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Journal of Pure and Applied Algebra 212 (2008) 1072–1085

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Formally smooth bimodules

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Received 22 January 2007; received in revised form 24 August 2007; accepted 1 September 2007
Available online 31 October 2007

Communicated by S. Donkin

Abstract

The notion of a formally smooth bimodule is introduced and its basic properties are analyzed. In particular it is proven that a B–A bimodule M which is a generator left B-module is formally smooth if and only if the M-Hochschild dimension of B is at most one. It is also shown that modules M which are generators in the category $\sigma[M]$ of M-subgenerated modules provide natural examples of formally smooth bimodules.

MSC: Primary: 16D20; secondary: 14A22

1. Introduction

The notion of a (formally) smooth algebra was introduced in [15]. It has been recognized in [6] that smooth (or quasi-free) algebras can be interpreted as functions on non-commutative nonsingular (smooth) affine varieties or as analogues of manifolds in non-commutative geometry. This point of view was then developed further in [10], where an approach to *smooth non-commutative geometry* was outlined. In [11] this gave rise to the introduction of formally smooth objects, morphisms and functors as main building blocks of non-commutative algebraic geometry. Following on, the non-commutative geometric aspects of smooth algebras (or, more generally, R-*rings* or *smooth algebra extensions*) such as tangent and cotangent bundles or symplectic structures were discussed in [7] (cf. [17]), in the framework of *double derivations*. A general algebraic approach to formal smoothness in monoidal abelian categories, including the cohomological aspects, was recently proposed in [2,3].

The aim of this paper is to find a common ground for the notions of formal smoothness which have attracted so much attention in recent literature. The basic idea for this goes back to [16], where it is observed that properties of an extension of algebras, such as separability, can be encoded more generally as properties of bimodules rather than algebra maps. We thus propose the definition of a *formally smooth bimodule*, and show that this notion encodes smooth

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.09.004

algebras and smooth extensions (which can be understood as smooth algebras in the monoidal category of bimodules). Furthermore we show that a smooth bimodule can be interpreted as a smooth object in the sense of [11]. The definition of a smooth bimodule is presented within the framework of relative homological algebra, making specific use of tools recently developed in [2], and, in particular, developing the *module–relative-Hochschild. cohomology*. With these tools we show that separable bimodules can be understood as (non-commutative, relative) "bundles of points" (objects with zero relative-Hochschild dimension), while the formally smooth (generator) bimodules can be viewed as (non-commutative, relative) "bundles of curves" or "line bundles" (objects with relative-Hochschild dimension at most one). On a more module-theoretic side, we show that given a left B-module M with endomorphism ring S, M is a separable B–S bimodule if and only if it is a generator of all left B-modules. On the other hand, if M is a generator in the category $\sigma[M]$ of M-subgenerated left B-modules, then M is a formally smooth B–S bimodule.

Module-theoretic conventions. By a ring we mean a unital associative ring. ${}_B\mathcal{M}$, $\mathcal{M}_A$, ${}_B\mathcal{M}_A$ denote categories of (unital) left B-modules, right A-modules and B–A bimodules. Morphisms in these categories are respectively denoted by ${}_B\mathrm{Hom}(-,-)$, $\mathrm{Hom}_A(-,-)$ and ${}_B\mathrm{Hom}_A(-,-)$. For a B–A bimodule M we often write ${}_BM$, M_A, ${}_BM_A$ to indicate the ring and module structures used. The arguments of left B-module maps are always written on the left. This induces a composition convention for the endomorphism ring $S := {}_B\mathrm{End}(M)$ of ${}_BM$, which makes M a B–S bimodule. Given bimodules ${}_BM_A$ and ${}_BN_T$, we view the abelian group ${}_B\mathrm{Hom}(M,N)$ as an A–T bimodule with multiplications defined by

$$(m)(aft) := (ma)ft, \quad \text{for all } f \in {}_B\mathrm{Hom}(M,N),\ a \in A,\ m \in M,\ t \in T.$$

For a B–A bimodule M, ${}^*M$ denotes the dual A–B bimodule ${}_B\mathrm{Hom}(M,B)$.

2. Relative projectivity and separable functors

2.1. Relative projectivity and injectivity

A convenient description and conceptual interpretation of formally smooth or separable bimodules is provided by relative cohomology. In this introductory section we recall the basic properties of relative derived functors. Most of the material presented here can be found in [8, Chapter IX].

Let $\mathfrak{C}$ be a category and let $\mathcal{H}$ be a class of morphisms in $\mathfrak{C}$. An object $P \in \mathfrak{C}$ is called *f-projective*, where $f : C_1 \to C_2$ is a morphism in $\mathfrak{C}$, if

$$\mathfrak{C}(P,f) : \mathfrak{C}(P,C_1) \to \mathfrak{C}(P,C_2), \qquad g \mapsto f \circ g$$

is surjective. P is said to be *$\mathcal{H}$-projective* if it is f-projective for every $f \in \mathcal{H}$.

The *closure* $\overline{\mathcal{H}}$ of the class of morphisms $\mathcal{H}$ is defined by

$$\overline{\mathcal{H}} := \{f \in \mathfrak{C} \mid \text{ if an object } P \in \mathfrak{C} \text{ is } \mathcal{H}\text{-projective, then } P \text{ is } f\text{-projective}\}.$$

Obviously, $\overline{\mathcal{H}}$ contains $\mathcal{H}$ as a subclass and $\mathcal{H}$ is said to be *closed* if $\overline{\mathcal{H}} = \mathcal{H}$. A closed class $\mathcal{H}$ is said to be *projective* if, for each object $C \in \mathfrak{C}$, there is a morphism $f : P \to C$ in $\mathcal{H}$ where P is $\mathcal{H}$-projective.

If $\mathfrak{C}$ is an abelian category and $\mathcal{H}$ is a *closed* class of morphisms in $\mathfrak{C}$, then a morphism $f \in \mathfrak{C}$ is called *$\mathcal{H}$-admissible* if in the canonical factorization $f = \mu \circ \xi$, where μ is a monomorphism and ξ is an epimorphism, ξ is an element of $\mathcal{H}$. An exact sequence in $\mathfrak{C}$ is called *$\mathcal{H}$-exact* if all its morphisms are $\mathcal{H}$-admissible. Finally, an *$\mathcal{H}$-projective resolution* of an object $C \in \mathfrak{C}$ is an $\mathcal{H}$-exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} C \longrightarrow 0,$$

such that P_n is $\mathcal{H}$-projective, for every $n \in \mathbb{N}$. If $\mathcal{H}$ is a projective class of epimorphisms in $\mathfrak{C}$, then every object in $\mathfrak{C}$ admits an $\mathcal{H}$-projective resolution.

Let $\mathfrak{B}$, $\mathfrak{C}$ be abelian categories and let $\mathcal{H}$ be a projective class of epimorphisms in $\mathfrak{B}$ (so that every object in $\mathfrak{B}$ admits an $\mathcal{H}$-projective resolution). Given a contravariant additive functor $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ and given an $\mathcal{H}$-projective resolution in $\mathfrak{B}$

$$\mathbf{P}_\bullet \longrightarrow B \longrightarrow 0$$

of B, the object $\mathbf{H}^n(\mathbf{T}(\mathbf{P}_\bullet))$ depends only on B and yields an additive functor

$$\mathrm{R}^n_{\mathcal{H}}\mathbf{T} : \mathfrak{B} \to \mathfrak{C}, \qquad \mathrm{R}^n_{\mathcal{H}}\mathbf{T}(B) := \mathbf{H}^n(\mathbf{T}(\mathbf{P}_\bullet)).$$

The functor $\mathrm{R}^n_{\mathcal{H}}\mathbf{T}$ is called the n-th *right $\mathcal{H}$-derived functor* of $\mathbf{T}$.

As in the absolute case, one can show that any short $\mathcal{H}$-exact sequence

$$0 \to B_1 \to B_2 \to B_3 \to 0$$

in $\mathfrak{B}$ yields a long exact sequence

$$0 \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}(B_3) \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}(B_2) \to \mathrm{R}^0_{\mathcal{H}}\mathbf{T}(B_1) \to \mathrm{R}^1_{\mathcal{H}}\mathbf{T}(B_3) \to \cdots$$
$$\cdots \to \mathrm{R}^n_{\mathcal{H}}\mathbf{T}(B_3) \to \mathrm{R}^n_{\mathcal{H}}\mathbf{T}(B_2) \to \mathrm{R}^n_{\mathcal{H}}\mathbf{T}(B_1) \to \mathrm{R}^{n+1}_{\mathcal{H}}\mathbf{T}(B_3) \to \cdots$$

of $\mathcal{H}$-derived functors (cf. [8, Theorem 2.1, page 309]).

Let $\mathfrak{B}, \mathfrak{C}$ be abelian categories and let $\mathcal{H}$ be a projective class of epimorphisms in $\mathfrak{B}$. A contravariant functor $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$ is said to be *left $\mathcal{H}$-exact* if, for every $\mathcal{H}$-exact sequence $B_1 \to B_2 \to B_3 \to 0$, the sequence $0 \to \mathbf{T}(B_3) \to \mathbf{T}(B_2) \to \mathbf{T}(B_1)$ is exact. By [8, pages 311–312] a contravariant left $\mathcal{H}$-exact functor $\mathbf{T} : \mathfrak{B} \to \mathfrak{C}$, is additive and naturally isomorphic to $\mathrm{R}^0_{\mathcal{H}}\mathbf{T}$. Furthermore, $\mathrm{R}^n_{\mathcal{H}}\mathbf{T}(P) = 0$, for every $n > 0$ and for every $\mathcal{H}$-projective object P.

We now provide the main example of a closed projective class that we are interested in.

With any functor $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ we associate the class of $\mathbb{H}$-relatively split morphisms:

$$\mathcal{E}_{\mathbb{H}} := \{f \in \mathfrak{B} \mid \mathbb{H}(f) \text{ splits in } \mathfrak{A}\}.$$

Theorem 2.1 (*[2, Theorem 2.2]*). *Let $(\mathbb{T}, \mathbb{H})$ be an adjunction between the categories $\mathfrak{A}$ and $\mathfrak{B}$, with counit $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$. For any object $P \in \mathfrak{B}$, the following assertions are equivalent:*

(a) *P is $\mathcal{E}_{\mathbb{H}}$-projective.*
(b) *Every morphism $f : B \to P$ in $\mathcal{E}_{\mathbb{H}}$ has a section.*
(c) *The counit $\varepsilon_P : \mathbb{T}\mathbb{H}(P) \to P$ has a section.*
(d) *There is a splitting morphism $\pi : \mathbb{T}(X) \to P$ for a suitable object $X \in \mathfrak{A}$.*

In particular all objects of the form $\mathbb{T}(X)$, $X \in \mathfrak{A}$, are $\mathcal{E}_{\mathbb{H}}$-projective. Moreover $\mathcal{E}_{\mathbb{H}}$ is a closed projective class.

Thus $\mathcal{E}_{\mathbb{H}}$ is a projective class. Note that since, for any object $Y \in \mathfrak{B}$, the morphism $\mathbb{H}(\varepsilon_Y)$ is split by $\eta_{\mathbb{H}(Y)}$, the counit of adjunction ε_Y is in the class $\mathcal{E}_{\mathbb{H}}$. To apply the derived functors one needs to determine when $\mathcal{E}_{\mathbb{H}}$ is a class of epimorphisms (in which case any object in $\mathfrak{B}$ admits an $\mathcal{E}_{\mathbb{H}}$-projective resolution). The necessary and sufficient conditions for this are given in the next proposition, which is the only (mildly) new result in this section.

Proposition 2.2. *Let $(\mathbb{T}, \mathbb{H})$ be an adjunction, where $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ is a covariant functor. Let $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$ be the counit of the adjunction.*

The following assertions are equivalent:

(a) *$\mathcal{E}_{\mathbb{H}}$ is a class of epimorphisms.*
(b) *The counit $\varepsilon_Y : \mathbb{T}\mathbb{H}(Y) \to Y$ is an epimorphism for every $Y \in \mathfrak{B}$.*
(c) *$\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ is faithful.*

Proof. (a) $\Rightarrow$ (b) For all objects $Y \in \mathfrak{B}$, $\varepsilon_Y \in \mathcal{E}_{\mathbb{H}}$. Since $\mathcal{E}_{\mathbb{H}}$ is assumed to be a class of epimorphisms, ε_Y is an epimorphism.

(b) $\Leftrightarrow$ (c) This is a standard description of a right adjoint faithful functor; see e.g. [13, Section 2.12, Proposition 3].

(c) $\Rightarrow$ (a) It follows by the fact that faithful functors reflect epimorphisms. □

By Theorem 2.1, $\mathcal{E}_{\mathbb{H}}$ is always a projective class, and it is a class of epimorphisms, provided the equivalent conditions of Proposition 2.2 hold. In this case any object in $\mathfrak{B}$ admits an $\mathcal{E}_{\mathbb{H}}$-projective resolution which is unique up to a homotopy. Thus, for every $B' \in \mathfrak{B}$, one can consider the right $\mathcal{E}_{\mathbb{H}}$-derived functors $\mathrm{R}^\bullet_{\mathcal{E}_{\mathbb{H}}}\mathbf{F}_{B'}$ of $\mathbf{F}_{B'} := \mathfrak{B}(-, B') : \mathfrak{B} \to \mathfrak{Ab}$. These functors play a special role in what follows.

Definition 2.3. Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian categories. Let $(\mathbb{T}, \mathbb{H})$ be an adjunction, where $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ is a covariant functor. If $\mathcal{E}_{\mathbb{H}}$ is a class of epimorphisms and the functor $\mathbf{F}_{B'} := \mathfrak{B}(-, B')$ is left $\mathcal{E}_{\mathbb{H}}$-exact for every $B' \in \mathfrak{B}$, then for every $B, B' \in \mathfrak{B}$, we set

$$\mathrm{Ext}^{\bullet}_{\mathcal{E}_{\mathbb{H}}}(B, B') = \mathrm{R}^{\bullet}_{\mathcal{E}_{\mathbb{H}}} \mathbf{F}_{B'}(B).$$

The study of relative injectivity can be carried out in a dual way, i.e. working in the opposite category of $\mathfrak{C}$ (note that if a category is abelian, so is its opposite category). In particular, the dual of Theorem 2.1, [2, Theorem 2.3], states that the class of relatively cosplit morphisms,

$$\mathcal{I}_{\mathbb{T}} := \{g \in \mathfrak{A} \mid \mathbb{T}(g) \text{ cosplits in } \mathfrak{B}\},$$

is a closed injective class. Dualizing Proposition 2.2 one concludes that $\mathcal{I}_{\mathbb{T}}$ is a class of monomorphisms iff $\mathbb{T}$ is a faithful functor.

2.2. *Separable functors*

The notion of a separable functor was introduced in [12]. Following the formulation in [14], a covariant functor $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ is said to be *separable* if and only if the transformation $\mathfrak{B}(-, -) \to \mathfrak{A}(\mathbb{H}(-), \mathbb{H}(-))$, $f \mapsto \mathbb{H}(f)$, is a split natural monomorphism.

As explained in [12, Lemma 1.1], any equivalence of categories is separable, and a composition of separable functors is separable. Furthermore if a functor $\mathbb{H} \circ \mathbb{T}$ is separable, then so is $\mathbb{T}$. By [12, Proposition 1.2], a separable functor reflects split monomorphisms and split epimorphisms. This then implies that, for a pair of functors $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ and $\mathbb{H} : \mathfrak{B} \to \mathfrak{C}$, with $\mathbb{H}$ separable, the class of $\mathbb{H} \circ \mathbb{T}$-relatively split morphisms (resp. $\mathbb{H} \circ \mathbb{T}$-relatively cosplit morphisms) is the same as the class of $\mathbb{T}$-relatively split morphisms (resp. $\mathbb{T}$-relatively cosplit morphisms), i.e.

$$\mathcal{E}_{\mathbb{H}\circ\mathbb{T}} = \mathcal{E}_{\mathbb{T}} \quad (\text{ resp. } \mathcal{I}_{\mathbb{H}\circ\mathbb{T}} = \mathcal{I}_{\mathbb{T}}).$$

A particularly useful criterion of separability of a functor with an adjoint is provided by the Rafael theorem:

Theorem 2.4 (*[14, Theorem 1.2]*). *Let $\mathbb{T}$ be a left adjoint of a covariant functor $\mathbb{H}$.*

(1) *$\mathbb{T}$ is separable if and only if the unit of the adjunction is a natural section.*
(2) *$\mathbb{H}$ is separable if and only if the counit of the adjunction is a natural retraction.*

Combining Theorem 2.4 with Theorem 2.1 (and its dual) we obtain

Corollary 2.5. *Let $\mathbb{T} : \mathfrak{A} \to \mathfrak{B}$ be a covariant functor with right adjoint $\mathbb{H}$.*

(1) *If $\mathbb{H}$ separable, then any object in $\mathfrak{B}$ is $\mathcal{E}_{\mathbb{H}}$-projective.*
(2) *If $\mathbb{T}$ separable, then any object in $\mathfrak{A}$ is $\mathcal{I}_{\mathbb{T}}$-injective.*

3. Module–relative-Hochschild cohomology

In this section we introduce and compute (in a special case) the Hochschild cohomology relative to a bimodule. This cohomology is used in the description of separable and formally smooth bimodules.

Let A, B and T be rings. Given a bimodule ${}_BM_A$, consider the following adjunction:

$$\mathbb{L}_T : {}_A\mathcal{M}_T \to {}_B\mathcal{M}_T, \qquad \mathbb{R}_T : {}_B\mathcal{M}_T \to {}_A\mathcal{M}_T$$
$$\mathbb{L}_T(X) = M \otimes_A X, \qquad \mathbb{R}_T(Y) = {}_B\mathrm{Hom}(M, Y).$$

Note that the counit ε^T of this adjunction is, for all $Y \in {}_B\mathcal{M}_T$,

$$\varepsilon^T_Y : M \otimes_A {}_B\mathrm{Hom}(M, Y) \to Y, \qquad m \otimes_A f \mapsto (m)f.$$

We would like to compute the cohomology relative to the class

$$\mathcal{E}_{M,T} := \mathcal{E}_{\mathbb{R}_T} = \{f \in {}_B\mathcal{M}_T \mid {}_B\mathrm{Hom}(M, f) \text{ splits in } {}_A\mathcal{M}_T\}$$

To apply the derived functors we need to determine when $\mathcal{E}_{M,T}$ is a class of epimorphisms.

Proposition 3.1. *Let $\varepsilon^T : \mathbb{L}_T\mathbb{R}_T \to \mathrm{Id}_{{}_B\mathcal{M}_T}$ be the counit of the adjunction $(\mathbb{L}_T, \mathbb{R}_T)$. The following assertions are equivalent:*

(a) *$\mathcal{E}_{M,T}$ is a class of epimorphisms for every ring T.*
(a$'$) *$\mathcal{E}_{M,B}$ is a class of epimorphisms.*
(a$''$) *$\mathcal{E}_{M,\mathbb{Z}}$ is a class of epimorphisms.*
(b) *The counit $\varepsilon^T_Y : \mathbb{L}_T\mathbb{R}_T(Y) \to Y$ is an epimorphism for every ring T and for every $Y \in {}_B\mathcal{M}_T$.*
(b$'$) *The counit $\varepsilon^B_Y : \mathbb{L}_B\mathbb{R}_B(Y) \to Y$ is an epimorphism for every $Y \in {}_B\mathcal{M}_B$.*
(b$''$) *The counit $\varepsilon^{\mathbb{Z}}_Y : \mathbb{L}_{\mathbb{Z}}\mathbb{R}_{\mathbb{Z}}(Y) \to Y$ is an epimorphism for every $Y \in {}_B\mathcal{M}_{\mathbb{Z}} = {}_B\mathcal{M}$.*
(c) *$\mathbb{R}_T : {}_B\mathcal{M}_T \to {}_A\mathcal{M}_T$ is faithful for every ring T.*
(c$'$) *$\mathbb{R}_B : {}_B\mathcal{M}_B \to {}_A\mathcal{M}_B$ is faithful.*
(c$''$) *$\mathbb{R}_{\mathbb{Z}} : {}_B\mathcal{M} \to {}_A\mathcal{M}$ is faithful.*
(d) *The evaluation map*

$$\mathrm{ev}_M : M \otimes_A {}^*M \to B, \qquad \mathrm{ev}_M(m \otimes_A f) = (m)f,$$

where ${}^*M := {}_B\mathrm{Hom}\,(M, B)$ is an epimorphism (of B-bimodules).
(e) *M is a generator in ${}_B\mathcal{M}$.*

Proof. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), (a$'$) $\Leftrightarrow$ (b$'$) $\Leftrightarrow$ (c$'$) and (a$''$) $\Leftrightarrow$ (b$''$) $\Leftrightarrow$ (c$''$) follow by Proposition 2.2. The implication (c) $\Rightarrow$ (c$'$) is obvious, while (a$'$) $\Rightarrow$ (d) follows by identifying ev_M with the counit of adjunction (at B) $\varepsilon^B_B \in \mathcal{E}_{M,B}$. The latter is in the class $\mathcal{E}_{M,B}$, and hence is an epimorphism (by assumption (a$'$)). The equivalences (c$''$) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) are standard characterizations of generators in the category of modules (cf. [19, 13.7]). Finally, since, for all $f \in {}_B\mathrm{Hom}_T(Y, Y')$, $\mathbb{R}_{\mathbb{Z}}$ (f) $= {}_B\mathrm{Hom}\,(M, f) = \mathbb{R}_T\,(f)$, the condition (c$''$) implies (c). $\square$

Clearly, for every $Y' \in_B \mathcal{M}_B$, the functor $\mathbf{F}_{Y'} := {}_B\mathrm{Hom}_B(-, Y') : {}_B\mathcal{M}_B \to \mathfrak{Ab}$ is left $\mathcal{E}_{M,B}$-exact so, in view of equivalent conditions in Proposition 3.1 we can propose the following:

Definition 3.2. Let M be a B–A bimodule which is a generator as a left B-module, and let $\mathcal{E}_{M,B}$ be the class of all B-bimodule maps f, such that ${}_B\mathrm{Hom}(M, f)$ splits as an A–B bimodule map. The *M-Hochschild cohomology of B with coefficients in a B-bimodule N* is defined by

$$\mathrm{H}^\bullet_M(B, N) := \mathrm{Ext}^\bullet_{\mathcal{E}_{M,B}}(B, N),$$

(cf. Definition 2.3 for the explanation of the relative Ext-functor).

If the number

$$\min\left\{n \in \mathbb{N} \cup \{0\} \mid \mathrm{H}^{n+1}_M(B, N) = 0 \text{ for every } N \in {}_B\mathcal{M}_B\right\}$$

exists, then it is called the *M-Hochschild dimension of B* and is denoted by $\mathrm{Hdim}_M(B)$. Otherwise B is said to have infinite M-Hochschild dimension.

Similarly to the non-relative case, M-Hochschild cohomology can be equivalently described as the cohomology of a complex associated with the standard resolution. The standard resolution can be described in general as follows. Start with an additive functor $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ of abelian categories with a left adjoint $\mathbb{T}$. This defines a comonad $F := \mathbb{T}\mathbb{H}$ on $\mathfrak{B}$ with the counit given by the counit of adjunction $(\mathbb{T}, \mathbb{H})$, $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$. For an object $B \in \mathfrak{B}$, one considers the *associated augmented chain complex*

$$\cdots \xrightarrow{d_3} F^3(B) \xrightarrow{d_2} F^2(B) \xrightarrow{d_1} F(B) \xrightarrow{d_0} F^0(B) := B \to 0,$$

where

$$d_n = \sum_{i=0}^{n} (-1)^i\, F^i\left(\varepsilon_{F^{n-i}(B)}\right)$$

(see [21, 8.6.4, page 280]).

Proposition 3.3. *Let $\mathfrak{A}$, $\mathfrak{B}$ be abelian categories. Let $\mathbb{H} : \mathfrak{B} \to \mathfrak{A}$ be a faithful covariant functor with a left adjoint $\mathbb{T}$. For all objects of $\mathfrak{B}$, the associated augmented chain complex is an $\mathcal{E}_{\mathbb{H}}$-exact sequence.*

Proof. Let $\varepsilon : \mathbb{T}\mathbb{H} \to \mathrm{Id}_{\mathfrak{B}}$ be the counit of the adjunction and let $\eta : \mathrm{Id}_{\mathfrak{A}} \to \mathbb{H}\mathbb{T}$ be the unit of the adjunction. For all integers $n \geq -1$, define

$$s_n := \eta_{\mathbb{H}F^{n+1}(B)} : \mathbb{H}F^{n+1}(B) \to \mathbb{H}F^{n+2}(B).$$

Then, $\mathbb{H}(d_0) \circ s_{-1} = \mathbb{H}(\varepsilon_B) \circ \eta_{\mathbb{H}(B)} = \mathrm{Id}_{\mathbb{H}(B)}$. Furthermore, for all $n \geq 0$, $d_n = \varepsilon_{F^n(B)} - F(d_{n-1})$, so that

$$\begin{aligned} \mathbb{H}(d_{n+1}) \circ s_n &= \mathbb{H}\left(\varepsilon_{F^{n+1}(B)}\right) \circ \eta_{\mathbb{H}F^{n+1}(B)} - \mathbb{H}F(d_n) \circ \eta_{\mathbb{H}F^{n+1}(B)} \\ &= \mathbb{H}\left(\varepsilon_{F^{n+1}(B)}\right) \circ \eta_{\mathbb{H}F^{n+1}(B)} - \eta_{\mathbb{H}F^{n}(B)} \circ \mathbb{H}(d_n) \\ &= \mathrm{Id}_{\mathbb{H}F^{n+1}(B)} - s_{n-1} \circ \mathbb{H}(d_n), \end{aligned}$$

where the second equality follows by the naturality of the unit of adjunction. Hence $s_\bullet$ is a contracting homotopy for the complex $(\mathbb{H}(F^\bullet(B)), \mathbb{H}(d_\bullet))$, which implies that the augmented chain complex $(F^\bullet(B), d_\bullet)$ is an $\mathcal{E}_{\mathbb{H}}$-exact sequence. □

In the case of the adjunction $(\mathbb{L}_B, \mathbb{R}_B)$, the comonad is $F = M \otimes_A {}_B\mathrm{Hom}(M, -)$. Application of the functor ${}_B\mathrm{Hom}_B(-, N) : {}_B\mathcal{M}_B \to \mathfrak{Ab}$ to the associated augmented chain complex results in the cochain complex

$$({}_B\mathrm{Hom}_B(F^\bullet(B), N), d^\bullet := {}_B\mathrm{Hom}_B(d_\bullet, N)),$$

whose cohomology is $\mathrm{H}^\bullet_M(B, N)$.

The M-Hochschild cohomology has a particularly simple description in the case where M is a progenerator left B-module. In this case it can be identified with a (relative-)Hochschild cohomology of the endomorphism ring of M. This can be described as follows.

Given a ring extension $A \to S$ (or an A-ring S), the A-relative-Hochschild cohomology of S with values in an S-bimodule W [9], $\mathrm{H}^\bullet(S|A, W)$, is defined as the cohomology of the cochain complex

$$0 \to {}_A\mathrm{Hom}_A(A, W) \xrightarrow{b^0} {}_A\mathrm{Hom}_A(S, W) \xrightarrow{b^1} {}_A\mathrm{Hom}_A(S^{\otimes_A 2}, W) \xrightarrow{b^2} {}_A\mathrm{Hom}_A(S^{\otimes_A 3}, W) \xrightarrow{b^3} \cdots,$$

where, for all $f \in {}_A\mathrm{Hom}_A(S^{\otimes_A n}, W)$, $n = 0, 1, 2, \ldots$,

$$b^n(f) = \mu^l_W \circ (S \otimes_A f) + \sum_{i=1}^{n} (-1)^i f \circ (S^{\otimes i-1} \otimes_A m_S \otimes_A S^{\otimes n-i}) + (-1)^{n+1} \mu^r_W \circ (f \otimes_A S).$$

Here μ^l_W, μ^r_W denote left and right S-multiplication, respectively, on W and $m_S : S \otimes_A S \to S$ is the product map. Also, in the case $n = 0$, the obvious isomorphisms $A \otimes_A S \simeq S \otimes_A A \simeq S$ are implicitly used. $\mathrm{H}^\bullet(S|A, W)$ can be understood as the Hochschild cohomology of the algebra S in monoidal category of A-bimodules (cf. [3, Theorem 4.42]). The *Hochschild dimension of S over A* is then defined by

$$\mathrm{Hdim}(S|A) := \min\left\{n \in \mathbb{N} \cup \{0\} \mid \mathrm{H}^{n+1}(S|A, W) = 0 \text{ for every } W \in {}_S\mathcal{M}_S\right\},$$

provided that the minimum on the right hand side exists.

Theorem 3.4. *Let A, B be rings. Consider a bimodule ${}_BM_A$ such that ${}_BM$ is a progenerator. Let S be the endomorphism ring of the left B-module M. Then, for all B-bimodules N,*

$$\mathrm{H}^\bullet_M(B, N) = \mathrm{H}^\bullet\left(S|A, {}^*M \otimes_B N \otimes_B M\right).$$

Furthermore, for a fixed $n \in \mathbb{N}$, the following assertions are equivalent:

(a) $\mathrm{H}^n_M(B, N) = 0$, *for every B-bimodule N.*
(b) $\mathrm{H}^n(S|A, W) = 0$, *for every S-bimodule W.*

In particular

$$\mathrm{Hdim}_M(B) = \mathrm{Hdim}(S|A).$$

Proof. Since M is a finitely generated and projective left B-module, the functor $\mathbb{R}_B$ is isomorphic to ${}^*M \otimes_B (-)_B$. The comonad comes out as

$$F(N) = \mathbb{L}_B\mathbb{R}_B(N) \equiv C \otimes_B N, \quad \text{for all } N \in {}_B\mathcal{M}_B,$$

where $C := M \otimes_A {}^*M \equiv \mathbb{L}_B\mathbb{R}_B(B)$. The counit of adjunction at B is simply the evaluation map $\mathrm{ev}_M : M \otimes_A {}^*M \to B$, $m \otimes_A f \mapsto (m)f$. Using the standard isomorphisms $C \otimes_B B \simeq C$, and applying the Hom-functor to the augmented chain complex associated with B, we can identify $\mathrm{H}^\bullet_M(B, N)$ with the cohomology of the cochain complex

$$0 \to {}_B\mathrm{Hom}_B(B, N) \xrightarrow{d_0^*} {}_B\mathrm{Hom}_B(C, N) \xrightarrow{d_1^*} {}_B\mathrm{Hom}_B(C^{\otimes_B 2}, N) \xrightarrow{d_2^*} {}_B\mathrm{Hom}_B(C^{\otimes_B 3}, N) \xrightarrow{d_3^*} \cdots,$$

where, for all $f \in {}_B\mathrm{Hom}_B(C^{\otimes_B n}, N)$, $n = 0, 1, 2, \ldots$,

$$d_n^*(f) = \sum_{i=0}^{n} (-1)^i\, f \circ (C^{\otimes_B i} \otimes_B \mathrm{ev}_M \otimes_B C^{\otimes_B n-i}).$$

Since M is a finitely generated and projective left B-module, S can be identified with ${}^*M \otimes_B M$. Under this identification, the product is given by ${}^*M \otimes_B \mathrm{ev}_M \otimes_B M$ and the unit is the dual basis element $\sum_{a\in I} {}^*e_a \otimes_B e_a$. Furthermore, one can consider the isomorphisms

$$\Phi_n : {}_B\mathrm{Hom}_B(C^{\otimes_B n+1}, N) \to {}_A\mathrm{Hom}_A(S^{\otimes_A n}, {}^*M \otimes_B N \otimes_B M),$$

defined by

$$[\Phi_n(f)](x) = \left({}^*M \otimes_B f \otimes_B M\right)(1_S \otimes_A x \otimes_A 1_S), \quad \text{for every } x \in S^{\otimes_A n}.$$

Using the definitions of cochain maps and above identification of S, one easily checks that these isomorphisms fit into the commutative diagrams

$$\begin{array}{ccc} {}_B\mathrm{Hom}_B(B, N) & \xrightarrow{d_0^*} & {}_B\mathrm{Hom}_B(C, N) \\ \simeq\downarrow & & \downarrow \Phi_0 \\ N^B = \{n \in N \mid bn = nb, \forall b \in B\} & \xrightarrow{b^{-1}} & {}_A\mathrm{Hom}_A(A, {}^*M \otimes_B N \otimes_B M) \end{array}$$

and

$$\begin{array}{ccc} {}_B\mathrm{Hom}_B(C^{\otimes_B n}, N) & \xrightarrow{d_n^*} & {}_B\mathrm{Hom}_B(C^{\otimes_B n+1}, N) \\ \Phi_{n-1}\downarrow & & \downarrow \Phi_n \\ {}_A\mathrm{Hom}_A(S^{\otimes_A n-1}, {}^*M \otimes_B N \otimes_B M) & \xrightarrow{b^{n-1}} & {}_A\mathrm{Hom}_A(S^{\otimes_A n}, {}^*M \otimes_B N \otimes_B M) \end{array}$$

This immediately implies that

$$\mathrm{H}^\bullet_M(B, N) = \mathrm{H}^\bullet\left(S|A, {}^*M \otimes_B N \otimes_B M\right),$$

as required.

It remains to prove that the statements (a) and (b) are equivalent. The implication $(b) \Rightarrow (a)$ is obvious. To prove the converse, take any S-bimodule W and define a B-bimodule $N = M \otimes_S W \otimes_S {}^*M$. Then

$${}^*M \otimes_B N \otimes_B M = {}^*M \otimes_B M \otimes_S W \otimes_S {}^*M \otimes_B M = S \otimes_S W \otimes_S S \simeq W.$$

This completes the proof. □

4. Separable bimodules

The aim of the section is to supplement (and extend) the functorial description of separable bimodules in [4, Corollary 5.8] with the cohomological description of such bimodules. First recall from [16] the following:

Definition 4.1. Let A, B be rings. A B–A bimodule M is said to be *separable*, or B is said to be M*-separable over* A if the evaluation map

$$\mathrm{ev}_M : M \otimes_A {}^*M \to B, \qquad \mathrm{ev}_M(m \otimes_A f) = (m)f,$$

is a split epimorphism of B-bimodules.

Throughout this section, M is a B–A bimodule, and $\mathbb{L}_T$, $\mathbb{R}_T$, $\mathcal{E}_{M,T}$ are the functors and the class of morphisms (associated with M) described at the beginning of Section 3.

Proposition 4.2. *The following assertions are equivalent for a B–A bimodule M.*

(a) *M is a separable bimodule.*
(b) *For all rings T, $\mathbb{R}_T : {}_B\mathcal{M}_T \to {}_A\mathcal{M}_T$ is a separable functor.*
(c) *$\mathbb{R}_B : {}_B\mathcal{M}_B \to {}_A\mathcal{M}_B$ is a separable functor.*
(d) *$\mathbb{R}_{\mathbb{Z}} : {}_B\mathcal{M} \to {}_A\mathcal{M}$ is a separable functor.*
(e) *Any B-bimodule is $\mathcal{E}_{M,B}$-projective.*
(f) *B is $\mathcal{E}_{M,B}$-projective.*
(g) *M is a generator in ${}_B\mathcal{M}$ and $\mathrm{H}^n_M(B, N) = 0$, for every $N \in {}_B\mathcal{M}_B$ and for every $n \geq 1$.*
(h) *M is a generator in ${}_B\mathcal{M}$ and $\mathrm{H}^1_M(B, N) = 0$, for every $N \in_B \mathcal{M}_B$.*
(i) *M is a generator in ${}_B\mathcal{M}$ and $\mathrm{Hdim}_M(B) = 0$.*

Proof. (a) ⇒ (b) For any B–T bimodule Y, there is a (natural in Y) A–T bimodule map

$$\hat{\xi} : {}^*M \otimes_B Y \to {}_B\mathrm{Hom}(M, Y), \qquad {}^*m \otimes_B y \mapsto [m \mapsto (m)\,{}^*m\, y],$$

(cf. [1, Proposition 20.10]). Tensoring this map with M, we obtain a B–T bimodule map

$$\xi : M \otimes_A {}^*M \otimes_B Y \to M \otimes_A {}_B\mathrm{Hom}(M, Y).$$

It is clear from the definition and naturality of $\hat{\xi}$ that, for all $x \in (M \otimes_A {}^*M)^B := \{x \in M \otimes_A {}^*M \mid \forall b \in B, xb = bx\}$, the map $\xi(x \otimes_B -) : Y \to M \otimes_A {}_B\mathrm{Hom}(M, Y)$ is natural in Y and B–T bilinear.

If M is a separable bimodule, then there exists $s \in (M \otimes_A {}^*M)^B$ such that $\mathrm{ev}_M(s) = 1_B$. One easily checks that $\xi(s \otimes_A -)$ is a natural splitting of the counit of the adjunction $(\mathbb{L}_T, \mathbb{R}_T)$. Hence, by Rafael's theorem (Theorem 2.4), $\mathbb{R}_T$ is a separable functor.

Implications (b) ⇒ (c) and (e) ⇒ (f) are obvious, while the equivalence (a) ⇔ (d) is proven in [4, Corollary 5.8]. The implication (c)⇒ (e) follows by Corollary 2.5.

Since ev_M is the same as the counit of adjunction $(\mathbb{L}_B, \mathbb{R}_B)$ evaluated at B, the implication (f) ⇒ (a) follows by Theorem 2.1. Thus, if B is $\mathcal{E}_{M,B}$-projective, then ev_M is an epimorphism, and hence M is a generator in ${}_B\mathcal{M}$ by Proposition 3.1. Therefore, the equivalences between (f), (g), (h) and (i) follow by the definitions of M-Hochschild cohomology of B with coefficients in N and M-Hochschild dimension of B, and by the properties of $\mathrm{Ext}^{\bullet}_{\mathcal{E}_{M,B}}(-,-)$. □

Recall that a ring morphism $A \to S$ is called a *separable extension* if the product map $m_S : S \otimes_A S \to S$ has an S-bimodule section.

Proposition 4.3 (*[16, Theorem 1]*). *Let A, B be rings. Consider a bimodule ${}_BM_A$ such that M is a finitely generated and projective left B-module. Let*

$$S = {}_B\mathrm{End}\,(M) = \mathbb{R}_A\,(M) \simeq {}^*M \otimes_B M.$$

The following assertions are equivalent:

(a) *${}_BM_A$ is a separable bimodule.*
(b) *M is a generator in ${}_B\mathcal{M}$ and the canonical morphism $i : A \to S$, $a \mapsto [m \mapsto ma]$ is a separable extension.*

Proof. By Proposition 4.2, a separable bimodule M is a generator in ${}_B\mathcal{M}$. Thus, in both cases, M is a progenerator and, by Theorem 3.4, $\mathrm{Hdim}_M(B) = \mathrm{Hdim}\,(S|A)$. Since the extension $A \to S$ is separable if and only if $\mathrm{Hdim}\,(S|A) = 0$ (cf. [3, Theorem 4.43]), the assertion follows by Proposition 4.2.

The proposition can also be proven directly as follows. In both cases M is a progenerator left B-module, and hence S is isomorphic to ${}^*M \otimes_B M$ (${}_BM$ is finitely generated and projective) and B is isomorphic to $M \otimes_S {}^*M$ (${}_BM$ is a generator). These isomorphisms allow one to identify the product m_S in S with ${}^*M \otimes_B \mathrm{ev}_M \otimes_B M$, and ev_M with $M \otimes_S m_S \otimes_S {}^*M$. With these identifications, the mutual equivalence of statements (a) and (b) is clear. □

Remark 4.4. For a left B-module M, let $S = {}_B\mathrm{End}\,(M)$. For the B–S bimodule M, the map i of Proposition 4.3 is the identity and hence it trivially defines a separable extension. Still ${}_BM_S$ needs not be a separable bimodule unless ${}_BM$ is a generator in ${}_B\mathcal{M}$ (see Corollary 5.11).

5. Formally smooth bimodules

In this section we introduce the notion of a formally smooth bimodule, give a cohomological interpretation and describe examples of such bimodules. Throughout this section, M is a B–A bimodule, and, for any ring T, $\mathbb{L}_T$, $\mathbb{R}_T$, $\mathcal{E}_{M,T}$ are the functors and the class of morphisms (associated with M) described at the beginning of Section 3.

Definition 5.1. Let A, B be rings. A B–A bimodule M is said to be *formally smooth* or B is said to be *M-smooth over A* whenever the kernel of the evaluation map

$$\mathrm{ev}_M : M \otimes_A {}^*M \to B, \qquad \mathrm{ev}_M(m \otimes_A f) = (m)f.$$

is an $\mathcal{E}_{M,B}$-projective B-bimodule.

Following [11] a pair of functors $\mathbb{U}_* : \bar{\mathfrak{A}} \to \mathfrak{A}$, $\mathbb{U}^* : \mathfrak{A} \to \bar{\mathfrak{A}}$ such that $\mathbb{U}^*$ is fully faithful and left adjoint to $\mathbb{U}_*$ is called a *Q-category*. As explained in [11, Section 2.5], with any category $\mathfrak{C}$ and any class of morphisms $\mathcal{H}$ in $\mathfrak{C}$ which contains all the identity morphisms, one can associate a Q-category as follows. First construct the category $\mathfrak{H}$, whose objects are elements f, g of $\mathcal{H}$ and morphisms are commutative squares

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & N \arrow[d, "g"] \\ M' \arrow[r] & N' \end{tikzcd}$$

where the horizontal arrows are in $\mathfrak{C}$. The *inverse image functor* $\mathbb{U}^* : \mathfrak{C} \to \mathfrak{H}$ is

$$\mathbb{U}^* : M \mapsto \mathrm{id}_M, \qquad \left(M \xrightarrow{f} N \right) \mapsto \left(\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mathrm{id}_M"'] & N \arrow[d, "\mathrm{id}_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \right).$$

The *direct image functor* $\mathbb{U}_* : \mathfrak{H} \to \mathfrak{C}$ is defined by

$$\mathbb{U}_* : \left(\begin{tikzcd} M \arrow[d, "f"] \\ M' \end{tikzcd} \right) \mapsto M, \qquad \left(\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & N \arrow[d, "g"] \\ M' \arrow[r] & N' \end{tikzcd} \right) \mapsto \left(M \longrightarrow N \right).$$

We denote this Q-category by $\mathfrak{A}_{\mathcal{H}}$ and call it a *Q-category induced by the class $\mathcal{H}$*. Following [11, Sections 3.7 & 4.5] an object $P \in \mathfrak{C}$ is said to be *formally $\mathfrak{A}_{\mathcal{H}}$-smooth* if and only if, for every $f \in \mathcal{H}$, the mapping $\mathfrak{C}(P, f)$ is a strict epimorphism (i.e. a surjective map) of sets. Thus P is formally $\mathfrak{A}_{\mathcal{H}}$-smooth if and only if P is $\mathcal{H}$-projective. This leads immediately to the following lemma, which also explains the choice of the terminology.

Lemma 5.2. *A bimodule ${}_BM_A$ is formally smooth if and only if* $\mathrm{Ker}(\mathrm{ev}_M)$ *is a formally smooth object in the Q-category induced by the class of morphisms $\mathcal{E}_{M,B}$.*

The following proposition gives the first examples of formally smooth bimodules.

Proposition 5.3. *A B–A bimodule M is formally smooth whenever one of the following conditions holds:*

(1) *M is a separable bimodule.*
(2) *The map ev_M is injective.*

Proof. (1) If M is a separable bimodule, then Proposition 4.2 implies that any B-bimodule is $\mathcal{E}_{M,B}$-projective. In particular $\mathrm{Ker}(\mathrm{ev}_M)$ is $\mathcal{E}_{M,B}$-projective.

(2) If ev_M is injective, then $\mathrm{Ker}(\mathrm{ev}_M)$ is trivial, and hence $\mathcal{E}_{M,B}$-projective. $\square$

The cohomological interpretation of formally smooth bimodules is provided by the following:

Proposition 5.4. *Let A, B be rings. Take a B–A bimodule M which is a generator in ${}_B\mathcal{M}$. Then the following assertions are equivalent:*

(a) *M is a formally smooth bimodule.*
(b) $\mathrm{H}^n_M(B, N) = 0$, *for all $N \in {}_B\mathcal{M}_B$ and all $n \geq 2$.*
(c) $\mathrm{H}^2_M(B, N) = 0$, *for all $N \in {}_B\mathcal{M}_B$.*
(d) $\mathrm{Hdim}_M(B) \leq 1$.

Proof. The equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) follow by the definitions of the M-Hochschild cohomology of B with coefficients in N and the M-Hochschild dimension of B, and by the properties of $\mathrm{Ext}^{\bullet}_{\mathcal{E}_{M,B}}(-,-)$.

(a) $\Leftrightarrow$ (c). Write (L, i_L) for the kernel of ev_M, and consider the following exact sequence of B-bimodules:

$$0 \longrightarrow L \xrightarrow{i_L} M \otimes_A {}^*M \xrightarrow{\mathrm{ev}_M} B \longrightarrow 0.$$

Note that ev_M is surjective as M is a generator in ${}_B\mathcal{M}$. Also, since ev_M is the same as the counit of adjunction $(\mathbb{L}_B, \mathbb{R}_B)$, $\mathrm{ev}_M = \varepsilon^B_B$, the map ev_M is in the class $\mathcal{E}_{M,B}$. Hence the above sequence is $\mathcal{E}_{M,B}$-admissible and, for any B-bimodule N, gives rise to a long exact sequence, a part of which is

$$\mathrm{Ext}^1_{\mathcal{E}_{M,B}}(M \otimes_A {}^*M, N) \to \mathrm{Ext}^1_{\mathcal{E}_{M,B}}(L, N) \to \mathrm{Ext}^2_{\mathcal{E}_{M,B}}(B, N) \to \mathrm{Ext}^2_{\mathcal{E}_{M,B}}(M \otimes_A {}^*M, N).$$

By Theorem 2.1, $M \otimes_A {}^*M = \mathbb{L}_B({}^*M)$ is $\mathcal{E}_{M,B}$-projective so that $\mathrm{Ext}^1_{\mathcal{E}_{M,B}}(L, N) \simeq \mathrm{Ext}^2_{\mathcal{E}_{M,B}}(B, N) = \mathrm{H}^2_M(B, N)$. Hence the $\mathcal{E}_{M,B}$-projectivity of $L = \mathrm{Ker}(\mathrm{ev}_M)$ is equivalent to the property (c). $\square$

Examples of formally smooth bimodules can be obtained from smooth extensions.

Definition 5.5. Let A, B be rings and let $i : A \to B$ be a ring homomorphism. Consider the adjunction

$$\mathbb{T} : {}_A\mathcal{M}_A \to {}_B\mathcal{M}_B, \qquad \mathbb{H} : {}_B\mathcal{M}_B \to {}_A\mathcal{M}_A$$
$$\mathbb{T}(X) = B \otimes_A X \otimes_A B, \qquad \mathbb{H}(Y) = Y.$$

Then i is called a *formally smooth extension* whenever $\mathrm{Ker}(m_B)$ is $\mathcal{E}_{\mathbb{H}}$-projective. Here $m_B : B \otimes_A B \to B$ is the multiplication map and $\mathcal{E}_{\mathbb{H}}$ is a class of $\mathbb{H}$-relatively split morphisms as in Theorem 2.1.

By [3, Corollary 3.12], ring extension $A \to B$ is formally smooth provided B is formally smooth when regarded as an algebra in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$.

Lemma 5.6. *Let A, B be rings and let M be a B–A bimodule that is finitely generated and projective as a left B-module. Let $S = {}_B\mathrm{End}(M) = \mathbb{R}_A(M) \simeq {}^*M \otimes_B M$ be the endomorphism ring. Write $m_S : S \otimes_A S \to S$ for the multiplication map and (L, i_L) for the kernel of ev_M. The sequence*

$$0 \longrightarrow {}^*M \otimes_B L \otimes_B M \xrightarrow{{}^*M \otimes_B i_L \otimes_B M} S \otimes_A S \xrightarrow{m_S} S \longrightarrow 0.$$

is exact.

Proof. Start with the exact sequence

$$0 \longrightarrow L \xrightarrow{i_L} M \otimes_A {}^*M \xrightarrow{\mathrm{ev}_M} B. \tag{$*$}$$

Since M is a finitely generated and projective left B-module, ${}^*M$ is a finitely generated and projective right B-module. By tensoring sequence $(*)$ on the left with ${}^*M$ and on the right with M, we obtain the following exact sequence:

$$0 \longrightarrow {}^*M \otimes_B L \otimes_B M \xrightarrow{{}^*M \otimes_B i_L \otimes_B M} {}^*M \otimes_B M \otimes_A {}^*M \otimes_B M \xrightarrow{{}^*M \otimes_B \mathrm{ev}_M \otimes_B M} {}^*M \otimes_B B \otimes_B M.$$

The isomorphisms ${}^*M \otimes_B B \otimes_B M \simeq {}^*M \otimes_B M \simeq S$ allow one to identify the map ${}^*M \otimes_B \mathrm{ev}_M \otimes_B M$ with m_S. Being a multiplication of unital rings the latter is surjective. $\square$

Proposition 5.7. *Let A, B be rings and let M be a B–A bimodule that is finitely generated and projective as a left B-module. Let $S = {}_B\mathrm{End}(M) = \mathbb{R}_A(M) \simeq {}^*M \otimes_B M$ be the endomorphism ring. Write i for the canonical ring map*

$$i : A \to S, \qquad a \mapsto [m \mapsto ma].$$

(1) *If the bimodule ${}_BM_A$ is formally smooth, then $i : A \to S$ is a formally smooth extension.*
(2) *If M is a generator in ${}_B\mathcal{M}$ and $i : A \to S$ is a formally smooth extension, then ${}_BM_A$ is a formally smooth bimodule.*

Proof. (1) In view of Theorem 2.1, to prove that $i : A \to S$ is a formally smooth extension, it suffices to prove that $\mathrm{Ker}(m_S)$ is a direct summand (in ${}_S\mathcal{M}_S$) of $S \otimes_A X \otimes_A S$, for a suitable object $X \in {}_A\mathcal{M}_A$. Write (L, i_L) for the kernel of ev_M. Since M is formally smooth, L is $\mathcal{E}_{M,B}$-projective. By Theorem 2.1, this means that the counit of the adjunction $(\mathbb{L}_B, \mathbb{R}_B)$ evaluated at L

$$\varepsilon_L^B : \mathbb{L}_B\mathbb{R}_B(L) \simeq M \otimes_A {}^*M \otimes_B L \to L$$

has a section $\sigma : L \to M \otimes_A {}^*M \otimes_B L$ in ${}_B\mathcal{M}_B$. Since M is a finitely generated and projective left B-module, the functor $\mathbb{R}_B$ can be naturally identified with the tensor functor ${}^*M \otimes_B -$. Furthermore, ev_M is in the class $\mathcal{E}_{M,B}$, and hence ${}^*M \otimes_B \mathrm{ev}_M \simeq \mathbb{R}_B(\mathrm{ev}_M)$ splits in ${}_A\mathcal{M}_B$. Thus applying $\mathbb{R}_B$ to the defining sequence of (L, i_L) we obtain the split exact sequence of A–B bimodules

$$0 \longrightarrow {}^*M \otimes_B L \xrightarrow{{}^*M \otimes_B i_L} {}^*M \otimes_B M \otimes_A {}^*M \xrightarrow{{}^*M \otimes_B \mathrm{ev}_M} {}^*M \longrightarrow 0$$

In particular ${}^*M \otimes_B i_L$ is a section in ${}_A\mathcal{M}_B$, and, consequently $M \otimes_A {}^*M \otimes_B i_L$ is a section in ${}_B\mathcal{M}_B$. Therefore, the map

$$\alpha := \left(L \xrightarrow{\sigma} M \otimes_A {}^*M \otimes_B L \xrightarrow{M \otimes_A {}^*M \otimes_B i_L} M \otimes_A {}^*M \otimes_B M \otimes_A {}^*M \right)$$

splits in ${}_B\mathcal{M}_B$. This implies that,

$${}^*M \otimes_B L \otimes_B M \xrightarrow{{}^*M \otimes_B \alpha \otimes_B M} {}^*M \otimes_B M \otimes_A {}^*M \otimes_B M \otimes_A {}^*M \otimes_B M \simeq S \otimes_A S \otimes_A S$$

splits in ${}_S\mathcal{M}_S$. In view of Lemma 5.6, $\mathrm{Ker}(m_S) \simeq {}^*M \otimes_B L \otimes_B M$, and hence ${}^*M \otimes_B \alpha \otimes_B M$ is the required S-bimodule section of a map $S \otimes_A S \otimes_A S \to \mathrm{Ker}(m_S)$.

(2) By [3, Theorem 3.8 & Theorem 4.42], if $i : A \to S$ is formally smooth, then $\mathrm{Hdim}(S|A) \leq 1$. Since M is a generator in ${}_B\mathcal{M}$, Theorem 3.4 implies that $\mathrm{Hdim}_M(B) \leq 1$. Proposition 5.4 then implies that M is a formally smooth bimodule. $\square$

Proposition 5.8. *Let B be an algebra over a commutative ring k. Consider the functor*

$$\mathbb{F} : \mathcal{M}_k \to {}_k\mathcal{M}_k : (V, \mu^r) \longmapsto \left(V, \mu^l, \mu^r\right),$$

where the left k-multiplication is defined by $\mu^l(\lambda \otimes_k v) := \mu^r(v \otimes_k \lambda)$, *for all* $\lambda \in k$ *and* $v \in V$. *Furthermore, consider the adjunction*

$$\mathbb{T}' : \mathcal{M}_k \to {}_B\mathcal{M}_B, \qquad \mathbb{H}' : {}_B\mathcal{M}_B \to \mathcal{M}_k$$
$$\mathbb{T}'(X) = B \otimes_k \mathbb{F}(X) \otimes_k B, \qquad \mathbb{H}'(Y) = Y.$$

The following assertions are equivalent.

(a) *The bimodule* ${}_BM_k = {}_BB_k$ *is formally smooth.*
(b) $\mathrm{Ker}(m_B)$ *is* $\mathcal{E}_{\mathbb{H}'}$*-projective.*
(c) *The extension* $k \to B$ *is formally smooth.*

Proof. Clearly the B-module B can be identified with both its dual and its endomorphism ring. With this identification the evaluation map $\mathrm{ev}_B = m_B$. Hence the equivalence (a) $\Leftrightarrow$ (c) follows by Proposition 5.7. The implication (b) $\Rightarrow$ (c) is an immediate consequence of the observation that any B-bimodule map that splits as a k-bimodule map splits also as a right k-module map (i.e. $\mathcal{E}_{\mathbb{H}} \subseteq \mathcal{E}_{\mathbb{H}'}$, where $\mathbb{H}$ is a functor in Definition 5.5).

(a) $\Rightarrow$ (b) We need to show that $L := \mathrm{Ker}(\mathrm{ev}_B)$ is $\mathcal{E}_{\mathbb{H}'}$-projective. The counit of the adjunction $(\mathbb{T}', \mathbb{H}')$ is given by the two-sided multiplication

$$\varepsilon'_N : B \otimes_k \mathbb{F}\mathbb{H}'(N) \otimes_k B \to N, \quad \text{for every } N \in_B \mathcal{M}_B.$$

Note that $\varepsilon'_N \in \mathcal{E}_{\mathbb{H}'}$ as it is the counit. Consider the adjunction

$$\mathbb{T}'' : \mathcal{M}_k \to \mathcal{M}_B, \qquad \mathbb{H}'' : \mathcal{M}_B \to \mathcal{M}_k$$
$$\mathbb{T}''(X) = X \otimes_k B, \qquad \mathbb{H}''(Y) = Y.$$

By the standard argument (cf. e.g. [6, Proposition 2.5]), $L \simeq \frac{B}{k} \otimes_k B = \mathbb{T}''(B/k)$. The latter is $\mathcal{E}_{\mathbb{H}''}$-projective by Theorem 2.1. Since $\varepsilon'_L \in \mathcal{E}_{\mathbb{H}'} \subseteq \mathcal{E}_{\mathbb{H}''}$ we conclude that ε'_L splits in $\mathcal{M}_B$, that is $\varepsilon'_L \in \mathcal{E}_{M,B}$ (note that $L = \mathbb{F}\mathbb{H}'(L)$ as it is a subbimodule of $B \otimes_k B$). By hypothesis L is $\mathcal{E}_{M,B}$-projective so that ε'_L splits in ${}_B\mathcal{M}_B$. By Theorem 2.1 (c)$\Rightarrow$(a), we thus conclude that L is $\mathcal{E}_{\mathbb{H}'}$-projective. $\square$

In view of Proposition 5.8 a formally smooth algebra B over a field k is a formally smooth (B, k)-bimodule. In this way one can construct examples of formally smooth bimodules which are not separable. For example, the tensor algebra $T_k(V)$ of a vector space V is formally smooth but not separable in view of Proposition 4.3. In fact it is well known that any separable extension of a field k is finite dimensional over k (cf. [18, Proposition 1.1]).

Let M be a left B-module and write S for its endomorphism ring. Recall that a left B-module N is said to be *M-static* provided the evaluation

$$\mathrm{ev}_{M,N} : M \otimes_S {}_B\mathrm{Hom}(M, N) \to N, \qquad m \otimes_A f \mapsto (m)f$$

is an isomorphism (see e.g. [20, 2.3]). Recall further that the image of the evaluation map $\mathrm{ev}_{M,N}$ is called the *trace of M in N* and is denoted by $\mathrm{Tr}_M(N)$. Finally, denote by $\sigma[M]$ the full subcategory of ${}_B\mathcal{M}$, whose objects are all modules subgenerated by M (cf. [19, Section 15]).

Proposition 5.9. *Let B be a ring, M be a left B-module and set $S =_B \mathrm{End}(M)$, so that M is a B–S bimodule. The following assertions are equivalent:*

(a) *The evaluation map* $\mathrm{ev}_M : M \otimes_S {}^*M \to B$ *is injective.*
(b) *The B-module* $\mathrm{Tr}_M(B)$ *is M-static.*

In particular these conditions hold whenever M is a generator in $\sigma[M]$.

Proof. Since

$${}_B\mathrm{Hom}(M, \mathrm{Tr}_M(B)) = {}_B\mathrm{Hom}(M, B),$$

the equivalence follows by observing that $\mathrm{ev}_{M,\mathrm{Tr}_M(B)}$ is exactly ev_M corestricted to its image. The last assertion follows by [22, Lemma 1.3]. $\square$

Combining Proposition 5.9 with Proposition 5.3 we immediately obtain

Corollary 5.10. *If a left B-module M with endomorphism ring S generates $\sigma[M]$, then M is a formally smooth B–S bimodule.*

Corollary 5.11. *Let B be a ring, M be a left B-module, and let $S = {}_B\mathrm{End}(M)$. The following assertions are equivalent:*

(a) ${}_BM_S$ *is a separable bimodule.*
(b) M *is a generator in* ${}_B\mathcal{M}$.
(c) *The evaluation map* $\mathrm{ev}_M : M \otimes_S {}^*M \to B$ *is an isomorphism.*

Proof. The implication (a) $\Rightarrow$ (b) follows by Proposition 4.2. If M is a generator of ${}_B\mathcal{M}$, then it is also a generator of $\sigma[M]$. By Proposition 5.9, $B = \mathrm{Tr}_M(B)$ is M-static. Hence ev_M is an isomorphism. This proves that (b) implies (c). The implication (c) $\Rightarrow$ (a) is obvious. $\square$

The following proposition explains how two formally smooth bimodules can be combined to give a formally smooth bimodule, and thus can be seen as module version of [6, Proposition 5.3].

Proposition 5.12. *Let A, B and T be rings. Let Y be a T–A bimodule and let X be a B–T bimodule such that the evaluation map $\mathrm{ev}_X : X \otimes_T {}^*X \to B$ is injective and that X is flat as a right T-module. Assume that one of the following conditions* (1) *or* (2) *is satisfied:*

(1) *Y is a separable T–A bimodule.*
(2) (i) *${}^*X$ is flat as a right T-module,*
 (ii) *Y is finitely generated and projective as a left T-module, and*
 (iii) *Y is a formally smooth T–A bimodule.*

Then

$$ {}_BM_A = {}_BX \otimes_T Y_A. $$

is a formally smooth bimodule.

In particular, if a left B-module X is a generator of $\sigma[X]$, $T = {}_B\mathrm{End}(X)$, and either (1) *or* (2) *hold, then $M = X \otimes_T Y$ is a formally smooth B–A bimodule.*

Proof. Associate with Y the tensor–Hom adjunction,

$$ \mathbb{T} : {}_A\mathcal{M}_B \to {}_T\mathcal{M}_B, \qquad \mathbb{H} : {}_T\mathcal{M}_B \to {}_A\mathcal{M}_B $$
$$ \mathbb{T}(U) = Y \otimes_A U, \qquad \mathbb{H}(W) = {}_T\mathrm{Hom}(Y, W), $$

and denote its counit (the evaluation map) by ε. Use the natural isomorphism

$$ \Phi : {}^*M = {}_B\mathrm{Hom}(X \otimes_T Y, B) \to {}_T\mathrm{Hom}(Y, {}^*X), \qquad f \mapsto [y \mapsto f(- \otimes_T y)], $$

to write the evaluation map $\mathrm{ev}_M : M \otimes_A {}^*M \to B$ as

$$ \mathrm{ev}_M = \mathrm{ev}_X \circ (X \otimes_T \varepsilon_{{}^*X}) \circ (M \otimes_A \Phi) . $$

Since, by assumption, X_T is flat and ev_X is injective, and since Φ is an isomorphism, there is an isomorphism of B-bimodules

$$ \mathrm{Ker}(\mathrm{ev}_M) \simeq X \otimes_T \mathrm{Ker}(\varepsilon_{{}^*X}). \tag{$*$} $$

Assume that condition (1) is satisfied, i.e. that Y is a separable bimodule. By Proposition 4.2, $\mathbb{H}$ is a separable functor, and hence, by Corollary 2.5, any object in ${}_T\mathcal{M}_B$ is $\mathcal{E}_{\mathbb{H}}$-projective. In particular $\mathrm{Ker}(\varepsilon_{{}^*X})$ is $\mathcal{E}_{\mathbb{H}}$-projective. By Theorem 2.1, $\mathrm{Ker}(\varepsilon_{{}^*X})$ is a direct summand in ${}_T\mathcal{M}_B$ of $\mathbb{T}(U)$ for some $U \in {}_A\mathcal{M}_B$, and hence $X \otimes_T \mathrm{Ker}(\varepsilon_{{}^*X})$ is a direct summand of

$$ X \otimes_T \mathbb{T}(U) = X \otimes_T Y \otimes_A U = M \otimes_A U = \mathbb{L}_B(U) $$

in ${}_B\mathcal{M}_B$. Theorem 2.1 implies that $\mathrm{Ker}(\mathrm{ev}_M)$ is $\mathcal{E}_{M,B}$-projective, so M is a formally smooth bimodule.

Assume that conditions (2) hold. Since Y is a finitely generated and projective left T-module, the functor $\mathbb{H}$ is naturally isomorphic to the tensor functor ${}^*Y \otimes_T -$, and the counit ε evaluated at W can be identified with $\mathrm{ev}_Y \otimes_T W$. In particular, $\mathrm{Ker}(\varepsilon_{{}^*X}) \simeq \mathrm{Ker}(\mathrm{ev}_Y \otimes_T {}^*X)$. Since ${}^*X$ is a flat left T-module, the isomorphism $(*)$ yields

$$\mathrm{Ker}(\mathrm{ev}_M) \simeq X \otimes_T \mathrm{Ker}(\mathrm{ev}_Y) \otimes_T {}^*X\,.$$

Since ${}_TY_A$ is a formally smooth bimodule, $\mathrm{Ker}(\mathrm{ev}_Y)$ is $\mathcal{E}_{\mathbb{H}}$-projective, which, as in the case (1), implies that M is a formally smooth B–A bimodule.

To prove the final statement observe that if ${}_BX$ is a generator of $\sigma[X]$ and $T = {}_B\mathrm{End}(X)$, then by Proposition 5.9, ev_X is injective. Furthermore by [19, Section 15.9], X_T is flat; hence the main assumptions about X are satisfied. □

In [5, Section 2] several ways of constructing separable bimodules are described. Combined with Proposition 5.12 these can provide a source of examples of smooth bimodules.

Acknowledgements

This paper was written when A. Ardizzoni and C. Menini were members of G.N.S.A.G.A. with partial financial support from M.I.U.R. The stay of T. Brzeziński at University of Ferrara was supported by I.N.D.A.M.

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