Lemma 2.1.

3.3. *Sublines and subplanes of* $\mathrm{PG}(2, q^n)$

We completely determine the representation of order-q-sublines and secant and tangent order-q-subplanes of $\mathrm{PG}(2, q^n)$ in $\mathrm{PG}(2n, q)$. As before, an *order-q-subplane* of $\mathrm{PG}(2, q^n)$ is a subplane B of $\mathrm{PG}(2, q^n)$ of order q. Every line of $\mathrm{PG}(2, q^n)$ meets B in 0, 1 or $q+1$ points; a line of B has $q+1$ points and is called an *order-q-subline* of $\mathrm{PG}(2, q^n)$. We generalize the results of Section 2.3. As before, there are six cases to consider, and we completely determine five of them. Three cases are stated in Theorems 3.2, 3.3, and 3.4. The proofs of these theorems are straightforward generalizations of those in Section 2.3, so we do not include them here. We provide sketch proofs for the cases a disjoint order-q-subline and tangent order-q-subplane in Theorems 3.5 and 3.6. As in the $n = 3$ case, we leave open the case of the exterior order-q-subplanes.

Theorem 3.2. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ of order q^n defined from an $(n-1)$-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(2n, q)$.

(a) *A plane of* $\mathrm{PG}(2n, q)\backslash\Sigma_\infty$ *that meets $q+1$ elements of $\mathcal{S}$ represents an order-q-subplane of $\mathcal{P}(\mathcal{S})$ secant to ℓ_∞.*
(b) *If* $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^n)$, *then every order-$q$-subplane of $\mathcal{P}(\mathcal{S})$ that is secant to ℓ_∞ is represented by a plane of* $\mathrm{PG}(2n, q)\backslash\Sigma_\infty$ *that meets $q+1$ elements of $\mathcal{S}$.*

Theorem 3.3. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ of order q^n defined from an $(n-1)$-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(2n, q)$.

(a) *A line of* $\mathrm{PG}(2n, q)\backslash\Sigma_\infty$ *represents an order-q-subline of $\mathcal{P}(\mathcal{S})$ tangent to ℓ_∞.*
(b) *If* $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^n)$, *then every order-$q$-subline of $\mathcal{P}(\mathcal{S})$ tangent to ℓ_∞ is represented by a line of* $\mathrm{PG}(2n, q)\backslash\Sigma_\infty$.

Theorem 3.4. *Consider the translation plane $\mathcal{P}(\mathcal{S})$ of order q^n defined from an $(n-1)$-spread $\mathcal{S}$ of a hyperplane Σ_∞ of* $\mathrm{PG}(2n, q)$.

(a) *A regulus $\mathcal{R}$ of $\mathcal{S}$ represents an order-q-subline of ℓ_∞ in $\mathcal{P}(\mathcal{S})$.*
(b) *If* $\mathcal{P}(\mathcal{S}) \cong \mathrm{PG}(2, q^n)$, *then every order-$q$-subline of ℓ_∞ in $\mathcal{P}(\mathcal{S})$ is represented by a regulus of $\mathcal{S}$.*

We now consider the remaining two cases: an order-q-subline disjoint from ℓ_∞, an order-q-subplane tangent to ℓ_∞. The results from Section 2.3 do generalize; however, the proofs are much more complex. We sketch the proofs for this general case.

Theorem 3.5.

(a) *Let ℓ be a line of* $\mathrm{PG}(2, q^n)$ *and let b be an order-q-subline of ℓ that is disjoint from ℓ_∞. Then, in* $\mathrm{PG}(2n, q)$, *b corresponds to a normal rational curve in the n-space Σ corresponding to ℓ.*
(b) *Let Σ be an n-space of* $\mathrm{PG}(2n, q)\backslash\Sigma_\infty$ *about a spread element and let $\mathcal{N}$ be a normal rational curve in Σ that is disjoint from Σ_∞. Then $\mathcal{N}$ corresponds to an order-q-subline of* $\mathrm{PG}(2, q^n)$ *if and only if, in the extension* $\mathrm{PG}(2n, q^n)$, *$\mathcal{N}$ meets the conjugate transversal lines $g, g^q, \ldots, g^{q^{n-1}}$ of the spread $\mathcal{S}$.*

Proof (Sketch only). We generalize the proof of Theorem 2.5. Using the notation of that proof, we can write a point $P_d=(d,d,d+\omega)$ of the subline ℓ_ω as $P_d=(\alpha,\alpha,c)=(d+\omega^q)\cdots(d+\omega^{q^{n-1}})(d,d,d+\omega)$. Now $c\in \mathrm{GF}(q)$, and c is a polynomial in d of degree n. We can write $\alpha=a_0+a_1\tau+\cdots+a_{n-1}\tau^{n-1}$ for $a_i\in\mathrm{GF}(q)$, where the a_i are polynomials in d of degree at most n, and a_0 has degree n. Hence we can find a homography that maps the points of $\mathrm{PG}(2n,q)$ corresponding to the points on ℓ_ω to the points $\{(d^n,d^{n-1},\dots,d,1)\mid d\in\mathrm{GF}(q)\cup\{\infty\}\}$. That is, ℓ_ω corresponds to a normal rational curve in an n-space of $\mathrm{PG}(2n,q)$.

To prove part (b), we show that $\boldsymbol{P}_{-\omega}=(-1)^n\alpha_\omega(p_0,\dots,p_{n-1},p_0,\dots,p_{n-1},0)$ where p_i is defined in Lemma 3.1 and α_ω satisfies

$$\left(\omega^q-\omega\right)\left(\omega^{q^2}-\omega\right)\cdots\left(\omega^{q^{n-1}}-\omega\right)=\left(\tau^q-\tau\right)\left(\tau^{q^2}-\tau\right)\cdots\left(\tau^{q^{n-1}}-\tau\right)\alpha_\omega.$$

In a generalization of Theorem 2.5 we write

$$\left(d+\omega^q\right)\left(d+\omega^{q^2}\right)\cdots\left(d+\omega^{q^{n-1}}\right)=z_0+z_1\tau+\cdots+z_{n-1}\tau^{n-1},$$

where $z_i=z_i(d):\mathrm{GF}(q^n)\to\mathrm{GF}(q^n)$ and $z_i(d)\in\mathrm{GF}(q)$ for $d\in\mathrm{GF}(q)$. We apply the field automorphism $x\mapsto x^q$ a further $n-1$ times to obtain a total of n equations, and then substitute $d=-\omega$ and arrive at the following n equations:

$$\left(\tau^q-\tau\right)\left(\tau^{q^2}-\tau\right)\cdots\left(\tau^{q^{n-1}}-\tau\right)=A_0+A_1\tau+\cdots+A_{n-1}\tau^{n-1}, \tag{20}$$

$$0=A_0+A_1\tau^q+\cdots+A_{n-1}\tau^{(n-1)q}, \tag{21}$$

$$\vdots$$

$$0=A_0+A_1\tau^q+\cdots+A_{n-1}\tau^{(n-1)q^{n-1}},$$

where the A_i have the same properties as the z_i earlier. Now

$$\begin{aligned}&\tau\left(\tau^q-\tau\right)\left(\tau^{q^2}-\tau\right)\cdots\left(\tau^{q^{n-1}-\tau}\right)\\&=\binom{T'}{n-1}\tau+\cdots+(-1)^{i-1}\binom{T'}{n-i}+\cdots+(-1)^{n-2}\binom{T'}{1}\tau^{n-1}+(-1)^{n-1}\tau^n\end{aligned}$$

where $T'=\{\tau^q,\tau^{q^2},\dots,\tau^{q^{n-1}}\}$, $T=T'\cup\{\tau\}$ and for example $\binom{T}{k}$ is the sum of the products of all the elements in each k subset of T, e.g.

$$\binom{T}{1}=\tau+\tau^q+\cdots+\tau^{q^{n-1}},\qquad \binom{T}{n}=\tau\tau^q\cdots\tau^{q^{n-1}}.$$

As $(x-\tau)(x-\tau^q)\cdots(x-\tau^{q^{n-1}})=x^n-t_{n-1}\tau^{n-1}-\cdots-t_1x-t_0$, it follows that

$$t_k=(-1)^{n-k+1}\binom{T}{n-k}\quad\text{and}\quad\binom{T}{k+1}=\tau\binom{T'}{k}+\binom{T'}{k+1}$$

for $0\leqslant k\leqslant n-1$. We show by backward induction $i=n,n-1,\dots,2,1$ that $p_i=(-1)^{n+i-1}\binom{T'}{n-i}$ by using above relationships, together with the relationship $t_{k-1}+p_{k-1}\tau=p_{k-2}$. This will be enough to prove that (20) holds for $d=-\omega$.

For (21), note that $(-1)^n\tau^{i+1}p_i=-t_0-t_1\tau-\cdots-t_i\tau^i$ for $0\leqslant i\leqslant n-1$ and then show that RHS $(21)\times\tau^n=0$.

So we have shown that $\boldsymbol{P}_{-\omega} \in g$. In a similar manner to the case $n = 3$, we show that $\boldsymbol{P}_{-\omega^q} = (\boldsymbol{P}_{-\omega})^q$ and so $\boldsymbol{P}_{-\omega^q} \in g^q$ and so on.

Hence we can conclude that the normal rational curve meets g and all the transversals of $\mathcal{S}$. Conversely, let Q_1, Q_2, Q_3 be non-collinear points of $\Sigma \backslash \Sigma_\infty$, and let $\boldsymbol{P}_i = g^{q^{i-1}} \cap \overline{\Sigma}$. Then $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_n$, Q_1, Q_2, Q_3 are a set of $n+3$ points in an n-space Σ, no $n+1$ in an $(n-1)$-space, and so lie in a unique normal rational curve. Hence the counting argument in the proof of Theorem 2.5 generalizes. □

Theorem 3.6. *Let B be an order-q-subplane of* $\mathrm{PG}(2, q^n)$ *that is tangent to ℓ_∞ in the point T. Let Σ_T be the spread element corresponding to T. Then B determines a set $\boldsymbol{B}$ of points in* $\mathrm{PG}(2n, q)$ (*where the affine points of B correspond to the affine points of $\boldsymbol{B}$*) *such that*:

(a) *$\boldsymbol{B}$ is a ruled surface with normal rational curve directrix $\mathcal{C}$ contained in the $(n-1)$-space $\Sigma_T \in \mathcal{S}$, and normal rational curve directrix $\mathcal{N}$ contained in an n-space Σ that meets Σ_∞ in a spread element* (*distinct from Σ_T*). *The points of $\boldsymbol{B}$ lie on $q+1$ pairwise disjoint generator lines joining $\mathcal{C}$ to $\mathcal{N}$.*
(b) *The $q+1$ generator lines of $\boldsymbol{B}$ joining $\mathcal{C}$ to $\mathcal{N}$ are determined by a projectivity from $\mathcal{C}$ to $\mathcal{N}$.*
(c) *When we extend $\boldsymbol{B}$ from* $\mathrm{PG}(2n, q)$ *to* $\mathrm{PG}(2n, q^n)$, *it contains the* (*conjugate skew*) *transversal lines $g, g^q, \ldots, g^{q^{n-1}}$ of the spread $\mathcal{S}$.*
(d) *$\boldsymbol{B}$ is the intersection of $3n$ quadrics in* $\mathrm{PG}(2n, q)$.

Proof. We generalize the proof of Theorem 2.7. Using the notation from that proof, in this more general setting, we have that the points $\boldsymbol{Q}_d$ lie in a spread element Σ_T, and have coordinates that are polynomials in d of degree $n-1$. Hence they can be mapped to a normal rational curve in an $(n-1)$-space. So parts (a) and (b) hold. Note that we can naturally extend $\boldsymbol{B}$ to $\overline{\boldsymbol{B}}$ in $\mathrm{PG}(2n, q^n)$ as follows. We extend the normal rational curves $\mathcal{C}$ and $\mathcal{N}$ in $\mathrm{PG}(2n, q)$ to $\overline{\mathcal{C}}$ and $\overline{\mathcal{N}}$ in $\mathrm{PG}(2n, q^n)$ (respectively), and the projectivity $\mathcal{C} \to \mathcal{N}$ is extended to $\overline{\mathcal{C}} \to \overline{\mathcal{N}}$, then $\boldsymbol{B}$ is extended to $\overline{\boldsymbol{B}}$ in $\mathrm{PG}(2n, q^n)$. Part (c) follows by computing the coordinates of $\boldsymbol{Q}_{-\omega}$ to be $\boldsymbol{Q}_{-\omega} = (p_0, \ldots, p_{n-1}, 0, \ldots, 0, 0)$ where p_i are defined in Lemma 3.1, hence $\boldsymbol{Q}_{-\omega}$ lies on g, and so $\overline{\boldsymbol{B}}$ contains g. Finally the proof of Theorem 2.7(d) generalizes immediately to show that $\boldsymbol{B}$ is the intersection of $3n$ quadrics in $\mathrm{PG}(2n, q)$. Hence $\boldsymbol{B}$ is an algebraic variety. □

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