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A note on some equalities for frames in Hilbert spaces

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ABSTRACT

Some equalities for frames involving the real parts of some complex numbers have been recently established in [P. Găvruta, On some identities and inequalities for frames in Hilbert spaces, J. Math. Anal. Appl., 321 (2006) 469–478]. In the current note, we generalize the equalities to a more general form which does not involve the real parts of the complex numbers.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [1] to address some very deep problems in nonharmonic Fourier series (see [2]). Basically, Duffin and Schaeffer abstracted the fundamental notion of Gabor frames for studying signal processing. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossmann and Meyer [3]. Since then the theory of frames began to be more widely studied (see [4,5]). Frames provide unconditional basis-like, but generally nonunique, representations of vectors in a Hilbert space. The redundancy and flexibility offered by frames has spurred their application in a variety of areas throughout mathematics and engineering, such as wireless communications [6], σ–δ quantization [7] and image processing [8].

We need recall the definition and some properties of frames in Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space and J be a countable index set. A frame for $\mathcal{H}$ is a sequence $\{f_j : j \in J\}$ such that there are two positive constants A and B satisfying

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le B\|f\|^2 \tag{1}$$

for all $f \in \mathcal{H}$. The constants A and B are called lower and upper frame bounds, respectively. If $A = B$, then this frame is called an A-tight frame, and if $A = B = 1$, then it is called a Parseval frame.

Associated with each frame $\{f_j : j \in J\}$ there are three linear and bounded operators:

$$\begin{array}{llll} \text{synthesis operator} & T : l^2(J) \longrightarrow \mathcal{H}, & T(\{c_j\}_{j\in J}) = \sum_{j\in J} c_j f_j, \\ \text{analysis operator} & T^* : \mathcal{H} \longrightarrow l^2(J), & T^* f = \{\langle f, f_j\rangle\}_{j\in J}, \\ \text{frame operator} & S : \mathcal{H} \longrightarrow \mathcal{H}, & Sf = TT^* f = \sum_{j\in J} \langle f, f_j\rangle f_j. \end{array}$$

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Moreover, T^* is the adjoint of T and S is a self-adjoint positive invertible operator in $\mathcal{H}$. The frame operator S leads to the frame reconstruction formula

$$f = \sum_{j\in J}\langle f, f_j\rangle S^{-1}f_j = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j, \quad \forall f \in \mathcal{H}, \tag{2}$$

where the collection $\{\tilde{f}_j \equiv S^{-1}f_j : j \in J\}$ is also a frame for H, which is called the canonical dual frame of $\{f_j : j \in J\}$.

In general, the frame $\{g_j : j \in J\}$ for $\mathcal{H}$ is called an alternate dual frame of $\{f_j : j \in J\}$ if $\forall f \in \mathcal{H}$,

$$f = \sum_{j\in J}\langle f, g_j\rangle f_j. \tag{3}$$

For basic results on frames, see [2,4,5,9,10].

In [11], the authors verified a longstanding conjecture of the signal processing community: a signal can be reconstructed without information about the phase. While working on efficient algorithms for signal reconstruction, the authors of [12] established the remarkable Parseval frame equality given below.

Theorem 1.1. *If $\{f_j : j \in J\}$ is a Parseval frame for $\mathcal{H}$, then $\forall K \subset J$ and $\forall f \in \mathcal{H}$,*

$$\sum_{j\in K}|\langle f, f_j\rangle|^2 - \left\|\sum_{j\in K}\langle f, f_j\rangle f_j\right\|^2 = \sum_{j\in K^c}|\langle f, f_j\rangle|^2 - \left\|\sum_{j\in K^c}\langle f, f_j\rangle f_j\right\|^2,$$

where $K^c = J \setminus K$.

Recently, Theorem 1.1 was generalized to alternate dual frames [13]. The following form was given in [13].

Theorem 1.2. *If $\{f_j : j \in J\}$ is a frame for $\mathcal{H}$ and $\{g_j : j \in J\}$ is an alternate dual frame of $\{f_j : j \in J\}$, then $\forall K \subset J$ and $\forall f \in \mathcal{H}$,*

$$Re\left(\sum_{j\in K}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in K}\langle f, g_j\rangle f_j\right\|^2 = Re\left(\sum_{j\in K^c}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in K^c}\langle f, g_j\rangle f_j\right\|^2. \tag{4}$$

In this note, we generalize the equality (4) to a more general form which does not involve the real parts of the complex numbers.

2. The main result and its proof

We first give a simple result for operators.

Lemma 2.1. *Let P and Q be two linear bounded operators on $\mathcal{H}$ such that $P + Q = I$; then*

$$P - P^*P = Q^* - Q^*Q,$$

where I denotes the identity operator on $\mathcal{H}$.

Proof. A simple computation shows that

$$P - P^*P = (I - P^*)P = Q^*(I - Q) = Q^* - Q^*Q. \quad \square$$

Now, the main result of this note is stated as follows.

Theorem 2.2. *Let $\{f_j : j \in J\}$ be a frame for $\mathcal{H}$ and $\{g_j : j \in J\}$ be an alternate dual frame of $\{f_j : j \in J\}$; then $\forall K \subset J$ and $\forall f \in \mathcal{H}$,*

$$\left(\sum_{j\in K}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in K}\langle f, g_j\rangle f_j\right\|^2 = \overline{\left(\sum_{j\in K^c}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right)} - \left\|\sum_{j\in K^c}\langle f, g_j\rangle f_j\right\|^2. \tag{5}$$

Proof. For $K \subset J$, the operator U_K is defined by

$$U_K f = \sum_{j\in K}\langle f, g_j\rangle f_j, \quad f \in \mathcal{H}.$$

Then it is easy to prove that the operator U_K is well defined and the series $\sum_{j\in K}\langle f, g_j\rangle f_j$ converges unconditionally. By (3), $U_K + U_{K^c} = I$. Thus, by Lemma 2.1 we have

$$\begin{aligned}\left(\sum_{j\in K}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in K}\langle f, g_j\rangle f_j\right\|^2 &= \langle U_K f, f\rangle - \langle U_K^* U_K f, f\rangle \\ &= \langle U_{K^c}^* f, f\rangle - \langle U_{K^c}^* U_{K^c} f, f\rangle \\ &= \overline{\langle f, U_{K^c} f\rangle - \|U_{K^c} f\|^2} \\ &= \overline{\left(\sum_{j\in K^c}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right)} - \left\|\sum_{j\in K^c}\langle f, g_j\rangle f_j\right\|^2.\end{aligned}$$

Hence (5) holds. The proof is completed. □

It is easy to see that the result from Theorem 1.2 is obtained if we take the real part on both sides of (5). In fact, we can give a more general result.

Theorem 2.3. *Let $\{f_j : j \in J\}$ be a frame for $\mathcal{H}$ and $\{g_j : j \in J\}$ be an alternate dual frame of $\{f_j : j \in J\}$; then for every bounded sequence $\{b_j : j \in J\}$ and $\forall f \in \mathcal{H}$,*

$$\left(\sum_{j\in J} b_j\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in J} b_j\langle f, g_j\rangle f_j\right\|^2 = \overline{\left(\sum_{j\in J}(1-b_j)\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right)} - \left\|\sum_{j\in J}(1-b_j)\langle f, g_j\rangle f_j\right\|^2. \tag{6}$$

The proof of Theorem 2.3 is immediate. Obviously, for every $K \subset J$, taking $b_j = 1$ if $j \in K$ and 0 if $j \in K^c$ in (6), we then obtain the equality (5). On the other hand, we can get the equality in Theorem 3.3 of [13] by taking the real part on both sides of (6).

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References

[1] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341–366.
[2] R. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New Work, 1980.
[3] I. Daubechies, A. Grossman, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.
[4] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[5] P.G. Casazza, The art of frame theory, Taiwanese J. Math. 4 (2000) 129–201.
[6] T. Strohmer, R.W. Heath Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
[7] J.J. Benedetto, A. Powell, O. Yilmaz, Sigma–Delta ($\Sigma\Delta$) quantization and finite frames, IEEE Trans. Inform. Theory 52 (2006) 1990–2005.
[8] E.J. Candès, D.L. Donoho, Continuous curvet transform: II. Discretization and frames, Appl. Comput. Harmon. Anal. 19 (2005) 198–222.
[9] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.
[10] D. Han, D.R. Larson, Frames, bases and group representations, Mem. Amer. Math. Soc. 147 (2000).
[11] R. Balan, P.G. Casazza, D. Edidin, On signal reconstruction without phase, Appl. Comput. Harmon. Anal. 20 (2006) 345–356.
[12] R. Balan, P.G. Casazza, D. Edidin, G. Kutyniok, A new identity for Parseval frames, Proc. Amer. Math. Soc. 135 (2007) 1007–1015.
[13] P. Găvruta, On some identities and inequalities for frames in Hilbert spaces, J. Math. Anal. Appl. 321 (2006) 469–478.