Sublines and subplanes of PG(2, q^3) in the Bruck–Bose representation in PG(6, q)

S.G. Barwick*, Wen-Ai Jackson

Department of Pure Mathematics, University of Adelaide, Adelaide 5005, Australia

ARTICLE INFO

Article history:
Received 15 February 2011
Revised 30 June 2011
Accepted 4 July 2011
Available online 29 July 2011
Communicated by Simeon Ball

MSC:
51E20

Keywords:
Bruck–Bose representation
Baer subplanes

ABSTRACT

In this article we look at the Bruck–Bose representation of PG(2, q^3) in PG(6, q). We look at sublines and subplanes of order q in PG(2, q^3) and describe their representation in PG(6, q). We then show how these results can be generalized to the Bruck–Bose representation of PG(2, q^n) in PG(2n, q).

© 2011 Elsevier Inc. All rights reserved.

doi:10.1016/j.ffa.2011.07.001

1. Introduction

The Bruck–Bose representation of PG(2, q^2) in PG(4, q) has been studied in great detail. Many authors have investigated the representation of Baer sublines, subplanes and unitals of PG(2, q^2) in PG(4, q) (see [2] for a survey and proofs of many of these results). In this article, we investigate a cubic extension, namely the Bruck–Bose representation of PG(2, q^3) in PG(6, q). We study sublines and secant and tangent subplanes of PG(2, q^3) of order q and determine their representation in PG(6, q). We then generalize these results to the Bruck–Bose representation of PG(2, q^n) in PG(2n, q), in that we completely determine the representation in PG(2n, q) of sublines and secant and tangent subplanes of PG(2, q^n) of order q. The results in this paper form a foundation for further work by the authors investigating the representation in PG(6, q) of unitals and Baer subplanes of PG(2, q^2) when q is square.

* Corresponding author. Fax: +61 8 8303 3696.
E-mail address: susan.barwick@adelaide.edu.au (S.G. Barwick).
2. The Bruck–Bose representation of PG(2, q^3) in PG(6, q)

2.1. The 2-spreads and 2-reguli of PG(5, q)

A 2-spread of PG(5, q) is a set of q^3 + 1 planes that partition PG(5, q). A 2-regulus of PG(5, q) is the system of maximal 2-spaces of a Segre variety S_{1,2} (see [7, Section 25.5] for full details on Segre varieties). That is, a 2-regulus R is a set of q + 1 mutually disjoint planes π_1, ..., π_{q+1} with the property that if a line meets three of the planes, then it meets all q + 1 of them. Thus there are q^2 + q + 1 mutually disjoint lines associated with R (these are the maximal 1-spaces of S_{1,2}). Three mutually disjoint planes in PG(5, q) lie on a unique 2-regulus. A 2-spread S is regular if for any three planes in S, the 2-regulus containing them is contained in S. In a regular 2-spread, any q + 1 spread elements meeting a line form a 2-regulus.

The following construction of a regular 2-spread of PG(5, q) will also be useful. Embed PG(5, q) in PG(5, q^3) and let g be a line of PG(5, q^3) disjoint from PG(5, q). Let g^q, g^{q^2} be the conjugate lines of g, both of these are disjoint from PG(5, q). Let P_i be a point on g; then the plane ⟨P_i, P_i^q, P_i^{q^2}⟩ meets PG(5, q) in a plane. As P_i ranges over all the points of g, we get q^3 + 1 planes of PG(5, q) that partition the space. These planes form a regular spread S of PG(5, q). The lines g, g^q, g^{q^2} are called the (conjugate skew) transversal lines of the spread S. Conversely, given a regular 2-spread in PG(5, q), there is a unique set of three (conjugate skew) transversal lines in PG(5, q^3) that generate S in this way. See [7, Section 25.6] for more information on 2-reguli and 2-spreads.

2.2. The Bruck–Bose representation

In this section we introduce the linear representation of a finite translation plane P of dimension at most three over its kernel, an idea which was developed independently by André [1] and Bruck and Bose [4,5]. We will use the vector space construction as developed by Bruck and Bose.

Let Σ_∞ be a hyperplane of PG(6, q) and let S be a 2-spread of Σ_∞. We use the phrase a subspace of PG(6, q) \ Σ_∞ to mean a subspace of PG(6, q) that is not contained in Σ_∞. Consider the following incidence structure: the points of A(S) are the points of PG(6, q) \ Σ_∞; the lines of A(S) are the 3-spaces of PG(6, q) \ Σ_∞ that contain an element of S; and incidence in A(S) is induced by incidence in PG(6, q). Fig. 1 illustrates this construction. Then the incidence structure A(S) is an affine plane of order q^3. We can complete A(S) to a projective plane P(S); the points on the line at infinity ℓ_∞ have a natural correspondence to the elements of the 2-spread S.

The projective plane P(S) is the Desarguesian plane PG(2, q^3) if and only if S is a regular 2-spread of Σ_∞ ≃ PG(5, q) [3].

In the case P(S) ≃ PG(2, q^3), we can relate the coordinates of PG(2, q^3) and PG(6, q) as follows. Let τ be a primitive element in GF(q^3) with primitive polynomial

\[ x^3 - t_2x^2 - t_1x - t_0. \]
Then every element $\alpha \in \text{GF}(q^3)$ can be uniquely written as $\alpha = a_0 + a_1 \tau + a_2 \tau^2$ with $a_0, a_1, a_2 \in \text{GF}(q)$. Points in $\text{PG}(2, q^3)$ have homogeneous coordinates $(x, y, z)$ with $x, y, z \in \text{GF}(q^3)$. Let the line at infinity $\ell_\infty$ have equation $z = 0$; so the affine points of $\text{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\text{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in \text{GF}(q)$. Let $\Sigma_\infty$ have equation $z = 0$. Let $P = (\alpha, \beta, 1)$ be a point of $\text{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1 \tau + a_2 \tau^2$ and $\beta = b_0 + b_1 \tau + b_2 \tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in \text{GF}(q)$. Then the map $\phi: \text{PG}(2, q^3) \setminus \ell_\infty \rightarrow \text{PG}(6, q) \setminus \Sigma_\infty$ such that $\phi(\alpha, \beta, 1) = (a_0, a_1, a_2, b_0, b_1, b_2, 1)$ is the Bruck–Bose map.

To complete this to a projective map, we generalize the construction of Bruck–Bose coordinates of $\text{PG}(2, q^2)$ in $\text{PG}(4, q)$ (see [2, Section 3.4.4] for full details of this 2-dimensional case). First define $\sigma(\alpha, \beta, 0) = (a_0, a_1, a_2, b_0, b_1, b_2, 0)$. Let $(1, \delta, 0)$ be a point of $\ell_\infty$ in $\text{PG}(2, q^2)$; then we can write $\delta = f_0 + f_1 \tau + f_2 \tau^2$ for unique $f_0, f_1, f_2 \in \text{GF}(q)$. Then the spread element of $\Sigma_\infty$ in $\text{PG}(6, q)$ corresponding to $(1, \delta, 0)$ is the plane spanned by the three points $D_0, D_1, D_2$ given by

$$D_0 = \sigma(1, \delta, 0) = (1, 0, 0, f_0, f_1, f_2, 0),$$
$$D_1 = \sigma(\tau, \delta, \tau) = (0, 1, 0, f_2t_0, f_0 + f_2t_1, f_1 + f_2t_2, 0),$$
$$D_2 = \sigma(\tau^2, \delta \tau^2, 0) = (0, 0, 1, f_1t_0 + f_2t_0t_2, f_2t_0 + f_2t_1t_2 + f_1t_1, f_0 + f_2t_1 + f_2t_2^2 + f_1t_2, 0).$$

Note also that the point $(0, 1, 0)$ of $\ell_\infty$ in $\text{PG}(2, q^3)$ corresponds to the spread element that is spanned by the three points $(0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 1, 0)$.

Later we will need the equation of the three (conjugate skew) transversals of $S$ in the cubic extension $\Sigma = \text{PG}(6, q^3)$; so we calculate their equations here.

**Lemma 2.1.** Let $g$ be the line of $\text{PG}(6, q^3)$ through the points $A_1 = (p_0, p_1, p_2, 0, 0, 0, 0)$ and $A_2 = (0, 0, 0, p_0, p_1, p_2, 0)$ where $p_0 = t_1 + t_2 \tau - \tau^2, p_1 = t_2 - \tau, p_2 = -1$. Then $g$ is one of the three (conjugate skew) transversals of the regular 2-spread $S$. (The remaining transversals are $g^q, g^{q^2}$.)

**Proof.** Label the planes of $S$ by $\pi_\delta, \delta \in \text{GF}(q^3) \cup \{\infty\}$, such that $\pi_\delta$ corresponds to the point $(1, \delta, 0)$ of $\ell_\infty$ in $\text{PG}(2, q^3)$. Note that $g$ is not a line of $\Sigma_\infty \equiv \text{PG}(5, q)$, but lies in the cubic extension. We need to show that the line $g$ meets each of the planes $\pi_\delta$ (considering $\pi_\delta$ as a plane in the cubic extension). Now $\pi_\infty$ corresponds to the point $(0, 1, 0)$ and so is the plane spanned by the points $(0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0)$. Clearly $\pi_\infty$ meets the line $g$ in the point $A_2$. The plane $\pi_\delta, \delta \in \text{GF}(q^3)$, is spanned by the three points $D_0, D_1, D_2$ calculated above. The point $P = A_1 + \delta A_2$ is on the line $g$. Moreover, $P = p_0 D_0 + p_1 D_1 + p_2 D_2$ and so $P$ is on the spread element $\pi_\delta$. Hence $g$ meets every spread element, and so $g$ and its conjugates $g^q, g^{q^2}$ are the unique transversals of the spread $S$. $\Box$

2.3. Sublines and subplanes of $\text{PG}(2, q^3)$

In any plane $\mathcal{P}$ of order $q^3$, a natural subplane $B$ to consider is an order-$q$-subplane, that is, a set of $q^2 + q + 1$ points where every line of $\mathcal{P}$ meets $B$ in 0, 1 or $q + 1$ points. Hence every line of the subplane $B$ has $q + 1$ points and we call these order-$q$-sublines. In particular, if $\mathcal{P} = \text{PG}(2, q^3)$, then every order-$q$-subplane is isomorphic to $\text{PG}(2, q)$.

We consider the representation of order-$q$-sublines and order-$q$-subplanes in the Bruck–Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$. There are six cases to consider: Theorem 2.4 looks at order-$q$-sublines contained in $\ell_\infty$; Theorem 2.3 looks at order-$q$-sublines tangent to $\ell_\infty$; Theorem 2.5 looks at order-$q$-sublines disjoint from $\ell_\infty$. Theorem 2.2 looks at order-$q$-subplanes secant to $\ell_\infty$ and Theorem 2.7 looks at order-$q$-subplanes tangent to $\ell_\infty$. Note that the case of order-$q$-subplanes exterior to $\ell_\infty$ is not covered in this paper and a succinct description of its representation in $\text{PG}(6, q)$ remains an open problem.
Theorem 2.2. Consider the translation plane $\mathcal{P}(S)$ defined from a 2-spread $S$ of a hyperplane $\Sigma_\infty$ of PG(6, q).

(a) A plane of PG(6, q) \ $\Sigma_\infty$ that meets $q + 1$ elements of $S$ represents an order-$q$-subplane of $\mathcal{P}(S)$ secant to $\ell_\infty$.

(b) If $\mathcal{P}(S) \cong$ PG(2, $q^3$), then every order-$q$-subplane of $\mathcal{P}(S)$ that is secant to $\ell_\infty$ is represented by a plane of PG(6, q) \ $\Sigma_\infty$ that meets $q + 1$ elements of $S$.

Proof. Let $\pi$ be a plane of PG(6, q) \ $\Sigma_\infty$, so $\pi$ meets $\Sigma_\infty$ in a line. We are interested in the case when this line is not contained in a spread element, so it meets $q + 1$ spread elements, each in a point. (Note that if $S$ were a regular spread then these $q + 1$ spread elements would form a 2-regulus of $S$.) Then, in $\mathcal{P}(S)$, $\pi$ corresponds to a set $A$ of $q^2$ affine points, and $q + 1$ points on $\ell_\infty$ (corresponding to the $q + 1$ spread elements that $\pi$ meets). In PG(6, q), a 3-space about a spread element meets $\pi$ in 0, 1 or $q + 1$ points; hence, in $\mathcal{P}(S)$, lines meet $A$ in 0, 1 or $q + 1$ points. Thus $A$ is a plane of order $q$ and (a) holds.

To prove (b), we count the number of order-$q$-subplanes secant to $\ell_\infty$ in PG(2, $q^3$). Since a quadrangle $(A, B, C, D)$ with $A, B \in \ell_\infty$ and $C, D \notin \ell_\infty$ uniquely determines an order-$q$-subplane, the number of order-$q$-subplanes secant to $\ell_\infty$ is

$$\frac{(q^3 + 1)(q^3)(q^6)(q^6 - 2q^3 + 1)}{(q + 1)(q)(q^2)(q^2 - 2q + 1)} = (q^2 - q + 1)q^6(q^2 + q + 1)^2.$$  (1)

We now count the number of planes of PG(6, q) that meet $\Sigma_\infty$ in a line which does not lie in a spread element. We first count the number of lines of PG(5, q); it is

$$\frac{q^6 - 1}{q - 1} \frac{q^6 - 1}{q - 1} - 1 = (q^4 + q^2 + 1) \left(\frac{q^5 - 1}{q - 1}\right).$$

The number of lines in one spread element is $q^2 + q + 1$; so we subtract $(q^3 + 1)(q^3 + q + 1)$ to find the number of lines not in any spread element. Given that any such line determines $q^6/q^2 = q^4$ planes of PG(6, q) \ $\Sigma_\infty$ it is easy to show that the required number (1) follows. Hence, when $\mathcal{P}(S) \cong$ PG(2, $q^3$), the correspondence of part (a) is exact. □

As an immediate corollary, we have the representation of order-$q$-sublines that are tangent to $\ell_\infty$.

Theorem 2.3. Consider the translation plane $\mathcal{P}(S)$ defined from a 2-spread $S$ of a hyperplane $\Sigma_\infty$ of PG(6, q).

(a) A line of PG(6, q) \ $\Sigma_\infty$ represents an order-$q$-subline of $\mathcal{P}(S)$ tangent to $\ell_\infty$.

(b) If $\mathcal{P}(S) \cong$ PG(2, $q^3$), then every order-$q$-subline of $\mathcal{P}(S)$ tangent to $\ell_\infty$ is represented by a line of PG(6, q) \ $\Sigma_\infty$.

Theorem 2.4. Consider the translation plane $\mathcal{P}(S)$ defined from a 2-spread $S$ of a hyperplane $\Sigma_\infty$ of PG(6, q).

(a) A 2-regulus $\mathcal{R}$ of $S$ represents an order-$q$-subline of $\ell_\infty$ in $\mathcal{P}(S)$.

(b) If $\mathcal{P}(S) \cong$ PG(2, $q^3$), then every order-$q$-subline of $\ell_\infty$ in $\mathcal{P}(S)$ is represented by a 2-regulus of $S$.

Proof. Part (a) is an immediate corollary of Theorem 2.2. To prove (b) we first note that three points on $\ell_\infty$ in PG(2, $q^3$) are contained in a unique subline of order $q$. Hence the number of sublines of $\ell_\infty$ of order $q$ is $(q^6 + 1)/(q^3 + 1) = q^2(q^4 + q^2 + 1)$. Alternatively, suppose $S$ is a regular 2-spread in $\Sigma_\infty \cong$ PG(5, q). Then the number of 2-reguli in $S$ is $q^3(q^4 + q^2 + 1)$ by [7, Theorem 25.6.6]. As these two numbers are equal, if $\mathcal{P}(S) \cong$ PG(2, $q^3$), then the correspondence in (a) is exact. □
Theorem 2.5.

(a) Let \( \ell \) be a line of \( \text{PG}(2, q^3) \) and let \( b \) be an order-\( q \)-subline of \( \ell \) that is disjoint from \( \ell_\infty \). Then in \( \text{PG}(6, q) \), \( b \) corresponds to a normal rational curve in the 3-space \( \Sigma \) corresponding to \( \ell \).

(b) Let \( \Sigma \) be a 3-space of \( \text{PG}(6, q) \backslash \Sigma_\infty \), about a spread element and let \( \mathcal{N} \) be a normal rational curve in \( \Sigma \) that is disjoint from \( \Sigma_\infty \). Then \( \mathcal{N} \) corresponds to an order-\( q \)-subline of \( \text{PG}(2, q^3) \) disjoint from \( \ell_\infty \) if and only if in the cubic extension \( \text{PG}(6, q^3) \), the cubic extension \( \mathcal{N} \) of \( \mathcal{N} \) meets the conjugate transversal lines \( g, g^q, g^{q^2} \) of the spread \( \mathcal{S} \).

Proof. We use coordinates to prove (a), generalizing the argument in [2, Theorem 3.17]. Let \( \ell \) be the line of equation \( x = y \) in \( \text{PG}(2, q^3) \) and let \( \omega \in \text{GF}(q^3) \backslash \text{GF}(q) \). We find an order-\( q \)-subline of \( \ell \) disjoint from \( \ell_\infty \) as follows. The set \( \{(1, d, 0) \mid d \in \text{GF}(q) \cup \{\infty\}\} \) is clearly an order-\( q \)-subline of \( \ell_\infty \). The homography with matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 1 & 1 \\
\omega & 1 & 0 
\end{pmatrix}
\]

maps this to the set \( \ell_\omega = \{P_d = (d, d + \omega) \mid d \in \text{GF}(q) \cup \{\infty\}\}. \) So \( \ell_\omega \) is an order-\( q \)-subline of \( \ell \) which is clearly disjoint from \( \ell_\infty \).

As we are using homogeneous coordinates, for all \( d \in \text{GF}(q) \) we can write the points \( P_d \) of \( \ell_\omega \) as

\[
P_d = (\alpha, \alpha, c) = (d(d + \omega^q)(d + \omega^q), (d + \omega^q), (d + \omega)(d + \omega^q)(d + \omega^q^2)).
\]

We claim that \( c = c(d) = (d + \omega^q)(d + \omega^q) \) is a polynomial in \( d \) with \( c : \text{GF}(q) \to \text{GF}(q) \). We have \( c(d) \in \text{GF}(q) \) as \( c(d)^q = c(d) \) for all \( d \in \text{GF}(q) \). So we can write \( c(d) = c_0 + c_1d + c_2d^2 + d^3 \) for some constants \( c_0, c_1, c_2 \in \text{GF}(q) \). Write \( \omega = w_0 + w_1\tau + w_2\tau^2 \) for constants \( w_0, w_1, w_2 \in \text{GF}(q) \). Now recall that \( \tau \) is a solution to \( \chi^2 - t_2x^2 - t_1x - t_0 \) and so the other two roots are \( \tau^q, \tau^q^2 \). Hence \( t_2 = \tau + \tau^q + \tau^q^2, \quad t_1 = \tau^q + \tau^q^2 + \tau^q^2\tau^2 \) and \( t_0 = \tau^q\tau^q^2 \). Many relationships follow, for example \( t_2^2 + 2t_1 = \tau^2 + \tau^q^2 + \tau^q^2 \) and \( t_1^2 - 2t_0t_2 = \tau^q\tau^q^2 + \tau^q^2 + \tau^q^2 \). Using these and other relationships we can show that

\[
c_0 = w_0^2 + w_0^2w_1t_2 + w_0^2w_2(t_2^2 + 2t_1) - w_0w_1w_2(3t_0 + t_1t_2)
\]

\[
+ w_0w_2^2(t_1^2 - 2t_0t_2) + w_1^2t_0 + w_1^2w_2t_0t_2 - w_1w_2^2t_0t_1 + w_2^2t_0^2,
\]

\[
c_1 = 3w_0^2 + 2w_0w_1t_2 + w_0w_2(2t_2^2 + 4t_1) - w_1^2t_1 - w_1w_2(3t_0 + t_1t_2) + w_2^2(t_1^2 - 2t_0t_2),
\]

\[
c_2 = 3w_0 + w_1t_2 + w_2(t_2^2 + 2t_1).
\]

However, the element \( \alpha = d(d + \omega^q)(d + \omega^q^2) \) is in \( \text{GF}(q^3) \) and not necessarily \( \text{GF}(q) \), so we will write it as \( \alpha = a_0 + a_1\tau + a_2\tau^2 \) for unique functions \( a_0, a_1, a_2 : \text{GF}(q) \to \text{GF}(q) \) of \( d \). Using similar methods as for the \( c_i \) we calculate

\[
a_0(d) = d^3 + d^2(2w_0 + w_1t_2 + w_2(t_2^2 + 2t_1))
\]

\[
+ d(w_0^2 + w_0w_1t_2 + w_0w_2(t_2^2 + 2t_1) - w_1^2t_1 - w_1w_2(t_0 + t_1t_2) + w_2^2(t_1^2 - t_0t_2)),
\]

\[
a_1(d) = -d^2w_1 + d(-w_0w_1 - w_1^2t_2 - w_1w_2t_2^2 + w_2^2(t_0 + t_1t_2)),
\]

\[
a_2(d) = -d^2w_2 + d(-w_0w_2 + w_1^2 + w_1w_2t_2 - w_1^2t_1).
\]
In particular, we have \( a_0(d) \) is a cubic in \( d \), and \( a_1(d), a_2(d) \) are quadratics in \( d \), that is,

\[
a_0(d) = a_{00} + a_{01}d + a_{02}d^2 + d^3, \quad a_1(d) = a_{10} + a_{11}d + a_{12}d^2, \quad a_2(d) = a_{20} + a_{21}d + a_{22}d^2.
\]

for constants \( a_{ij} \in \text{GF}(q) \), and where \( a_{00} = a_{10} = a_{20} = 0 \).

Hence the point \( P_d = (\alpha, \alpha, c) \) in \( \text{PG}(2, q^3) \) corresponds to the point

\[
P_d = (a_0(d), a_1(d), a_2(d), a_0(d), a_1(d), a_2(d), c(d))
\]

in \( \text{PG}(6, q) \).

To show that \( \mathcal{N} = \{ P_d \mid d \in \text{GF}(q) \cup \{ \infty \} \} \) is a normal rational curve in \( \Sigma \), it is sufficient to exhibit a homography that maps the point \( (d^3, d^2, d, 1) \) to the point \( (a_0(d), a_1(d), a_2(d), c(d)) \). Consider the matrix

\[
B = \begin{pmatrix}
1 & a_{02} & a_{01} & a_{00} \\
0 & a_{12} & a_{11} & a_{10} \\
0 & a_{22} & a_{21} & a_{20} \\
1 & c_2 & c_1 & c_0
\end{pmatrix}, \quad \text{where} \quad \begin{pmatrix}
a_0(d) \\
a_1(d) \\
a_2(d) \\
c(d)
\end{pmatrix} = B \begin{pmatrix}
d^3 \\
d^2 \\
d \\
1
\end{pmatrix}.
\]

To show that \( B \) corresponds to a homography, we need to show that the determinant of \( B \) is non-zero. It is sufficient to show that there are four distinct values of \( d \) for which the points \( P_d \) are linearly independent, since this would mean that \( B \) has full rank.

We will do this in the following way. First, we show that for \( d \in \text{GF}(q) \) we can obtain the coordinates of \( P_d \) by solving a set of simultaneous linear equations. Then we can extend this definition to \( d \in \text{GF}(q^3) \), so the extended curve \( \mathcal{N} = \{ P_d \mid d \in \text{GF}(q^3) \cup \{ \infty \} \} \) contains \( \mathcal{N} \). It is sufficient to find four values of \( d \in \text{GF}(q^3) \) with points \( P_d \) being linearly independent. The points we will consider correspond to \( d = 0, -\omega, -\omega^2, -\omega^3 \).

Recall the definition of \( P_d \) \( (d \in \text{GF}(q)) \) from (2). So we have

\[
d(d + \omega^q)(d + \omega^q^2) = a_0(d) + a_1(d)\tau + a_2(d)\tau^2
\]

where the \( a_i : \text{GF}(q) \to \text{GF}(q) \). How can we determine, without explicit calculation, the values of the \( a_i(d) \) for a given value of \( d \)? As above, for \( n = 3 \) we can explicitly calculate the \( a_i \) and hence \( a_i(d) \) (for \( d \in \text{GF}(q) \)) but for general \( n \) this will be a difficult problem.

Eq. (5) determines uniquely the values of \( a_i(d) \) for \( d \in \text{GF}(q) \). However, when we extend to \( d \in \text{GF}(q^3) \), then one equation will not uniquely determine the values of \( a_i(d) \) for a given value of \( d \). As there are three unknowns \( a_0(d), a_1(d), a_2(d) \), we will need three linearly independent equations to uniquely determine the values of \( a_i(d) \). Applying the field automorphism to (5), and then again to the resultant equation, we obtain for all \( d \in \text{GF}(q) \),

\[
d(d + \omega^q^2)(d + \omega) = a_0(d) + a_1(d)\tau^q + a_2(d)\tau^{2q},
\]

\[
d(d + \omega)(d + \omega^q) = a_0(d) + a_1(d)\tau^{q^2} + a_2(d)\tau^{2q^2}.
\]

Adding our last equation from the definition of \( P_d \) we obtain the system \( D = TR_d \), where \( D, T, R_d \) are defined below

\[
D = \begin{pmatrix}
d(d + \omega^q)(d + \omega^q^2) \\
d(d + \omega^q^2)(d + \omega) \\
d(d + \omega)(d + \omega^q) \\
(d + \omega)(d + \omega^q)(d + \omega^q^2)
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & \tau & \tau^2 & 0 \\
1 & \tau^q & \tau^{2q} & 0 \\
1 & \tau^{q^2} & \tau^{2q^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad R_d = \begin{pmatrix}
a_0(d) \\
a_1(d) \\
a_2(d) \\
c(d)
\end{pmatrix}.
Note that the determinant of $T$ is $1$ times the determinant of a Vandermonde matrix, and so $|T| \neq 0$. Hence we have obtained expressions for $a_0(d), a_1(d), a_2(d)$ and $c(d)$. The expression for $P_d$ given in (3) is defined for $d \in GF(q)$, but extends in the natural way for $d \in GF(q^3)$ using our expression $R_d = T^{-1}D$. We will now consider the value $d = -\omega$.

We work in $PG(6, q^3)$ and show that

$$P_{-\omega} = \alpha_\omega \left( \tau^2 - t_2 \tau - t_1, \tau - t_2, 1, \tau^2 - t_2 \tau - t_1, \tau - t_2, 1, 0 \right),$$

where $\alpha_\omega \in GF(q^3)$ (defined below) is a constant depending on $\omega$. For $d = -\omega$, the last coordinate $c(d)$ of $P_d$ is zero, so we only need to show that our choice of $P_{-\omega}$ satisfies (5), (6), (7). First note that

$$-\omega(\omega^q - \omega)(\omega^{q^2} - \omega) = (\tau - \tau^q)(\tau^{q^2} - \tau)\alpha_\omega, \quad (8)$$

where $\alpha_\omega = -\omega(w_1 + w_2(\tau^q + \tau))(w_1 + w_2(\tau^{q^2} + \tau)) \neq 0$ since the LHS of (8) is non-zero. So substituting $d = -\omega$ into (5), (6) and (7) and letting $A_i = a_i/\alpha_\omega : GF(q) \rightarrow GF(q^3)$ ($1 \leq i \leq 3$), we obtain

$$-\omega(-\omega + \omega^q)(-\omega + \omega^{q^2}) = \alpha_\omega \left( A_0 + A_1 \tau + A_2 \tau^2 \right),$$

$$0 = \alpha_\omega \left( A_0 + A_1 \tau^q + A_2 \tau^{q^2} \right),$$

$$0 = \alpha_\omega \left( A_0 + A_1 \tau^{q^2} + A_2 \tau^{q^3} \right); \quad (11)$$

that is

$$(-\tau + \tau^q)(-\tau + \tau^{q^2}) = A_0 + A_1 \tau + A_2 \tau^2, \quad (12)$$

$$0 = A_0 + A_1 \tau^q + A_2 \tau^{q^2}, \quad (13)$$

$$0 = A_0 + A_1 \tau^{q^2} + A_2 \tau^{q^3}. \quad (14)$$

Now we check that RHS (12) = $(\tau^2 - t_2 \tau - t_1) + (\tau - t_2) + \tau^2 = 3\tau^2 - 2t_2 \tau - t_1$. Also, LHS (12) = $\tau^q \tau^{q^2} - \tau \tau^q + \tau^2 = (\tau^2 - t_2 \tau - t_1) + (\tau - t_2) + \tau^2 = 3\tau^2 - 2t_2 \tau - t_1 = RHS (12)$. Further, RHS (13) = $(\tau^2 - t_2 \tau - t_1) + (\tau - t_2) + \tau^q + \tau^{q^2} = (\tau^2 - t_2 \tau - t_1) + (\tau - t_2) + \tau^q + \tau^{q^2} = (\tau^2 - t_2 \tau - t_1) + \frac{1}{\tau} = \frac{1}{\tau}(\tau^2 - t_2 \tau - t_1)$. Hence

$$(P_{-\omega})^q = \left( a_0(-\omega)^q, a_1(-\omega)^q, a_2(-\omega)^q, a_0(-\omega)^q, a_1(-\omega)^q, a_2(-\omega)^q, 0 \right)$$
$$= \left( a_0(-\omega^q), a_1(-\omega^q), a_2(-\omega^q), a_0(-\omega^q), a_1(-\omega^q), a_2(-\omega^q), 0 \right)$$
$$= P_{-\omega^q}.$$
Note that we have shown above that $N$ meets the plane $\Sigma \cap \Sigma_\infty$ when $c = 0$, and in the cubic extension $PG(6, q^3)$, $N$ meets the plane $\Sigma \cap \Sigma_\infty$ in the three points $P_\omega$, $P_{-\omega}$, $P_{-\omega^2}$.

We now prove part (b). Let $\ell$ be a line of $PG(2, q^3)$ and let $\Sigma$ be the 3-space of $PG(6, q)$ corresponding to $\ell$. We now count to show that the number of normal rational curves in $\Sigma$ that when extended to $PG(6, q^3)$ meet the transversals of $\Sigma$, is equal to the number of order-$q$-sublines of $\ell$ that are disjoint from $\ell_\infty$. In $PG(6, q^2)$, let $P_1 = g \cap \Sigma$, $P_2 = g^2 \cap \Sigma$, and $P_3 = g^3 \cap \Sigma$. Now $(P_1, P_2, P_3)$ is a plane that meets $PG(6, q)$ in the spread element $\pi = \Sigma \cap \Sigma_\infty$. Further, if the extension of a plane of $PG(6, q)$ meets one of the $P_i$, then it contains them all. Hence if $Q_1$, $Q_2$, $Q_3$ are three non-collinear points of $\Sigma \setminus \pi$, then in $PG(6, q^3)$, $(P_1, P_2, P_3, Q_1, Q_2, Q_3)$ is a set of six points, no four coplanar. Hence, by [6, Theorem 21.1.1], they lie in a unique normal rational curve $N$ of $\Sigma \subset PG(6, q^3)$. That is, $Q_1$, $Q_2$, $Q_3$ lie in a unique normal rational curve of $\Sigma$ which contains the points $P_1$, $P_2$, $P_3$ in the cubic extension $PG(6, q^3)$.

Let $Q_1$, $Q_2$, $Q_3$ be three points of $\ell \setminus \ell_\infty$. They lie in a unique order-$q$-subline of $\ell$. This subline meets $\ell_\infty$ if and only if, in $PG(6, q)$, the points $Q_1$, $Q_2$, $Q_3$ are collinear. If $Q_1$, $Q_2$, $Q_3$ are not collinear in $PG(6, q)$, then, by the above argument, they lie in a unique normal rational curve of $\Sigma$ that contains $P_1$, $P_2$, $P_3$ in the cubic extension $PG(6, q^3)$. Hence the number of order-$q$-sublines of $\ell$ that are disjoint from $\ell_\infty$ is equal to the number of normal rational curves of $\Sigma$ that contain $P_1$, $P_2$, $P_3$ in the cubic extension $PG(6, q^3)$, proving part (b). \(\Box\)

We next look at the representation of subplanes that are tangent to $\ell_\infty$. We will need to use coordinates; so we first present a lemma that calculates results about the coordinates of a representative order-$q$-subplane that is tangent to $\ell_\infty$ in $PG(2, q^3)$. Note that $PG(2, q) = \{ (x, y, z) \mid x, y, z \in GF(q) \}$, not all zero) is an order-$q$-subplane that is secant to $\ell_\infty$. We find a homography that maps this to a tangent order-$q$-subplane. Note also that the point $(x, y, 1)$ lies in $PG(2, q)$ if and only if $x^q = x$ and $y^q = y$.

**Lemma 2.6.** In $PG(2, q^3)$, let $B$ be the unique order-$q$-subplane containing the quadrangle $(1, 0, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ and $(1 + \omega, 1, 1 + \omega)$ for some fixed $\omega \in GF(q^3) \setminus GF(q)$. Then $B$ is tangent to $\ell_\infty$, and $B$ contains the points $(1, 0, 1)$, $(1, 1, 1 + \omega)$, and the order-$q$-subline $\ell_\omega = \{ P_d = (d, d + \omega) \mid d \in GF(q) \cup \{ \infty \} \}$. Further, $B$ is the image of the order-$q$-subplane $PG(2, q)$ under the homography $\sigma_1$ with matrix $A_1$; and also under the homography $\sigma_2$ with matrix $A_2$ where

\[
A_1 = \begin{pmatrix}
-\omega & 1 + \omega & 0 \\
0 & 1 & 0 \\
0 & 1 + \omega & -\omega
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\omega & 1 & -\omega \\
0 & 1 & 0 \\
\omega & 1 & 0
\end{pmatrix}
\]

such that $\sigma_1 : PG(2, q) \rightarrow B$ is given by $\sigma_1(X) = A_1X$, writing each point $X$ as a column vector ($1 \leq i \leq 2$).

**Proof.** There is a unique homography $\sigma_1$ satisfying $\sigma_1(1, 0, 0) = (1, 0, 0)$, $\sigma_1(0, 1, 0) = (1 + \omega, 1, 1 + \omega)$, $\sigma_1(0, 0, 1) = (0, 0, 1)$, and $\sigma_1(1, 1, 1) = (1, 1, 1)$, namely the homography with matrix $A_1$ given above. So $B$ is the image of $PG(2, q)$ under the homography $\sigma_1$. Note that $\sigma_1^{-1} : B \rightarrow PG(2, q)$ such that $\sigma_1^{-1}(X) = A_1^T X$ has matrix given by

\[
A_1^T = \begin{pmatrix}
-1 & 1 + \omega & 0 \\
0 & \omega & 0 \\
0 & 1 + \omega & -1
\end{pmatrix}.
\]

To show that $B$ is tangent to $\ell_\infty$ (of equation $z = 0$), we show that $(1, 0, 0)$ is the only point of $B$ which is on $\ell_\infty$. Now, a point $(x, y, z) \in PG(2, q)$ maps to the point $\sigma_1(x, y, z) = (-x\omega + y(1 + \omega), y, y(1 + \omega) - z\omega)$ of $B$, which is on $\ell_\infty$ if and only if $y(1 + \omega) - z\omega = 0$ for some $y, z \in GF(q)$, if and only if $y = z = 0$, giving the point $(1, 0, 0)$. Hence $B$ meets $\ell_\infty$ in the point $(1, 0, 0)$. It is easy to check that $\sigma_1^{-1}(1, 0, 1)$, $\sigma_1^{-1}(1, 1, 1 + \omega)$, and $\sigma_1^{-1}(d, d, d + \omega)$ for $d \in GF(q) \cup \{ \infty \}$ all lie in $PG(2, q)$. Hence $(1, 0, 1)$, $(1, 1, 1 + \omega)$ and $\ell_\omega$ all lie in $B$. 
Alternatively, we can map $\PG(2, q)$ to an order-$q$-subplane $B'$ via the homography $\sigma_2$ uniquely determined by $\sigma_2(1, 0, 0) = (1, 0, 1)$, $\sigma_2(0, 1, 0) = (1, 1, 1)$, $\sigma_2(0, 0, 1) = (1, 0, 0)$, and $\sigma_2(1, 1, 1) = (1, 1, 1 + \omega)$. We calculate $\sigma_2(X) = A_2X$ where $A_2$ is given in the statement of the lemma. As $B$ and $B'$ both contain the quadrangle $(1, 0, 1), (1, 1, 1), (1, 0, 0)$, and $(1, 1, 1 + \omega)$, we have $B = B'$. Note also that $\sigma_2^{-1}$ has matrix $A'_2$ given by

$$A'_2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & \omega & 0 \\ -1 & 0 & 1 \end{pmatrix}. \quad \Box$$

**Theorem 2.7.** Let $B$ be an order-$q$-subplane of $\PG(2, q^2)$ that is tangent to $\ell_\infty$ in the point $T$. Let $\pi_T$ be the spread element corresponding to $T$. Then $B$ determines a set $\mathcal{B}$ of points in $\PG(6, q)$ (where the affine points of $B$ correspond to the affine points of $\mathcal{B}$) such that:

(a) $B$ is a ruled surface with conic directrix $C$ contained in the plane $\pi_T \in S$, and normal rational curve directrix $N$ contained in a 3-space $\Sigma$ that meets $\Sigma_\infty$ in a spread element (distinct from $\pi_T$). The points of $\mathcal{B}$ lie on $q + 1$ pairwise disjoint generator lines joining $C$ to $N$.

(b) The $q + 1$ generator lines of $B$ joining $C$ to $N$ are determined by a projectivity from $C$ to $N$.

(c) When we extend $B$ to $\PG(6, q^2)$, it contains the conjugate transversal lines $g, g^4, g^{q^2}$ of the spread $S$.

(d) $B$ is the intersection of nine quadrics in $\PG(6, q)$.

**Proof.** Let $B$ be an order-$q$-subplane of $\PG(2, q^2)$ that meets $\ell_\infty$ in the point $T$. Consider the $q + 1$ order-$q$-sublines of $B$ through $T$. By Theorem 2.3(b), these correspond to $q + 1$ lines $m_1, \ldots, m_{q+1}$ of $\PG(6, q) \setminus \Sigma_\infty$. Let $\mathcal{B}$ be the set of points that lie on these lines, so the affine part of $B$ corresponds to the affine part of the order-$q$-subplane $B$. Let $m_i \cap \pi_T = M_i, i = 1, \ldots, q + 1$. For $i 

First note that $M_1, \ldots, M_{q+1}$ are distinct. Since if $M_i = M_j$, then $\{m_i, m_j\}$ is a line and either $M_1 \cap \pi_T = M_i$, $i = 1, \ldots, q + 1$.

We will show that the points $\{M_1, \ldots, M_{q+1}\}$ form a conic. Hence we can conclude that $B$ consists of $q + 1$ mutually disjoint lines $m_1, \ldots, m_{q+1}$ joining the conic $C = \{M_1, \ldots, M_{q+1}\}$ in $\pi_T$, and the normal rational curve $N$. We use coordinates to show that the points $\{M_1, \ldots, M_{q+1}\}$ form a conic. As in the proof of Theorem 2.5, we may $\omega \in GF(q^2) \setminus GF(q)$ and consider the point set of $\PG(2, q^2)$ given by $\ell_\omega = \{P_d = (d, d, d + \omega) : d \in GF(q) \cup \{\infty\}\}$. Then $\ell_\omega$ is an order-$q$-subplane of the line $x = y$ which is disjoint from $\ell_\infty$. Let $B$ be the unique order-$q$-subplane containing $\ell_\omega$. $T = (1, 0, 0)$ and $A = (1, 0, 1)$ (as calculated in Lemma 2.6). Note that $B$ is tangent to $\ell_\infty$ at $T$. The line $x = z$ has two points in $B$, so $B$ meets $B$ in an order-$q$-subline. Let $R_d, d \in GF(q) \cup \{\infty\}$ be the intersection of the line $x = z$ and the line $T P_d$, so $R_d = (d + \omega, d, d + \omega)$, see Fig. 2. Using the notation from Theorem 2.5, the coordinates of $P_d$, $R_d$ in $\PG(6, q)$, are $P_d = (a_0(d), a_1(d), a_2(d), a_0(d), a_1(d), a_2(d), c(d))$ and $R_d = (c(d), 0, 0, a_0(d), a_1(d), a_2(d), c(d))$. If $d \in GF(q)$, the line joining $P_d$ and $R_d$ meets $\Sigma_\infty$ at the point $Q_d = (a_0(d) - c(d), a_1(d), a_2(d), 0, 0, 0, 0)$. Now consider the case $d = \infty$. In $\PG(2, q^2)$, $S_\infty = P_\infty = (1, 1, 1)$, so to find the final line through $T$ we need to find another point of $B$ on the line through $T = (1, 0, 0)$ and $P_\infty = (1, 1, 1)$, that is, the line of equation $y = z$. We note that $P_1 = (1, 1, 1 + \omega)$ and $(1, 0, 1)$ are points of $B$, so the line joining them is a line of $B$ and hence meets $y = z$ in a point of $B$; namely the point with coordinates $F = (1 - \omega, 1, 1)$. The two points $P_\infty$ and $F$ on the line $y = z$ are represented in $\PG(6, q)$ by the points $P_\infty = (1, 0, 0, 1, 0, 0, 1)$ and $F = (1 - w_0, -w_1, -w_2, 1, 0, 0, 0, 0)$. The line joining $P_\infty$ and $F$ meets $\Sigma_\infty$ in the point $(-w_0, -w_1, -w_2, 0, 0, 0, 0)$, which is the point $Q_\infty$. Hence we
have \( \{M_1, \ldots, M_{q+1}\} = \{ Q_d \mid d \in GF(q) \cup \{ \infty \} \} \). Now \( a_0(d) - c(d), a_1(d), a_2(d) \) are quadratics in \( d \) over \( GF(q) \) with "leading terms" \(-d^2 w_0, -d^2 w_1, -d^2 w_2 \) respectively. Hence we need to find a homography that maps the set \( \{ Q_d \mid d \in GF(q) \cup \{ \infty \} \} \) to the set of points \( \{ (d^2, d, 1) \mid d \in GF(q) \cup \{ \infty \} \} \). In the proof of Theorem 2.5, the matrix \( B \) was defined as follows, and now suppose \( C \) is the matrix with

\[
\begin{pmatrix}
a_0(d) \\
a_1(d) \\
a_2(d)
\end{pmatrix} = B
\begin{pmatrix}
d^3 \\
d^2 \\
d
\end{pmatrix},
\begin{pmatrix}
a_0(d) - c(d) \\
a_1(d) \\
a_2(d)
\end{pmatrix} = C
\begin{pmatrix}
d^2 \\
d \\
1
\end{pmatrix}.
\]

In Theorem 2.5 we showed that the four points \( P_d \) with \( d = 0, -\omega, -\omega^2, -\omega^3 \) were independent. Note that for \( d = -\omega, -\omega^2, -\omega^3 \), \( c(d) = 0 \), hence it follows that \( Q_d \) for \( d = -\omega, -\omega^2, -\omega^3 \) are three independent points. Thus \( C \) represents a homography, and hence the points \( \{M_1, \ldots, M_{q+1}\} \) form a conic in \( \pi_T \).

Hence we have shown that \( B \) is a ruled surface with conic directrix \( C = \{ Q_d \mid d \in GF(q) \cup \{ \infty \} \} \) in \( \pi_T \) and normal rational curve directrix \( N = \{ P_d \mid d \in GF(q) \cup \{ \infty \} \} \) in \( \Sigma \), with \( q+1 \) generator lines \( Q_d P_d, d \in GF(q) \cup \{ \infty \} \), proving (a). From the coordinates, we have a natural projectivity from the points \( Q_d \) of \( C \) to the points \( P_d \) of \( N \), proving (b).

Now consider the cubic extension \( PG(6, q^3) \). We can naturally extend the conic \( C \) and normal rational curve \( N \) in \( PG(6, q) \) to \( \overline{C} \) and \( \overline{N} \) in \( PG(6, q^3) \). Hence the projectivity \( C \to N \) is extended to \( \overline{C} \to \overline{N} \), and so \( B \) is naturally extended to \( \overline{B} \) in \( PG(6, q^3) \). We have from Theorem 2.5 that \( \overline{N} \) contains the points \( g \cap \overline{\Sigma} = P_{-\omega}, g^2 \cap \overline{\Sigma} = P_{-\omega^2}, g^3 \cap \overline{\Sigma} = P_{-\omega^3} \). Similarly, \( \overline{C} \) contains the point \( Q_{-\omega} \) which we calculate to be \( Q_{-\omega} = (\tau^2 - t_2 \tau - t_1, \tau - t_2, 1, 0, 0, 0, 0) \), which lies on \( g \) (using Lemma 2.1). It follows that \( \overline{C} \) contains the three points \( g \cap \overline{\pi_T} = Q_{-\omega}, g^2 \cap \overline{\pi_T} = Q_{-\omega^2}, g^3 \cap \overline{\pi_T} = Q_{-\omega^3} \). Thus \( \overline{B} \) contains the three transversal lines \( g = P_{-\omega} Q_{-\omega}, g^2 = P_{-\omega^2} Q_{-\omega^2}, g^3 = P_{-\omega^3} Q_{-\omega^3} \), proving (c).

Finally we show that \( B \) is an algebraic variety by showing that it is the intersection of nine quadrics. We continue to work with the order-\( q \)-subplane \( B \) with coordinates given in Lemma 2.6. So the affine points of \( B \) are the points \( (x, y, 1) \) satisfying \( \sigma_1^{-1}(x, y, 1) \in PG(2, q) \) (where \( \sigma_1 \) is specified in Lemma 2.6). That is, we require \( P = ((1 + \omega)y - x, \omega y, (1 + \omega)y - 1) \in PG(2, q) \).
We first consider the case \((1 + \omega) y - 1 \neq 0\). So \(P \equiv (((1 + \omega) y - x)/(1 + \omega) y - 1), \omega y/(1 + \omega) y - 1, 1\), and so \(P \in \text{PG}(2, q)\) if and only if

\[
\frac{(1 + \omega) y - x}{(1 + \omega) y - 1} = \frac{(1 + \omega) y - x}{(1 + \omega) y - 1} \quad \text{and} \quad \frac{\omega y}{(1 + \omega) y - 1} = \frac{\omega y}{(1 + \omega) y - 1}.
\]

The second equation is

\[
((1 + \omega) y - 1)(\omega y)^q - ((1 + \omega) y - 1)^q \omega y = 0. \tag{15}
\]

We simplify the first equation as follows:

\[
\frac{(1 + \omega) y - 1 + 1 - x}{(1 + \omega) y - 1} = \frac{(1 + \omega) y - 1 + 1 - x}{(1 + \omega) y - 1},
\]

\[
(1 - x)^q((1 + \omega) y - 1) = (1 - x)((1 + \omega) y - 1)^q. \tag{16}
\]

We use Eq. (15) in (16) to obtain

\[
(1 - x)(\omega y)^q - (1 - x)^q \omega y = 0, \tag{17}
\]

which we will need later.

Consider the two equations (15) and (16) in \(\text{PG}(2, q^3)\). If we write \(x = x_0 + x_1 \tau + x_2 \tau^2, y = y_0 + y_1 \tau + y_2 \tau^2, \) and \(\omega = w_0 + w_1 \tau + w_2 \tau^2\) for \(x_i, y_i, w_i \in \text{GF}(q)\), then simplify and equate coefficients of powers of \(\tau\), we will obtain the equations of six affine quadrics in \(\text{PG}(6, q)\). We can homogenize these, and so the points of \(B\) in \(\text{PG}(2, q^3)\) correspond to points in \(\text{PG}(6, q)\) that lie on all six quadrics. The intersection is not exact though, as we need to consider the case \((1 + \omega) y - 1 = 0\), that is, \(y = 1/(1 + \omega)\). The points of \(\text{PG}(2, q^3)\) that satisfy \(y = 1/(1 + \omega)\) lie on a line \(\ell_1\) that corresponds to a 3-space \(\Sigma_1\) of \(\text{PG}(6, q)\). The points on \(\ell_1\) satisfy Eqs. (15) and (16), and so the points of \(\Sigma_1\) will lie on all six quadrics, that is, all six quadrics will contain \(\Sigma_1\).

We can show that \(B\) is the precise intersection of nine quadrics by considering a second representation of \(B\) in \(\text{PG}(2, q^3)\). That is, we repeat the above argument using \(\sigma_2\) from Lemma 2.6. So we have the affine points of \(B\) are the points \((x, y, 1)\) satisfying \(\sigma_2^{-1}(x, y, 1) \in \text{PG}(2, q)\). That is, we require \(P = (1 - y, \omega y, 1 - x) \in \text{PG}(2, q)\).

Consider the case \(x \neq 1\). So \(P \equiv ((1 - y)/(1 - x), \omega y/(1 - x), 1), \) and so \(P \in \text{PG}(2, q)\) if and only if

\[
\frac{(1 - y)/(1 - x)^q}{1 - x} = \frac{(1 - y)/(1 - x)^q}{1 - x} \quad \text{and} \quad \frac{(\omega y)/(1 - x)^q}{1 - x} = \frac{\omega y}{1 - x}.
\]

Rearranging these two equations in \(\text{PG}(2, q^3)\) yields

\[
(1 - y)^q(1 - x) - (1 - y)(1 - x)^q = 0, \tag{18}
\]

\[
(1 - x)(\omega y)^q - (1 - x)^q \omega y = 0. \tag{19}
\]

As before, writing \(x = x_0 + x_1 \tau + x_2 \tau^2, y = y_0 + y_1 \tau + y_2 \tau^2, \) and \(\omega = w_0 + w_1 \tau + w_2 \tau^2\), simplifying, and equating coefficients of powers of \(\tau\) yields the equations of six affine quadrics in \(\text{PG}(6, q)\). These six quadrics all contain the 3-space \(\Sigma_2\) corresponding to the line \(\ell_2\) of \(\text{PG}(2, q^3)\) with affine equation \(x = 1\), so \(B\) is the residual intersection of these six quadrics.

Putting these two sets of six quadrics together, we have \(B\) contained in twelve quadrics. As the 3-space \(\Sigma_1\) meets the 3-space \(\Sigma_2\) in an affine point corresponding to the point \((1, 1, 1+\omega, 1)\) of
PG(2, q^2), we have that B is the exact intersection of all twelve quadrics. However, as Eq. (19) is the same as Eq. (17), B is the exact intersection of nine quadrics in PG(6, q).  

Note that in proving part (d) of the above theorem, we have shown that, in PG(6, q), B is the (residual) intersection of six quadrics that each contain a common 3-space. This generalizes the 2-dimensional result that a tangent Baer subplane of PG(2, q^2) corresponds in PG(4, q) to the (residual) intersection of two quadrics that contain a common plane (see Vincenti [9], and Quinn and Casse [8]). The argument in the proof of part (d) above can be generalized to the case of the Bruck–Bose representation of PG(2, q^2) in PG(4, q). In this case, Eqs. (15) and (16) yield four quadrics in PG(4, q). However, on closer inspection, in this special case the four quadrics collapse into the two quadrics given by Vincenti. Hence the proof of part (d) generalizes to this case to show that a tangent Baer subplane corresponds to the residual intersection of two quadrics with a common plane; and to the exact intersection of three quadrics. Moreover, the proof by Quinn and Casse in [8, Lemma 2.6] gives a geometric construction of these two quadrics. We note that it is straightforward to generalize the proof of Quinn and Casse to obtain a geometric construction of the six quadrics of B in PG(6, q).

3. The Bruck–Bose representation of PG(2, q^n) in PG(2n, q)

We can generalize all the results for the case n = 3 to the Bruck–Bose representation of PG(2, q^n) in PG(2n, q) for general n \geq 4. That is, we fully determine the representation of order-q-sublines and secant and tangent order-q-subplanes of PG(2, q^n) in PG(2n, q) for n \geq 4.

3.1. The t-spreads and t-reguli of PG(2t + 1, q)

A t-spread of PG(2t + 1, q) is a set of q^t + 1 t-spaces that partition PG(2t + 1, q). A t-regulus R of PG(2t + 1, q) is a set of q + 1 mutually disjoint t-spaces with the property that if a line meets three of the t-spaces in R, then it meets all q + 1 of them. Three mutually disjoint t-spaces in PG(2t + 1, q) lie on a unique t-regulus. A t-spread S is regular if for any three t-spaces in S, the t-regulus containing them is contained in S. In a regular t-spread, any q + 1 spread elements meeting a line form a t-regulus.

A regular t-spread of PG(2t + 1, q) has a set of t transversal lines that lie in PG(2t + 1, q) \{PG(2t + 1, q)\}. Embed PG(2t + 1, q) in PG(2t + 1, q^t) and let g be a line of PG(2t + 1, q^t) disjoint from PG(2t + 1, q). Let g^t, \ldots, g^{t-1} be the conjugate lines of g. Let P_i be a point on g, then the t-space \langle P_i, P_i, \ldots, P_i^{t-1} \rangle meets PG(2t + 1, q) in a t-space. As P_i ranges over all the points of g, we get q^t + 1 t-spaces of PG(2t + 1, q) that partition the space. These t-spaces form a regular spread S of PG(2t + 1, q). The lines g, g^t, \ldots, g^{t-1} are called the (conjugate skew) transversal lines of the spread S. Given a regular t-spread in PG(2t + 1, q), there is a unique set of t conjugate transversal lines in PG(2t + 1, q^t) that generate S in this way. See [7, Section 25.6] for more information on t-reguli and t-spreads.

3.2. The Bruck–Bose representation

Let S_\infty be a hyperplane of PG(2n, q) and let S be an (n - 1)-spread of S_\infty. Consider the following incidence structure: the points of A(S) are the points of PG(2n, q) \{S_\infty\}; the lines of A(S) are the n-spaces of PG(2n, q) \{S_\infty\} that contain an element of S; and incidence in A(S) is induced by incidence in PG(2n, q). Then the incidence structure A(S) is an affine plane of order q^n. We can complete A(S) to a projective plane P(S) where the points on the line at infinity \ell_\infty have a natural correspondence to the elements of the (n - 1)-spread S. Further, P(S) is Desarguesian if and only if S is regular.

It is straightforward to generalize the coordinates from Section 2.2 to PG(2n, q). In this case, \tau is a primitive element of GF(q^n) with primitive polynomial

\[ x^n - t_{n-1}x^{n-1} - \cdots - t_1x - t_0. \]
We can calculate the transversal lines of the regular \((n-1)\)-spread \(S\).

**Lemma 3.1.** Let \(g\) be the line of \(\text{PG}(2n, q^n)\) through the points \((p_0, \ldots, p_{n-1}, 0, \ldots, 0, 0)\), and \((0, \ldots, 0, p_0, \ldots, p_{n-1}, 0)\), where \(p_i = t_i + 1 + t_{i+2} \tau + \cdots + t_{n-1} \tau^{n-2-i} - \tau^{n-1-i}, i = 0, n-2,\) and \(p_{n-1} = -1\). Then \(g\) is one of the conjugate skew transversals of the regular \((n-1)\)-spread \(S\).

We omit the proof which involves induction to generalize the proof of Lemma 2.1.

3.3. Sublines and subplanes of \(\text{PG}(2, q^n)\)

We completely determine the representation of order-\(q\)-sublines and secant and tangent order-\(q\)-subplanes of \(\text{PG}(2, q^n)\) in \(\text{PG}(2n, q)\). As before, an order-\(q\)-subplane of \(\text{PG}(2, q^n)\) is a subplane \(B\) of \(\text{PG}(2, q^n)\) of order \(q\). Every line of \(\text{PG}(2, q^n)\) meets \(B\) in 0, 1 or \(q + 1\) points; a line of \(B\) has \(q + 1\) points and is called an order-\(q\)-subline of \(\text{PG}(2, q^n)\). We generalize the results of Section 2.3. As before, there are six cases to consider, and we completely determine five of them. Three cases are stated in Theorems 3.2, 3.3, and 3.4. The proofs of these theorems are straightforward generalizations of those in Section 2.3, so we do not include them here. We provide sketch proofs for the cases a disjoint order-\(q\)-subline and tangent order-\(q\)-subplane in Theorems 3.5 and 3.6. As in the \(n = 3\) case, we leave open the case of the exterior order-\(q\)-subplanes.

**Theorem 3.2.** Consider the translation plane \(\mathcal{P}(S)\) of order \(q^n\) defined from an \((n-1)\)-spread \(S\) of a hyperplane \(\Sigma_\infty\) of \(\text{PG}(2n, q)\).

(a) A plane of \(\text{PG}(2n, q) \setminus \Sigma_\infty\) that meets \(q + 1\) elements of \(S\) represents an order-\(q\)-subplane of \(\mathcal{P}(S)\) secant to \(\ell_\infty\).

(b) If \(\mathcal{P}(S) \cong \text{PG}(2, q^n)\), then every order-\(q\)-subplane of \(\mathcal{P}(S)\) that is secant to \(\ell_\infty\) is represented by a plane of \(\text{PG}(2n, q) \setminus \Sigma_\infty\) that meets \(q + 1\) elements of \(S\).

**Theorem 3.3.** Consider the translation plane \(\mathcal{P}(S)\) of order \(q^n\) defined from an \((n-1)\)-spread \(S\) of a hyperplane \(\Sigma_\infty\) of \(\text{PG}(2n, q)\).

(a) A line of \(\text{PG}(2n, q) \setminus \Sigma_\infty\) represents an order-\(q\)-subline of \(\mathcal{P}(S)\) tangent to \(\ell_\infty\).

(b) If \(\mathcal{P}(S) \cong \text{PG}(2, q^n)\), then every order-\(q\)-subline of \(\mathcal{P}(S)\) tangent to \(\ell_\infty\) is represented by a line of \(\text{PG}(2n, q) \setminus \Sigma_\infty\).

**Theorem 3.4.** Consider the translation plane \(\mathcal{P}(S)\) of order \(q^n\) defined from an \((n-1)\)-spread \(S\) of a hyperplane \(\Sigma_\infty\) of \(\text{PG}(2n, q)\).

(a) A regulus \(\mathcal{R}\) of \(S\) represents an order-\(q\)-subline of \(\ell_\infty\) in \(\mathcal{P}(S)\).

(b) If \(\mathcal{P}(S) \cong \text{PG}(2, q^n)\), then every order-\(q\)-subline of \(\ell_\infty\) in \(\mathcal{P}(S)\) is represented by a regulus of \(S\).

We now consider the remaining two cases: an order-\(q\)-subline disjoint from \(\ell_\infty\), an order-\(q\)-subplane tangent to \(\ell_\infty\). The results from Section 2.3 do generalize; however, the proofs are much more complex. We sketch the proofs for this general case.

**Theorem 3.5.**

(a) Let \(\ell\) be a line of \(\text{PG}(2, q^n)\) and let \(b\) be an order-\(q\)-subline of \(\ell\) that is disjoint from \(\ell_\infty\). Then, in \(\text{PG}(2n, q)\), \(b\) corresponds to a normal rational curve in the \(n\)-space \(\Sigma\) corresponding to \(\ell\).

(b) Let \(\Sigma\) be an \(n\)-space of \(\text{PG}(2n, q)\), \(\Sigma_\infty\) about a spread element and let \(\mathcal{N}\) be a normal rational curve in \(\Sigma\) that is disjoint from \(\Sigma_\infty\). Then \(\mathcal{N}\) corresponds to an order-\(q\)-subline of \(\text{PG}(2, q^n)\) if and only if, in the extension \(\text{PG}(2n, q^n)\), \(\mathcal{N}\) meets the conjugate transversal lines \(g^0, \ldots, g^{q^n-1}\) of the spread \(S\).
Proof (Sketch only). We generalize the proof of Theorem 2.5. Using the notation of that proof, we can write a point $P_d = (d, d, d+\omega)$ of the subline $\ell_\omega$ as $P_d = (\omega, \alpha, c) = (d+\omega^q) \cdots (d+\omega^{n-1})(d, d+\omega)$. Now $c \in \text{GF}(q)$, and $c$ is a polynomial in $d$ of degree $n$. We can write $\alpha = a_0 + a_1 \tau + \cdots + a_{n-1} \tau^{n-1}$ for $a_i \in \text{GF}(q)$, where the $a_i$ are polynomials in $d$ of degree at most $n$, and $a_0$ has degree $n$. Hence we can find a homography that maps the points of $PG(2n, q)$ corresponding to the points on $\ell_\omega$ to the points $\{(d^n, d^{n-1}, \ldots, d, 1) \mid d \in \text{GF}(q) \cup \{\infty\}\}$. That is, $\ell_\omega$ corresponds to a normal rational curve in an $n$-space of $PG(2n, q)$.

To prove part (b), we show that $P_{-\omega} = (-1)^n \alpha_\omega(p_0, \ldots, p_{n-1}, p_0, \ldots, p_{n-1}, 0)$ where $p_i$ is defined in Lemma 3.1 and $\alpha_\omega$ satisfies

$$(\omega^q - \omega)(\omega^{q^2} - \omega) \cdots (\omega^{q^{n-1}} - \omega) = (\tau^q - \tau)(\tau^{q^2} - \tau) \cdots (\tau^{q^{n-1}} - \tau) \alpha_\omega.$$  

In a generalization of Theorem 2.5 we write

$$(d + \omega^q)(d + \omega^{q^2}) \cdots (d + \omega^{q^{n-1}}) = z_0 + z_1 \tau + \cdots + z_{n-1} \tau^{n-1},$$

where $z_i = (z_i(d) : \text{GF}(q^n) \rightarrow \text{GF}(q^n))$ and $z_i(d) \in \text{GF}(q)$ for $d \in \text{GF}(q)$. We apply the field automorphism $x \mapsto x^q$ a further $n - 1$ times to obtain a total of $n$ equations, and then substitute $d = -\omega$ and arrive at the following $n$ equations:

$$(\tau^q - \tau)(\tau^{q^2} - \tau) \cdots (\tau^{q^{n-1}} - \tau) = A_0 + A_1 \tau + \cdots + A_{n-1} \tau^{n-1},$$  

$$(20)$$

$$0 = A_0 + A_1 \tau q^1 + \cdots + A_{n-1} \tau^{(n-1)q},$$  

$$(21)$$

$$\vdots$$

$$0 = A_0 + A_1 \tau q^1 + \cdots + A_{n-1} \tau^{(n-1)q^{n-1}},$$

where the $A_i$ have the same properties as the $z_i$ earlier. Now

$$\tau^{(\tau^q + \tau^q + \cdots + \tau^{q^{n-1}})}$$

$$= \left( \frac{T'}{n-1} \right) \tau + \cdots + (-1)^{i-1} \left( \frac{T'}{n-i} \right) \tau^{n-i-2} \left( \frac{T'}{1} \right) \tau^{n-1} + (-1)^{n-1} \tau^n$$

where $T' = \{\tau^q, \tau^{q^2}, \ldots, \tau^{q^{n-1}}\}$, $T = T' \cup \{\tau\}$ and for example $\left( \frac{T}{1} \right)$ is the sum of the products of all the elements in each $k$ subset of $T$, e.g.

$$\left( \frac{T}{1} \right) = \tau + \tau^q + \cdots + \tau^{q^{n-1}},$$

$$\left( \frac{T}{n} \right) = \tau \tau^q \cdots \tau^{q^{n-1}}.$$  

As $(x - \tau)(x - \tau q^1) \cdots (x - \tau^{q^{n-1}}) = x^n - t_{n-1} \tau^{n-1} - \cdots - t_1 x - t_0$, it follows that

$$t_k = (-1)^{n-k+1} \left( \frac{T}{n-k} \right)$$  

and

$$\left( \frac{T'}{k+1} \right) = \tau \left( \frac{T'}{k} \right) + \left( \frac{T'}{k+1} \right)$$

for $0 \leq k \leq n - 1$. We show by backward induction $i = n, n - 1, \ldots, 2, 1$ that $p_i = (-1)^{n-i+1} \left( \frac{T'}{n-i} \right)$ by using above relationships, together with the relationship $t_{k-1} + t_{k-1} \tau = p_{k-2}$. This will be enough to prove that (20) holds for $d = -\omega$.

For (21), note that $(-1)^n \tau^{i+1} p_i = -t_0 - t_1 \tau - \cdots - t_i \tau^i$ for $0 \leq i \leq n - 1$ and then show that RHS $(21) \times \tau^n = 0$. 

So we have shown that \( P_{-\omega} \in g \). In a similar manner to the case \( n = 3 \), we show that \( P_{-\omega^q} = (P_{-\omega})^q \) and so \( P_{-\omega^q} \in g^q \) and so on.

Hence we can conclude that the normal rational curve meets \( g \) and all the transversals of \( S \). Conversely, let \( Q_1, Q_2, Q_3 \) be non-collinear points of \( \Sigma\setminus\Sigma_{\infty} \), and let \( P_1 = g^{q^{d-1}} \cap \Sigma \). Then \( P_1, \ldots, P_n \), \( Q_1, Q_2, Q_3 \) are a set of \( n + 3 \) points in an \( n \)-space \( \Sigma \), no \( n + 1 \) in an \( (n - 1) \)-space, and so lie in a unique normal rational curve. Hence the counting argument in the proof of Theorem 2.5 generalizes. \( \square \)

**Theorem 3.6.** Let \( B \) be an order-\( q \)-subplane of \( \text{PG}(2, q^d) \) that is tangent to \( \ell_{\infty} \) in the point \( T \). Let \( \Sigma_T \) be the spread element corresponding to \( T \). Then \( B \) determines a set \( B \) of points in \( \text{PG}(2n, q) \) (where the affine points of \( B \) correspond to the affine points of \( B \)) such that:

(a) \( B \) is a ruled surface with normal rational curve directrix \( C \) contained in the \( (n - 1) \)-space \( \Sigma_T \subset S \), and normal rational curve directrix \( N \) contained in an \( n \)-space \( \Sigma \) that meets \( \Sigma_{\infty} \) in a spread element (distinct from \( \Sigma_T \)). The points of \( B \) lie on \( q + 1 \) pairwise disjoint generator lines joining \( C \) to \( N \).

(b) The \( q + 1 \) generator lines of \( B \) joining \( C \) to \( N \) are determined by a projectivity from \( C \) to \( N \).

(c) When we extend \( B \) from \( \text{PG}(2n, q) \) to \( \text{PG}(2n, q^d) \), it contains the (conjugate skew) transversal lines \( g, g^q, \ldots, g^{q^{d-1}} \) of the spread \( S \).

(d) \( B \) is the intersection of \( 3n \) quadrics in \( \text{PG}(2n, q) \).

**Proof.** We generalize the proof of Theorem 2.7. Using the notation from that proof, in this more general setting, we have that the points \( Q_d \) lie in a spread element \( \Sigma_T \), and have coordinates that are polynomials in \( d \) of degree \( n - 1 \). Hence they can be mapped to a normal rational curve in an \( (n - 1) \)-space. So parts (a) and (b) hold. Note that we can naturally extend \( B \) to \( \overline{B} \) in \( \text{PG}(2n, q^d) \) as follows. We extend the normal rational curves \( C \) and \( N \) in \( \text{PG}(2n, q) \) to \( \overline{C} \) and \( \overline{N} \) in \( \text{PG}(2n, q^d) \) (respectively), and the projectivity \( C \rightarrow N \) is extended to \( \overline{C} \rightarrow \overline{N} \), then \( B \) is extended to \( \overline{B} \) in \( \text{PG}(2n, q^d) \). Part (c) follows by computing the coordinates of \( Q_{-\omega} \) to be \( Q_{-\omega} = (p_0, \ldots, p_{n-1}, 0, \ldots, 0) \) where \( p_i \) are defined in Lemma 3.1, hence \( Q_{-\omega} \) lies on \( g \), and so \( \overline{B} \) contains \( g \). Finally the proof of Theorem 2.7(d) generalizes immediately to show that \( B \) is the intersection of \( 3n \) quadrics in \( \text{PG}(2n, q) \). Hence \( B \) is an algebraic variety. \( \square \)

**Acknowledgments**

We would like to thank the referees for their astute comments and suggestions for corrections.

**References**