Two-dimensional flow of a nonstationary micropolar fluid in the half-plane for which the shear stresses are given on the boundary

Ibrahim H. EL-SIRAFY
Department of Mathematics, University of Alexandria, Egypt

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Abstract: The object of this paper is to generalize the results of the solution of the homogeneous Navier-Stokes equations in the half-plane for the slow motion of viscous incompressible fluids to the class of the micropolar fluids for the case of the given shear stresses on the boundary. The no-spin boundary conditions for the microrotation vector has been used. Using the method of Laplace-Fourier transforms the solution is obtained by quadratures.

1. Introduction

The theory of micropolar fluids, advanced by Eringen [3] deals with a class of fluids which can support couple stresses and body couples and exhibit microrotational effects and microrotational inertia. This theory might serve as a satisfactory model for describing the flow properties of polymeric fluids, liquid crystals, fluids with certain additives and animal blood.

In a previous paper [2] Chernous and El-Sirafy solved some boundary value problems of slow flow of viscous incompressible fluid in the half-plane. One of these problems is the case when the tangential stresses and normal velocities are given on the boundary but the initial velocities vanish. This boundary value problem could be met in the study of the flow near a solid boundary which is not wet by the given fluid.

The object of this paper is to generalize the results of the mentioned problem of Chernous and El-Sirafy [2] to the class of the micropolar fluids for the case of the given shear stresses on the boundary \( y = 0 \). The no-spin boundary condition for the microrotation vector has been used. Also we consider the initial microrotation is zero. Using the method of Laplace-Fourier transforms, the solution is obtained by quadratures.

2. Formulation of the problem

The equations of slow motion of an incompressible micropolar fluid in the absence of both external forces and body forces [3] are

\[ \nabla \cdot \mathbf{v} = 0, \]  

(1)
\[ \rho \frac{\partial v}{\partial t} = - (\mu + k) \nabla \cdot v + k \nabla \times v - \nabla p, \quad (2) \]
\[ j \rho \frac{\partial \omega}{\partial t} = (a + \beta + \gamma) \nabla \cdot \omega - \gamma \nabla \times v + k \nabla \times v - 2k \nu, \quad (3) \]

where \( v, \omega \) and \( p \) are respectively the velocity, microrotation vectors and pressure. The symbols \( \rho \) and \( j \) denote the density and microinertia and \( (a, \beta, \gamma, \mu, k) \) are viscosity coefficients.

We consider the two-dimensional motion of a micropolar fluid in the half-plane \((-\infty < x < \infty, y > 0 | t > 0)\) and seek to determine the velocity, pressure and microrotation when the shear stresses \( xy \) and \( yx \) are given on the boundary under the condition that \( xy/xy = 1 + k/\mu \). The microrotation vanishes on the boundary. Initially the velocities and microrotation vanish. In this case

\[ v = (u, v, 0), \quad \nu = (0, 0, \nu). \]

The system \((1)-(3)\) becomes

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4) \]
\[ \rho \left( \frac{\partial w}{\partial t} \right) = (\mu + k) \nabla^2 w - 2(\partial p + i k \nu)/\partial \bar{z}, \quad (5) \]
\[ \rho j \left( \frac{\partial \omega}{\partial t} \right) = \gamma \nabla^2 \nu - 2k \nu + k(\partial v/\partial x - \partial u/\partial y), \quad (6) \]

where

\[ z = x + i y \quad \text{and} \quad w = u + i v. \]

The stresses and couple stresses are given by

\[
\begin{align*}
\bar{y}z = \bar{z}y = \bar{x}z = \bar{z}x = 0, \\
\bar{z}z = -p, \\
\bar{x}x + \bar{y}y &= -2p, \\
\bar{x}x - \bar{y}y &= 2(2\mu + k)(\partial u/\partial x), \\
yx &= (\mu + k)(\partial u/\partial y) + kv + \mu(\partial v/\partial x), \\
xy &= \mu(\partial u/\partial y) - kv + (\mu + k)(\partial v/\partial x)
\end{align*}
\]

and

\[
\begin{align*}
\frac{m_{yz}}{\gamma} = \frac{m_{zy}}{\beta} = \frac{\partial v}{\partial y}, \\
\frac{m_{xz}}{\gamma} = \frac{m_{zx}}{\beta} = \frac{\partial \nu}{\partial x}, \\
m_{xx} = m_{yy} = m_{zz} = m_{xy} = m_{yx} = 0.
\end{align*}
\]

3. Method of solution

From the equation of continuity \((4)\) we can write the function \( w \) in terms of two real functions \( \phi \) and \( \psi \) in the form

\[ w = 2\partial (\phi - i \psi)/\partial \bar{z}, \quad (9) \]

where

\[ \nabla^2 \phi = 0. \quad (10) \]
Inserting (9)-(10) in the system (4)-(6) we have
\[
(\mu + k) \nabla^2 \psi - \rho (\partial \psi / \partial t) = k \nu = 0, \tag{11}
\]
\[
\gamma \nabla^2 \nu - 2k \nu - k \nabla^2 \psi = \rho j(\partial \nu / \partial t). \tag{12}
\]
\[
p = -\rho (\partial \phi / \partial t). \tag{13}
\]
For the microrotation, we consider the no-spin boundary condition,
\[
\nu(x, 0, t) = 0. \tag{14}
\]
Here we need to solve the system (10)-(12) subject to the boundary conditions
\[
\partial u / \partial y = \lambda y / \mu = y / (\mu + k) = H(x, t), \quad \partial v / \partial x = 0 \quad \text{on } y = 0 \tag{15}
\]
where \( H(x, t) \) is a given function satisfying the condition
\[
H(x, t) = O(1/x) \quad \text{at } x \rightarrow \infty.
\]
Assuming that
\[
\nu(x, 0, t) = 0 \tag{16}
\]
and using (9), the boundary conditions now become
\[
\psi_{yy} + \phi_{xy} |_{y=0} = H(x, t), \quad \phi_y - \psi_x |_{y=0} = 0. \tag{17}
\]
And for the initial conditions we assume that
\[
\phi(x, y, 0) = \psi(x, y, 0) = \nu(x, y, 0) = 0. \tag{18}
\]
Taking the Laplace transform with time \( t \) and the Fourier transform with coordinate \( x \) of the system (10)-(12) and using conditions (14), (17), (18), we obtain
\[
\tilde{\phi}'' - q^2 \tilde{\phi} = 0,
\]
\[
(\mu + k) \tilde{\psi}'' - \left\{ (\mu + k) q^2 + \rho \lambda \right\} \tilde{\psi} + k \tilde{\nu} = 0, \tag{19}
\]
\[
\gamma \tilde{\nu}'' - (\gamma q^2 + 2k + \rho j \lambda) \tilde{\nu} - k \tilde{\psi}'' + k q^2 \tilde{\psi} = 0,
\]
\[
\tilde{\psi}'' + i q \tilde{\phi}' |_{y=0} = \tilde{H}, \quad \tilde{\phi}' - i q \tilde{\psi} |_{y=0} = 0, \tag{20}
\]
in which \( \tilde{\psi}(q, y, \lambda) \) is related to \( \psi(x, y, t) \) by
\[
\tilde{\psi}(q, y, \lambda) = \frac{1}{2\pi} \int_{0}^{\infty} e^{-\lambda t} \int_{-\infty}^{\infty} e^{-i q x} \psi(x, y, t) \, dx \, dt,
\]
\[
\psi(x, y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} \int_{-\infty}^{\infty} e^{i q x} \tilde{\psi}(q, y, \lambda) \, dq \, d\lambda, \quad c = \Re \lambda > 0,
\]
where accents denote differentiations with respect to \( y \) and similar relations could be written for the functions \( \tilde{\phi}, \phi, \tilde{\psi}, \psi \) respectively.

Now we may take as a trial solution
\[
\tilde{\psi} = A e^{-\alpha y}, \quad \tilde{\nu} = B e^{-\alpha y}.
\]
By substituting in (19) we get
\[
[(\mu + k)(r^2 - q^2) - \rho \lambda] A +kB = 0,
-k(r^2 - q^2) A + [\gamma(r^2 - q^2) - (2k + \rho j \lambda)] B = 0.
\]

Then the condition for a nonzero solution is
\[
\gamma(r^2 - q^2)^2 - \left(2k + \rho j \lambda + \frac{\rho \lambda \gamma - k^2}{\mu + k}\right)(r^2 - q^2) + \rho \lambda \frac{2k + \rho j \lambda}{\mu + k} = 0.
\]

Here we confine our study to a special class of fluids in which the microinertia coefficient \( j \) is given by
\[
j = 2\gamma/(2\mu + k).
\]

Thus after some calculations we obtain the solution of the system (19) in the form
\[
\tilde{\phi} (q, y, \lambda) = -i \frac{\mu + k}{\rho \lambda} \text{sign} \ q \cdot \tilde{H}(q, \lambda) \ e^{-i\gamma \gamma},
\]
\[
\tilde{\psi} (q, y, \lambda) = (\mu + k) \left[ \frac{\tilde{H}(q, \lambda)}{\rho \lambda} e^{-\gamma \gamma} - \frac{\gamma}{(2\mu + k)^2} \tilde{v}(q, y, \lambda) \right],
\]
\[
\tilde{v}(q, y, \lambda) = \frac{2(\mu + k)^2 \tilde{H}(q, \lambda)}{\gamma} \left( e^{-\gamma \gamma} - e^{-\gamma \gamma} \right),
\]

where
\[
r_1^2 = q^2 + a\rho \lambda, \quad r_2^2 = q^2 + b\rho \lambda + c,
\]
\[
a = 2/(2\mu + k), \quad b = 1/(\mu + k), \quad c = k(2\mu + k)/\gamma(\mu + k).
\]

Inverting the system (21) we get
\[
\phi(x, y, t) = \frac{\mu + k}{\rho \pi} \int_0^t d\tau \int_{-\infty}^{\infty} H(\xi, \tau) \frac{x - \xi}{(x - \xi)^2 + y^2} d\xi,
\]
\[
\psi(x, y, t) = -\frac{\gamma (\mu + k)}{(2\mu + k)^2} \nu(x, y, t)
\]
\[
+ \frac{\mu + k}{\rho \pi} y \int_0^t d\tau \int_{-\infty}^{\infty} \frac{H(\xi, \tau)}{(z - \xi)^2} e^{-a\rho \lambda |z - \xi|^2/4(z - \xi)^2} d\xi,
\]
\[
\nu(x, y, t) = \int_0^t d\tau \int_{-\infty}^{\infty} H(\xi, \tau) S(x - \xi, y, t - \tau) d\xi,
\]

where
\[
S(x, y, t) = \frac{c}{2\pi^2 i} \int_{a - i\infty}^{a + i\infty} e^{\lambda t} \frac{e^{\lambda t}}{(a - b) \rho \lambda - c} \int_0^\infty (e^{-\gamma \gamma} - e^{-\gamma \gamma}) \cos qx dq d\lambda.
\]
For the determination of \( S(x, y, t) \) we notice that (see [1, 1.4 (25)])

\[
\int_0^\infty e^{-\sqrt{x^2+y^2}\cos qx} dq = \beta y (x^2 + y^2)^{-1/2} K_1[\beta (x^2 + y^2)^{1/2}],
\]

\( \text{Re } \beta > 0, \quad y > 0, \)

where \( K_1(z) \) is the MacDonald function of first order of the argument \( z \). Thus with the aid of the convolution theorem, we get (see [1, 5.16 (37)])

\[
S(x, y, t) = \frac{c y}{4\pi(a-b)} e^{ct/(a-b)} [aE(x, y, t; a, c) - bE(x, y, t; b, ac/b)], \tag{25}
\]

where

\[
E(x, y, t; m, n) = \int_1^\infty e^{-pm|z|^2/4-n/(a-b)} d\lambda
\]

\[
= \frac{1}{t} \int_0^\infty e^{-pm|z|^2(1+\lambda)/4t-n(1+\lambda)^{-1}/(a-b)} d\lambda. \tag{26}
\]

The function \( E(x, y, t; m, n) \) can be expanded in the form (see [1, 4.2 (9)])

\[
E(x, y, t; m, n) = \frac{4e^{-pm|z|^2/4}}{\rho m|z|^2} \left[ 1 - \sum_{\nu=2}^\infty \sum_{r=1}^{\nu-1} \frac{\nu!(\nu-1)!2^{r-\nu}r^r(a-b)^r}{r!^\nu r!(\nu-1)!} \right] e^{-\rho m|z|^2/4t}
\]

\[
+ \frac{4}{\rho m|z|^2} \text{Ei} \left( -\frac{\rho m|z|^2/4t}{\rho m|z|^2} \right) \sum_{\nu=1}^\infty \frac{m^n n^r|z|^{2r}}{\nu!(\nu-1)!2^{2r}r^r(a-b)^r}, \tag{27}
\]

where

\[
\text{Ei}(-z) = -\int_z^\infty \frac{e^{-\lambda}}{\lambda} d\lambda, \quad |\text{arg } z| < \pi \tag{28}
\]

is the exponential integral function.

Using (9), (13), (22), and (23) we obtain the pressure \( p(x, y, t) \) and complex velocity \( w(x, y, t) \) in the forms

\[
p(x, y, t) = -\frac{\mu + k}{\pi} \int_{-\infty}^{\infty} H(\xi, t) \frac{x-\xi}{(x-\xi)^2 + y^2} d\xi, \tag{29}
\]

\[
\overline{w(x, y, t)} = \int_0^t d\tau \int_{-\infty}^{\infty} H(\xi, \tau) W(x-\xi, y, t-\tau) d\xi, \tag{30}
\]

where

\[
W(x, y, t) = \frac{\mu + k}{\rho \pi z^2} \left[ \left( 1 - \frac{i \rho y z}{2 t} \right) e^{-\rho (x^2+y^2)/4t} - 1 \right] - \frac{2i \gamma (\mu + k)}{(2 \mu + k)^2} \frac{\partial}{\partial z} S(x, y, t). \tag{31}
\]

In the limiting case as \( k, \gamma \to 0 \), the microrotation vanishes and we obtain the previous known
classical solution given by Chernous and El-Sirafy [2], namely

\[
p(x, y, t) = -\frac{\mu}{\pi}\int_{-\infty}^{\infty} H(\xi, t) \frac{x-\xi}{(x-\xi)^2 + y^2} d\xi.
\]

\[
\overline{w(x, y, t)} = \frac{\mu}{\rho\pi} \int_0^t d\tau \int_{-\infty}^{\infty} H(\xi, \tau) \left[1 - \frac{i\alpha y(z-\xi)}{2(t-\tau)}\right] e^{-\alpha(z-\xi)^2/(2(t-\tau)) - 1} d\xi.
\]

References

