NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF STATIONARY AND PERIODIC SOLUTIONS OF A STOCHASTIC DIFFERENCE EQUATION IN HILBERT SPACE

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1. INTRODUCTION

Random processes generated by stochastic difference equations are of interest for the theory of random processes, for statistical analysis and for various applications. The problem of existence of stationary solutions of difference equations has a long-standing history, see M. Arato [1], where further references may be found. Among the recent works one should mention the paper by A. Brandt [2]. The natural generalization of stationary process is periodic process introduced by A.Ya. Dorogovtsev [3]. The periodicity stands for periodicity of certain mean characteristics of the process. Periodic processes adequately describe real processes in radio engineering, season changes in applications, models in mathematical economics. Some problems of existence of periodic solutions of stochastic difference equations were studied by T. Morozan [5], [6].

The aim of the paper is to prove the criteria of the existence of stationary and periodic solutions of linear stochastic difference equation in a Hilbert space. We use some ideas of M. Arato [1]. The properties of periodic processes and conditions providing existence of periodic solutions of discrete stochastic systems in a Hilbert space were discussed in the thesis of Le Vigne Thuang. Some results of the paper were obtained jointly with him.

2. NOTATIONS AND DEFINITIONS

Let \((H, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space, \(\| \cdot \|\) and 0 are norm and zero element in \(H\). \(L(H)\) is the space of all linear bounded operators with the operator norm. Let \((\Omega, \mathcal{F}, P)\) be a probability space.

\(H\) - valued random elements \(x\) and \(y\) with \(E\|x\|^2 < +\infty, E\|y\|^2 < +\infty\) are called orthogonal if
\[E[x, z_1](y, z_2) = 0.\]

Let us note that mutual correlation operator of orthogonal elements is the zero operator and that orthogonality of mutually Gaussian elements implies their independence.

An \(H\) - valued random process \(\{x(n) : n \in \mathcal{Z}\}\) is called periodic of period \(\tau, \tau \in \mathcal{N}\), if
\[E[x(n + \tau)] = E[x(n)],\]
where \(\mathcal{B}(H)\) is the \(\sigma\)-field of Borel subsets of \(H\). A process which is periodic of period \(\tau = 1\) is called stationary.

Let \(\{x(n) : n \in \mathcal{Z}\}\) be an \(H\) - valued process with \(E\|x(n)\|^2 < +\infty\) for each \(n \in \mathcal{Z}\). Let \(T_{m,n}\) be the operator of mutual correlation between elements \(x(m)\) and \(x(n)\). The process \(\{x(n) : n \in \mathcal{Z}\}\) is called periodic in the wide sense of period \(\tau, \tau \in \mathcal{N}\), if
\[E(x(n + \tau)) = E(x(n)).\]
(where the expectation is Bochner integral with respect to the measure \( P \)), and

\[
\forall \{m, n\} \subset \mathcal{Z} : \quad T_{m+n, n+r} = T_{m, n}.
\]

A periodic in the wide sense process of period \( \tau = 1 \) is called **stationary in the wide sense.**

The process \( \{\xi(n) : n \in \mathcal{Z}\} \) is **Gaussian** if for all \( n \in \mathcal{N} \) and \( \{m_1, \ldots, m_n\} \subset \mathcal{Z}, m_i \neq m_j, i \neq j \) the vector \( [\xi(m_1), \ldots, \xi(m_n)] \) is a Gaussian \( H^n \)-valued random element.

The following proposition is proved with elementary techniques.

**Lemma 1.1.** An \( H \)-valued random process \( \{x(n) : n \in \mathcal{Z}\} \) is periodic of period \( \tau \) iff the \( H^\tau \)-valued process

\[
\{[x(\nu\tau), x(\nu\tau + 1), \ldots, x(\nu\tau + \tau - 1)] : \nu \in \mathcal{Z}\}
\]

is stationary.

### 3. EXISTENCE OF STATIONARY SOLUTIONS

**Theorem 3.1.** Let \( A \in \mathcal{L}(H) \). The equation

\[
x(n + 1) = Ax(n) + e(n), \quad n \in \mathcal{Z}
\]

has stationary in the wide sense solution \( \{x(n) : n \in \mathcal{Z}\} \) for every sequence \( \{e(n) : n \in \mathcal{Z}\} \) of pairwise orthogonal \( H \)-valued random elements such that for every \( n \in \mathcal{Z} \)

\[
Ee(n) = 0;
\]

\[
E[(e(n), z_1)(e(n), z_2)] = E[(e(0), z_1)(e(0), z_2)] =: (S_e z_1, z_2),
\]

iff for every orthonormal basis \( \{e_i : i \geq 1\} \) in \( H \) the following inequality is valid

\[
\sup_{i \geq 1} \sum_{k=0}^{\infty} \|A^k e_i\|^2 < +\infty.
\]

**Proof.** Necessity. Let \( \{x(n) : n \in \mathcal{Z}\} \) be the stationary in the wide sense solution of the equation (3.1) corresponding to the process \( \{e(n) : n \in \mathcal{Z}\} \) with correlation operator \( S_e \). As in the book by M. Arato [1] it is checked that the correlation operator \( S_x \) of the element \( x(0) \) satisfies the equality

\[
S_x = AS_x A^* + S_e,
\]

where \( A^* \) is the adjoint operator of \( A \). From (3.2), for every \( n \in \mathcal{N} \), we have

\[
S_x = A^{n+1} S_x A^{*(n+1)} + \sum_{k=0}^{n} A^k S_e A^{*k}.
\]

Hence, taking into account the properties of the correlation operators, the convergence of the following series is obtained

\[
\sum_{k=0}^{\infty} (A^k S_e A^{*k} z, z), \quad z \in H.
\]

By the condition of the theorem, this series is convergent for an arbitrary correlation operator \( S_e \). Thus

\[
(A^{n+1} S_x A^{*(n+1)} z, z) \to 0, \quad n \to \infty; \quad z \in H,
\]

and from (3.3) the following equality is obtained

\[
(S_x z, z) = \sum_{k=0}^{\infty} (A^k S_e A^{*k} z, z), \quad z \in H.
\]
Now let \( \{e_i : i \geq 1\} \) be an arbitrary orthonormal basis in \( H \). For every sequence of positive numbers such that
\[
\sum_{i=1}^{\infty} \lambda_i < +\infty
\]  
(3.5)
the operator
\[
S_\varepsilon := \sum_{i=1}^{\infty} \lambda_i \langle \cdot, e_i \rangle e_i
\]
is a correlation operator. For the corresponding operator \( S_x \) the equality (3.4) is true which implies the following equality for the trace of the operator \( S_x \)
\[
\text{tr } S_x = \sum_{i=1}^{\infty} (S_x e_i, e_i) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} (A^k S_x A^{*k} e_i, e_i) = \sum_{i=1}^{\infty} \lambda_i \left( \sum_{k=0}^{\infty} \|A^k e_i\|^2 \right).
\]
From the convergence of the last series for every sequence of positive numbers \( \{\lambda_i : i \geq 1\} \) satisfying (3.5), it follows that
\[
\sup_{j \geq 1} \sum_{k=0}^{\infty} \|A^k e_j\|^2 < +\infty.
\]

Sufficiency. Fix arbitrary \( n \in \mathbb{Z} \). Consider the following sequence of random elements in \( H \)
\[
\xi(m) := \sum_{j=0}^{m} A^j \varepsilon(n-1-j), \quad m \geq 1.
\]
The element \( \xi(m) \) has correlation operator
\[
T(m) = \sum_{j=0}^{m} A^j S_\varepsilon A^{*j}; \quad m \geq 1
\]
and for \( 1 \leq m < k \) the following equality holds
\[
E\|\xi(k) - \xi(m)\|^2 = \text{tr} \left( \sum_{j=m+1}^{k} A^j S_\varepsilon A^{*j} \right).
\]
(3.6)
For the orthonormal basis \( \{e_i : i \geq 1\} \) in \( H \) consisting of the eigenvectors of the operator \( S_\varepsilon \), we obtain from the representation (3.6)
\[
E\|\xi(k) - \xi(m)\|^2 = \sum_{i=1}^{\infty} \lambda_i \sum_{j=m+1}^{k} \|A^j e_i\|^2,
\]
where \( \{\lambda_i : i \geq 1\} \) are eigenvalues of the operator \( S_\varepsilon \), corresponding to the eigenvectors \( \{e_i, i \geq 1\} \). By the condition of the theorem, for every \( t \geq 1 \),
\[
\sum_{j=m+1}^{\infty} \|A^j e_t\|^2 \to 0, \quad m \to \infty.
\]
Furthermore, for \( m \geq 1 \)
\[
\sum_{j=m+1}^{\infty} \|A^j e_t\|^2 \leq \sup_{t \geq 1} \sum_{j=0}^{\infty} \|A^j e_t\|^2 < +\infty.
\]
Hence, the sequence \( \{x(m) : m \geq 1\} \) is Cauchy in the quadratic mean and there is a random element \( x(n) \) such that
\[
E\|x(n)\|^2 < +\infty, \quad x(n) = \sum_{j=0}^{\infty} A^j \varepsilon(n-j-1).
\]
The last series is convergent in the quadratic mean, with the correlation operator \( S_x \) of the element \( x(n) \) being
\[
S_x = \sum_{j=0}^{\infty} A^j S_x A^j,
\]
the series in (3.7) converging in the trace norm. Also for every \( n \in \mathbb{Z} \)
\[
E x(n) = 0,
\]
and the mutual correlation operator \( S_{x,n} \) of the elements \( x(n) \) and \( x(n) \) has the following representation
\[
(S_{x,n} z_1, z_2) = \sum_{j=0}^{\infty} (A^{n-k+j} S_x A^j z_1, z_2), \quad \{z_1, z_2\} \subset H.
\]
Hence,
\[
S_{x,n} = A^{n-k} S_x, \quad n \in \mathbb{Z}, k \in \mathbb{Z}, n \geq k.
\]
Thus, the process \( \{x(n) : n \in \mathbb{Z}\} \) is stationary in the wide sense in \( H \). Moreover, since \( A \in \mathcal{L}(H) \), with probability 1,
\[
Ax(n) + \varepsilon(n) = \sum_{j=0}^{\infty} A^{j+1} \varepsilon(n-j-1) + \varepsilon(n) = x(n+1)
\]
for every \( n \in \mathbb{Z} \). Theorem 3.1 is proved.

**Corollary 3.1.** Let \( \{\varepsilon(n) : n \in \mathbb{Z}\} \) be a sequence of Gaussian i.i.d. random elements in \( H \) with expectation 0 and correlation operator \( S_x \). Let the operator \( A \in \mathcal{L}(H) \) satisfy conditions of Theorem 3.1. Then there exists a unique stationary Gaussian Markov process \( \{x(n) : n \in \mathbb{Z}\} \) satisfying equation (3.1) with operators (3.8) as the correlation operators.

**Remark.** Let us mention simple sufficient conditions ensuring the conditions of Theorem 3.1.
1. If the spectral radius of the operator \( A \)
\[
r(A) := \lim_{n \to \infty} \|A^n\|^{1/n} < 1,
\]
then conditions of Theorem 3.1 are satisfied. Indeed,
\[
\sup_{\|e_1\| = 1} \sum_{k=0}^{\infty} \|A^k e_1\|^2 \leq \sum_{k=0}^{\infty} \|A^k\|^2
\]
with the series in the right converging by Cauchy’s theorem.
2. If \( \|A\| < 1 \) then conditions of Theorem 3.1 are satisfied.

Let us also mention uniqueness of solution of equation (2.1) under conditions of Theorem 3.1.

**4. Existence of Periodic Solutions**

Let \( \tau \in \mathcal{N} \) and \( \{A(n) : n \in \mathbb{Z}\} \subset \mathcal{L}(H) \) be such that
\[
\forall n \in \mathbb{Z} : \quad A(n + \tau) = A(n).
\]
Solutions of stochastic difference equations

Set

\[ B := A(\tau - 1)A(\tau - 2) \cdots A(1)A(0). \]

**Theorem 4.1.** Equation

\[ x(n + 1) = A(n)x(n) + \varepsilon(n), \quad n \in \mathbb{Z} \]

has periodic in the wide sense solution \( \{x(n) : n \in \mathbb{Z}\} \) of period \( \tau \) for every process \( \{\varepsilon(n) : n \in \mathbb{Z}\} \) which is periodic in the wide sense of period \( \tau \) and has orthogonal values with \( \mathbb{E}x(n) = 0, n \in \mathbb{Z} \), iff for every orthonormal basis \( \{e_i : i \geq 1\} \) in \( H \) the following inequality holds

\[
\sup_{i \geq 1} \sum_{k=0}^{\infty} \|B^k e_i\|^2 < +\infty. \tag{4.1}
\]

**Proof.** Necessity. It is easy to verify that for a process \( \{x(n) : n \in \mathbb{Z}\} \) in \( H \) which is periodic in the wide sense of period \( \tau \) for every \( i, 0 \leq i \leq \tau - 1 \) the process \( \{x(\nu \tau + i) : \nu \in \mathbb{Z}\} \) is stationary in the wide sense. Moreover, if the process \( \{x(n) : n \in \mathbb{Z}\} \) satisfies equation (4.1) then

\[ x((\nu + 1)\tau) = Bx(\nu \tau) + \xi(\nu), \quad \nu \in \mathbb{Z}, \tag{4.2} \]

where

\[
\xi(\nu) := \sum_{t=0}^{\tau-1} C_t \varepsilon((\nu + 1)\tau - 1 - t), \quad \nu \in \mathbb{Z};
\]

\[ C_0 := I, \quad C_t := A(\tau - 1) \cdots A(\tau - t), \quad 1 \leq t \leq \tau - 1. \]

Elements \( \{\xi(\nu) : \nu \in \mathbb{Z}\} \) are pairwise orthogonal and have expectation 0 and the same correlation operator

\[
\sum_{t=0}^{\tau-1} C_t S_{\tau-t-1} C^*_t, \tag{4.3}
\]

where \( S_{\tau-t} := S_{\tau k}, 0 \leq k \leq \tau - 1 \), and \( S_{\tau 0}, S_{\tau 1}, \ldots, S_{\tau \tau - 1} \) are the correlation operators of the elements \( \varepsilon(0), \varepsilon(1), \ldots, \varepsilon(\tau - 1) \), respectively. By Theorem 3.1, for every orthonormal basis \( \{e_i : i \geq 1\} \) in \( H \), the following inequality holds

\[
\sup_{i \geq 1} \sum_{k=0}^{\infty} \|B^k e_i\|^2 < +\infty.
\]

Sufficiency. Let \( \tau > 1 \). By Theorem 3.1 there is stationary in the wide sense solution \( \{x_0(\nu) : \nu \in \mathbb{Z}\} \) of the equation

\[ x_0(\nu + 1) = Bx_0(\nu) + \xi(\nu), \quad \nu \in \mathbb{Z}, \]

which has the following representation

\[ x_0(\nu) = \sum_{j=0}^{\infty} B^j \xi(\nu - j - 1), \quad \nu \in \mathbb{Z}. \]

The series for \( x_0(\nu) \) is convergent in the quadratic mean. Now define the process \( \{x(n) : n \in \mathbb{Z}\} \) in \( H \) in the following way

\[
x(\nu \tau) := x_0(\nu),
\]

\[
x(\nu \tau + 1) := A(\nu \tau)x(\nu \tau) + \varepsilon(\nu \tau) = A(0)x_0(\nu) + \varepsilon(\nu \tau),
\]

\[
x(\nu \tau + 2) := A(1)A(0)x_0(\nu) + A(1)\varepsilon(\nu \tau) + \varepsilon(\nu \tau + 1),
\]

\[ \ldots \]

\[
x(\nu \tau + i) := A(i - 1)A(i - 2) \cdots A(0)x_0(\nu) + \sum_{j=0}^{i-2} A(i - 1) \cdots A(j + 1) \varepsilon(\nu \tau + j) + \varepsilon(\nu \tau + i - 1), \quad 2 \leq i \leq \tau - 1.
\]
Each of the processes \( \{x(\nu r + i) : \nu \in \mathbb{Z} \}, 0 \leq i \leq r - 1 \) is stationary in the wide sense and the process \( \{x(n) : n \in \mathbb{Z} \} \) is periodic in the wide sense of period \( r \). This process satisfies
\[
x(n + 1) = A(n)x(n) + \varepsilon(n), \quad n \in \mathbb{Z}.
\]
For the proof it is only sufficient to prove equalities
\[
x(\nu r + r) = A(\nu r + r - 1)x(\nu r + r - 1) + \varepsilon(\nu r + r - 1),
\]
For fixed \( \nu \in \mathbb{Z} \) we have
\[
A(\nu r + r - 1)x(\nu r + r - 1) + \varepsilon(\nu r + r - 1) =
\]
\[
= A(r - 1)A(r - 2) \cdots A(0)x_0(\nu) + \sum_{j=0}^{r-3} A(r - 1) \cdots A(j + 1)\varepsilon(\nu r + j) +
\]
\[
+ A(r - 1)\varepsilon(\nu r + r - 2) + \varepsilon(\nu r + r - 1) =
\]
\[
= Bx_0(\nu) + \xi(\nu) = x_0(\nu + 1).
\]
Theorem 4.1 is proved.

Corollary 4.1. Let \( \{\varepsilon(n) : n \in \mathbb{Z} \} \) be a Gaussian periodical process in \( H \) of period \( r \) with independent values and \( E\varepsilon(n) = 0, n \in \mathbb{Z} \). Let \( \{A(n) : n \in \mathbb{Z} \} \subset \mathcal{L}(H) \),
\[
A(n + r) = A(n), \quad n \in \mathbb{Z}; \quad B = A(r - 1)A(r - 2) \cdots A(0),
\]
and let the operator \( B \) satisfy conditions of Theorem 4.1. Then there is a unique periodical Gaussian Markov process of period \( r \) satisfying
\[
x(n + 1) = A(n)x(n) + \varepsilon(n), \quad n \in \mathbb{Z}.
\] (4.4)

Theorem 4.1 admits the following generalization. Let \( p \in \mathcal{N} \cup \{0\} \) and \( \mathcal{P} \) be the class of all \( H \) - valued periodical in the wide sense sequences \( \{\varepsilon(n) : n \in \mathbb{Z} \} \) of period \( r \) such that \( E\varepsilon(n) = 0, n \in \mathbb{Z} \) and random elements \( \varepsilon(k) \) and \( \varepsilon(n) \) are orthogonal if \( |k - n| > p \).

Theorem 4.2. Let \( \{A(n) : n \in \mathbb{Z} \} \) and \( B \) be the operators in Corollary 4.1. Equation (4.4) has periodical in the wide sense solution \( \{x(n) : n \in \mathbb{Z} \} \) of period \( r \) for every process \( \{\varepsilon(n) : n \in \mathbb{Z} \} \) from class \( \mathcal{P} \), iff for every orthonormal basis \( \{e_i, i \geq 1\} \) in \( H \) inequality (4.1) holds.

The proof of Theorem 4.2 is similar to the proof of Theorem 4.1 and so it is omitted.

5. A SUFFICIENT CONDITION

Let \( \{A(n) : n \in \mathbb{Z} \} \) and \( B \) be the operators in Corollary 4.1 where the spectrum \( \sigma(B) \) of the operator \( B \) consists of two parts \( \sigma_- \) and \( \sigma_+ \) such that
\[
\sup(|\lambda| : \lambda \in \sigma_-) = r_- < 1,
\]
\[
\inf(|\lambda| : \lambda \in \sigma_+) = r_+ > 1.
\]
Theorem 5.1. Let operators \( \{A(n) : n \in \mathbb{Z} \} \) and \( B \) be as stated above and \( \{\varepsilon(n) : n \in \mathbb{Z} \} \) be a periodical process in \( H \) of period \( r \) such that
\[
E||\varepsilon(n)|| < +\infty, \quad E\varepsilon(n) = 0, \quad n \in \mathbb{Z}.
\]
Then there is a unique periodical process \( \{x(n) : n \in \mathbb{Z} \} \) of period \( r \) satisfying equation (4.4).

Proof. Use Lemma 1.1 and notation for \( \xi(\nu), \nu \in \mathbb{Z} \) of part 4. The process \( \{\xi(\nu) : \nu \in \mathbb{Z} \} \) is stationary and
\[
E||\xi(0)|| < +\infty, \quad E\xi(0) = 0.
\]
Let $P_-$ and $P_+$ be the projectors corresponding to spectral sets $\sigma_-$ and $\sigma_+$ (see [4]). We have

$$BP_- = P_- B, \quad BP_+ = P_+ B, \quad P_- + P_+ = I, \quad P_- P_+ = 0,$$

$$\sum_{j=-\infty}^{+\infty} \| (BP_-)^j \| < +\infty, \quad \sum_{j=-\infty}^{-1} \| (BP_+)^j \| < +\infty,$$

where $I$ and $0$ are identity and zero operators in $H$. Under the assumptions of Theorem 5.1, the process in $H$:

$$z_0(\nu) := \sum_{j=0}^{+\infty} (BP_-)^j \xi(\nu - 1 - j) - \sum_{j=-\infty}^{-1} (BP_+)^j \xi(\nu - 1 - j), \quad \nu \in \mathbb{Z},$$

(where $(BP_-)^0 := P_-$, $(BP_+)^0 := P_+$ and both the series are convergent in $H$ with probability 1) is stationary and

$$E\|z_0(0)\| < +\infty, \quad Ez_0(0) = 0.$$

Moreover, the following equality holds:

$$z_0(\nu + 1) = Bz_0(\nu) + \xi(\nu), \quad \nu \in \mathbb{Z}.$$

Further construction of the process $\{x(n) : n \in \mathbb{Z}\}$ in the statement of Theorem 5.1 essentially repeats the proof of sufficiency in Theorem 4.1 and so it is omitted. Theorem 5.1 is proved.

REFERENCES