Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 318 (2007) 1032–1038
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The strong Lefschetz property of the coinvariant ring of the Coxeter group of type H_4

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Received 1 March 2007
Available online 22 June 2007
Communicated by Meinolf Geck

Abstract

We prove that the coinvariant ring of the irreducible Coxeter group of type H_4 has the strong Lefschetz property.

Keywords: Coxeter group; Coinvariant ring; Strong Lefschetz property

1. Introduction

The strong Lefschetz property (Definition 1) is an abstraction of the hard Lefschetz theorem, which describes the behavior of the multiplication map by the Kähler form in the cohomology ring of a non-singular algebraic variety. We show that the coinvariant ring of the irreducible Coxeter group of type H_4 has the strong Lefschetz property (Theorem 5). This study is a supplement to [MNW06], which determines the set of all strong Lefschetz elements of the coinvariant rings of the irreducible Coxeter groups of types other than H_4, in terms of corresponding root systems (see Remark 4). For the coinvariant ring of type H_4, Stanley (below Theorem 3.1 of [Sta80]) and Hiller [Hil81, Remark on p. 70] left comments on the difficulty in proving the strong Lefschetz property for H_4. To the authors' knowledge it has not been proved up to now.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.06.016

The difficulty is caused by the complicated structure of the Coxeter group of type H_4, and computer algebra systems cannot give the answer under the natural realization of the coinvariant ring as a quotient ring of $K[x_1, x_2, x_3, x_4]$. The key to our main theorem is to transform the variables so that the last variable will be a Lefschetz element. By this technique, we obtain the theorem without heavy computation except for only a single computation of a Gröbner basis. The computation is executed by the computer algebra system Macaulay2 [GS]. The essence of the technique above is paraphrased as Lemma 2, which is a condition for the Lefschetz properties.

Note that to determine the set of all strong Lefschetz elements is much more difficult than to find a strong Lefschetz element, and we will study the set of the strong Lefschetz elements for type H_4 in a forthcoming paper.

This paper is organized as follows: In Section 2 we prove Lemma 2, which is the essence of the technique used in our main theorem. In Section 3 we show the main theorem. In Section 4 we summarize the techniques used in the computation from the viewpoint of computer algebra systems.

2. A condition for the Lefschetz properties

In this section we give a necessary and sufficient condition for graded rings to have the strong Lefschetz property (Lemma 2). This lemma is the essence of a technique in proving our main theorem. The strong or weak Lefschetz property is studied in [Wat87,Wat89,HW03, HW04,HMNW03], for instance, and there are also other conditions for the Lefschetz properties in terms of generic initial ideals or initial ideals by Wiebe [Wie04]. We see the relation between our lemma and Wiebe's conditions in the end of this section.

Let A be the polynomial ring $K[x_1, x_2, \dots, x_n]$ over a field K, and fix a term order on A. For an ideal I of A, let $\mathrm{in}(I)$ be the *initial ideal* of I, which is the monomial ideal generated by the initial monomials of the polynomials in I. If a monomial in A is not contained in the initial ideal $\mathrm{in}(I)$, then the monomial is called a *standard monomial* with respect to I. Note that the image of the set of the standard monomials, under the natural surjection $A \to A/I$, forms a linear basis of A/I.

We recall the Lefschetz properties.

Definition 1. (See Watanabe [Wat87], Iarrobino [Iar94].) Let R be a graded ring over a field K, and $R = \bigoplus_{i \geqslant 0} R_i$ its decomposition into homogeneous components with $\dim_K R_i < \infty$. The graded ring R is said to have the *strong* (*respectively weak*) *Lefschetz property*, if there exists an element $l \in R_1$ such that the multiplication map $\times l^s : R_i \to R_{i+s}$ $(f \mapsto l^s f)$ is full-rank for every $i \geqslant 0$ and $s > 0$ (respectively $s = 1$). In this case, l is called a *Lefschetz element*.

Suppose that the Hilbert function of the graded ring R is *symmetric*, that is, $R = \bigoplus_{i=0}^{c} R_i$ and $\dim_K R_i = \dim_K R_{c-i}$ for $i = 0, 1, \dots, \lfloor c/2 \rfloor$. In this case, it is clear that R has the strong Lefschetz property if and only if there exists $l \in R_1$ and $\times l^{c-2i} : R_i \to R_{c-i}$ is bijective for every $i = 0, 1, \dots, \lfloor c/2 \rfloor$. Remark that the coinvariant ring of type H_4, which is studied in the following section, has a symmetric Hilbert function.

The following lemma is the essence of a technique in proving our main theorem.

Lemma 2. *Let $I \subset A = K[x_1, x_2, \dots, x_n]$ be a homogeneous ideal. Take the graded reverse lexicographic order as a term order on A. Then the following two conditions are equivalent*:

(i) *The graded ring A/I has the strong (respectively weak) Lefschetz property, and x_n mod I is a Lefschetz element.*
(ii) *The graded ring $A/\operatorname{in}(I)$ has the strong (respectively weak) Lefschetz property, and x_n mod $\operatorname{in}(I)$ is a Lefschetz element.*

Proof. When x_n is not necessarily a Lefschetz element, we claim

$$\operatorname{in}\bigl(I : x_n^s\bigr) = \operatorname{in}(I) : x_n^s, \tag{1}$$

for $s > 0$. It is obvious that $\operatorname{in}(I : x_n^s) \subset \operatorname{in}(I) : x_n^s$, and we prove the other inclusion. For a monomial $m \in \operatorname{in}(I) : x_n^s$, we can take a homogeneous polynomial h such that $mx_n^s + h \in I$, where $\deg h = \deg(mx_n^s)$ and each term of h is smaller than mx_n^s. Thanks to the graded reverse lexicographic order, h is divisible by x_n^s, and hence $m + h/x_n^s \in I : x_n^s$. This shows that $\operatorname{in}(I : x_n^s) \supset \operatorname{in}(I) : x_n^s$. We thus have proved (1).

When x_n is not necessarily a Lefschetz element, we have the following formula using (1).

$$\begin{aligned}
\operatorname{rank}\bigl(\times x_n^s : (A/I)_i \to (A/I)_{i+s}\bigr) &= \dim_K (A/I)_i - \dim_K \bigl(I : x_n^s/I\bigr)_i \\
&= \dim_K A_i - \dim_K \bigl(I : x_n^s\bigr)_i \\
&= \dim_K A_i - \dim_K \bigl(\operatorname{in}(I) : x_n^s\bigr)_i \\
&= \operatorname{rank}\bigl(\times x_n^s : \bigl(A/\operatorname{in}(I)\bigr)_i \to \bigl(A/\operatorname{in}(I)\bigr)_{i+s}\bigr),
\end{aligned}$$

where the ith homogeneous components are denoted as $(A/I)_i$ and so on. For a homogeneous ideal J of A, the quotient ring A/J has the strong (respectively weak) Lefschetz property with a Lefschetz element x_n, if and only if the linear map $\times x_n^s : (A/J)_i \to (A/J)_{i+s}$ is full-rank for every $i \geqslant 0$ and $s > 0$ (respectively $s = 1$). Therefore it follows from the formula above that (i) and (ii) of the lemma are equivalent, using the fact that $\dim_K (A/J)_i = \dim_K (A/\operatorname{in}(J))_i$. □

In the rest of this section, we clarify the relation between the lemma and other conditions for the Lefschetz properties due to Wiebe. We recall the definition of generic initial ideals. Fix any term order on $A = K[x_1, x_2, \dots, x_n]$. For a homogeneous ideal I of A, there exists a Zariski open subset $U \subset GL(n; K)$ such that the initial ideals of $\varphi(I)$ are equal to each other for any $\varphi \in U$. This initial ideal is uniquely determined, called the *generic initial ideal* of I, and denoted by $\operatorname{Gin}(I)$ (see, e.g. [Eis95, 15.9]). The following proposition gives other conditions for the Lefschetz properties.

Proposition 3. *(See [Wie04, Lemma 2.7, Propositions 2.8 and 2.9].) Let $I \subset A = K[x_1, x_2, \dots, x_n]$ be a homogeneous ideal. We have the following:*

(i) *The graded ring A/I has the strong (respectively weak) Lefschetz property if and only if $A/\operatorname{Gin}(I)$ has the strong (respectively weak) Lefschetz property with respect to the graded reverse lexicographic order. In this case, x_n mod $\operatorname{Gin}(I)$ is a Lefschetz element of $A/\operatorname{Gin}(I)$.*
(ii) *The graded ring A/I has the strong (respectively weak) Lefschetz property if $A/\operatorname{in}(I)$ has the strong (respectively weak) Lefschetz property with respect to any term order.*

The conditions in Proposition 3 and the condition in Lemma 2 relate as follows: Lemma 2 is more practical than Proposition 3(i), since our lemma does not need generic initial ideals. Our lemma gives a necessary and sufficient condition in contrast to Proposition 3(ii). In particular, our lemma can be used for checking that an element is *not* a Lefschetz element.

3. The coinvariant ring of type H_4

The irreducible Coxeter group of type H_4 is of order 14,400 and its root system consists of 120 roots (see [Hum90] for details). In this section, we show that the coinvariant ring R of the irreducible Coxeter group of type H_4 has the strong Lefschetz property. To be concrete, we show that an element is a strong Lefschetz element of R by results calculated with the computer algebra system Macaulay2.

The *coinvariant ring* R has the natural realization

$$\mathbb{R}[x_1, x_2, x_3, x_4]/\big(I_{2k}(x_1, x_2, x_3, x_4) \bigm| k = 1, 6, 10, 15\big),$$

where $I_{2k}(x_1, x_2, x_3, x_4)$ are the polynomials defined as the sum of the $2k$th powers of 60 positive roots, and $I_2, I_{12}, I_{20}, I_{30}$ are the fundamental invariants of H_4 [Meh88, 2.7]. Note that the root system of type H_4 can be realized in $R_1 = \mathbb{R}x_1 + \mathbb{R}x_2 + \mathbb{R}x_3 + \mathbb{R}x_4$. Let

$$\begin{aligned}
\nu_1 &= x_1,\\
\nu_2 &= \tau^2 x_1 + x_2,\\
\nu_3 &= \tau^4 x_1 + \tau^2 x_2 + x_3,\\
\nu_4 &= \big(\tau^3 + \tau\big)x_1 + \tau x_2 + x_4,
\end{aligned}$$

where τ is a root of $\tau^2 - \tau - 1$. We take our candidate λ of a strong Lefschetz element of R as

$$\lambda = \nu_1 + \nu_2 + \nu_3 + \nu_4.$$

Remark 4. When one realizes the root system of type H_4 in R_1, the polynomials ν_1, ν_2, ν_3 and ν_4 span the fundamental Weyl chamber with positive coefficients. In particular, λ is in the fundamental Weyl chamber. We also remark that [MNW06] proves that the set of all strong Lefschetz elements is equal to the union of all Weyl chambers for the irreducible Coxeter groups of types other than H_4, and for type H_4 [MNW06] only proves that any strong Lefschetz element, if it exists, is in a Weyl chamber.

If we use the natural realization of the coinvariant ring as above, then our computation is too complicated for computer algebra systems to give the answer. Thus we transform the ideal $(I_{2k}(x_1, x_2, x_3, x_4) \mid k = 1, 6, 10, 15)$ by the transformation defined by $\mathbb{R}[x_1, x_2, x_3, x_4] \to \mathbb{R}[v_1, v_2, v_3, l]$ $((\nu_1, \nu_2, \nu_3, \lambda) \mapsto (v_1, v_2, v_3, l))$, and let I be the obtained ideal. The ideal I is generated by

$$\begin{aligned}
&I_2\big(v_1, v_2 - \tau^2 v_1, v_3 - \tau^2 v_2, l - v_3 - (\tau + 1)v_2 - (\tau + 1)v_1\big),\\
&I_{12}\big(v_1, v_2 - \tau^2 v_1, v_3 - \tau^2 v_2, l - v_3 - (\tau + 1)v_2 - (\tau + 1)v_1\big),\\
&I_{20}\big(v_1, v_2 - \tau^2 v_1, v_3 - \tau^2 v_2, l - v_3 - (\tau + 1)v_2 - (\tau + 1)v_1\big),\\
&I_{30}\big(v_1, v_2 - \tau^2 v_1, v_3 - \tau^2 v_2, l - v_3 - (\tau + 1)v_2 - (\tau + 1)v_1\big)
\end{aligned}$$

in the polynomial ring $A = \mathbb{R}[v_1, v_2, v_3, l]$. Note that the coinvariant ring R is isomorphic to A/I as graded rings.

We take the reverse lexicographic order such that $v_1 > v_2 > v_3 > l$ as a term order on A. Let S be the set of standard monomials with respect to I, and S_i the set of monomials of degree i in S. By Lemma 2 and the second paragraph of Definition 1, it is enough to show that

$$l^{60-2i} S_i = S_{60-i} \quad \text{for all } i < 30. \tag{2}$$

We would like to prove Eq. (2), but it is difficult to compute Gröbner bases due to intermediate coefficient swells. To avoid this bottleneck, we use the finite field $\mathbb{F}_{13^2}$, i.e., the field obtained by adjoining a root τ of $\tau^2 - \tau - 1$ to $\mathbb{F}_{13}$. Namely we prove Eq. (2) for the ideal I in $\mathbb{F}_{13^2}[v_1, v_2, v_3, l]$, and the following Macaulay2 session verifies Eq. (2):

```
i1: K =  GF(ZZ/13[tau]/(tau^2-tau-1), Variable => tau);
i2: A = K[v1, v2, v3, l];
i3: v4 = l-v1-v2-v3;
i4: x1 =                             v1;
i5: x2 =                v2 -tau*tau*v1;
i6: x3 =    v3 -tau*tau*v2;
i7: x4 =v4         -tau*v2     -tau*v1;
i8: INVs = { 2*x1, 2*x2, 2*x3, 2*x4,
            x1+x2+x3+x4, x1+x2+x3-x4, x1+x2-x3+x4, x1+x2-x3-x4,
            x1-x2+x3+x4, x1-x2+x3-x4, x1-x2-x3+x4, x1-x2-x3-x4,
            tau*x1+(1/tau)*x2+x3, tau*x1+(1/tau)*x2-x3,
            tau*x1-(1/tau)*x2+x3, tau*x1-(1/tau)*x2-x3,
            tau*x1+(1/tau)*x3+x4, tau*x1+(1/tau)*x3-x4,
            tau*x1-(1/tau)*x3+x4, tau*x1-(1/tau)*x3-x4,
            tau*x1+(1/tau)*x4+x2, tau*x1+(1/tau)*x4-x2,
            tau*x1-(1/tau)*x4+x2, tau*x1-(1/tau)*x4-x2,
            tau*x2+(1/tau)*x4+x3, tau*x2+(1/tau)*x4-x3,
            tau*x2-(1/tau)*x4+x3, tau*x2-(1/tau)*x4-x3,
            (1/tau)*x1+x2+tau*x3, (1/tau)*x1+x2-tau*x3,
            (1/tau)*x1-x2+tau*x3, (1/tau)*x1-x2-tau*x3,
            (1/tau)*x1+x3+tau*x4, (1/tau)*x1+x3-tau*x4,
            (1/tau)*x1-x3+tau*x4, (1/tau)*x1-x3-tau*x4,
            (1/tau)*x1+x4+tau*x2, (1/tau)*x1+x4-tau*x2,
            (1/tau)*x1-x4+tau*x2, (1/tau)*x1-x4-tau*x2,
            (1/tau)*x2+x4+tau*x3, (1/tau)*x2+x4-tau*x3,
            (1/tau)*x2-x4+tau*x3, (1/tau)*x2-x4-tau*x3,
            x1+tau*x2+(1/tau)*x3, x1+tau*x2-(1/tau)*x3,
            x1-tau*x2+(1/tau)*x3, x1-tau*x2-(1/tau)*x3,
            x1+tau*x3+(1/tau)*x4, x1+tau*x3-(1/tau)*x4,
            x1-tau*x3+(1/tau)*x4, x1-tau*x3-(1/tau)*x4,
            x1+tau*x4+(1/tau)*x2, x1+tau*x4-(1/tau)*x2,
            x1-tau*x4+(1/tau)*x2, x1-tau*x4-(1/tau)*x2,
            x2+tau*x4+(1/tau)*x3, x2+tau*x4-(1/tau)*x3,
            x2-tau*x4+(1/tau)*x3, x2-tau*x4-(1/tau)*x3 };
```

```
i9: I2 = k ->(sum (set apply(INVs, lf-> lf^(2*k))));
i10: I = ideal (I2(1), I2(6), I2(10), I2(15));
i11: R = A/I;
i12: S = apply(61,k -> first entries(basis({k},R)));
i13: scan(30,i ->(
        S' = apply(S_i, m -> m*(l^(60-2*i)) );
        << "  l^" << 60-2*i << " S_" << i << " = S_" << 60-i;
        << " is " << ( S' == S_(60-i) ) << endl;
     ));
```

In this session, we compute the following: In `i1`, we define the field K obtained by adjoining a root τ of $\tau^2 - \tau - 1$ to $\mathbb{F}_{13}$. In `i2`, we define the polynomial ring A. From `i3` to `i10`, we define the ideal I. In `i11`, we define the coinvariant ring $R = A/I$. In `i12`, we define the set S_k of the standard monomials of degree k. In `i13`, for each $i < 30$, we calculate $S' = l^{60-2i} S_i$ and compare S' with S_{60-i}.

It is easy to see that the strong Lefschetz property for the coefficient ring $\mathbb{F}_{13^2}$ yields that for $\mathbb{Z}[\tau]$, and then that for $\mathbb{R}$. Thus we have our main result.

Theorem 5. *The coinvariant ring of the irreducible Coxeter group of type* H_4 *has the strong Lefschetz property.*

4. Final remarks

Here we summarize the techniques that we used for the computation of the main theorem. As stated in Section 3, the main technique is to take a Lefschetz element as the last variable under the graded reverse lexicographic order. By this technique we do not need heavy computations except for a single computation of a Gröbner basis. Otherwise we need many reductions of polynomials of degrees up to 60 and computations of large determinants with rational entries. In addition, we use the finite field $\mathbb{F}_{13^2}$ instead of the rational number field, which is a common technique in computer algebra systems. When we use none or one of these techniques, the computation is too complicated for computers. The computer algebra system Macaulay2 returns the answer only when we use both techniques, and the computation takes less than 10 seconds in this case.

Acknowledgment

The authors would like to express their sincere thanks to Junzo Watanabe for his valuable comments during preparation of this paper.

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