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# JEU DE TAQUIN AND CONNECTED STANDARD SKEW TABLEAUX

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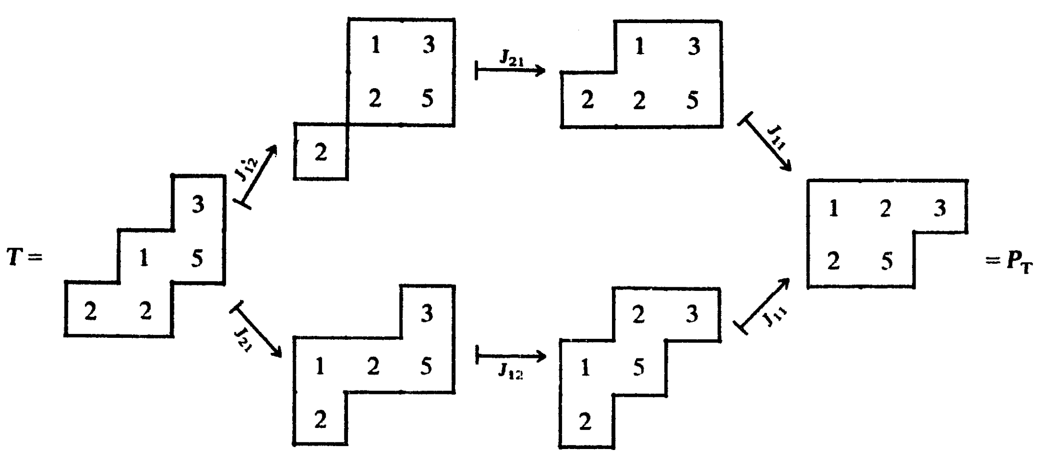
## 1. Introduction

Schützenberger's 'jeu de taquin' is a construction which step by step reduces a standard skew tableau $T$ to a standard tableau $P_T$ [1; 2]. This reduction process is not unique and depends on the choices of upper corner squares [2, p. 108]. Nevertheless, Schützenberger and Thomas independently proved that the resulting tableau $P_T$ is uniquely determined by $T$.

If we perform the jeu de taquin subject to a standard skew tableau $T$ and an upper corner square $(i, k)$ of $T$, we get a new standard skew tableau, which will be denoted by $J_{ik}(T)$.

**Example.** See Fig. 1.



When we started to study Schützenberger's construction, we had the vague feeling that in every case the taquin game transports connectedness, i.e. if $T$ is a connected standard skew tableau, then $J_{ik}(T)$ is connected, for every upper corner square $(i, k)$ of $T$. But the above $T$ provides a counterexample: $T$ is connected, whereas $J_{12}(T)$ has two connected components.

A suitable modification of our conjecture led us to the following.

**Theorem.** *If $T$ is a connected standard skew tableau, then there exists an upper corner square $(i, k)$ of $T$ such that $J_{ik}(T)$ is connected.*

We prepare the proof of this theorem by some definitions.

## 2. Definitions

A *skew tableau* $T$ is a mapping, whose domain, denoted by $|T|$, is the set-theoretic difference of two Young diagrams [2] and whose range is some finite set of positive integers. $T$ is *standard*, iff the entries in $T$ are weakly increasing from left to right in each row and strictly increasing from top to bottom in each column of $T$.

The transition from a standard skew tableau $T$ to $J_{ik}(T)$ specifies a finite sequence

$$(*) \qquad (i, k) =: (i_0, k_0), (i_1, k_1) \ldots, (i_r, k_r),$$

which can be interpreted as the path of a pawn starting at $(i, k)$ and traversing the standard skew tableau $T$ subject to the 'taquin rules' [1; 2, p. 109]. Such a sequence will be called the *taquin path* of $T$ with respect to the upper corner square $(i, k)$, or for short: the *$(i, k)$-taquin path of* $T$. In such a path $(*)$, $(i_{p+1}, k_{p+1})$ equals either $(i_p + 1, k_p)$ or $(i_p, k_p + 1)$, $0 \leq p < r$. We shall call a taquin path $(*)$

*horizontal*, iff $i_0 = \cdots = i_r$,
*vertical*, iff $k_0 = \cdots = k_r$, and
*zigzag*, iff $i_0 \neq i_r$ and $k_0 \neq k_r$.

In the above example $((1, 2), (2, 2), (3, 2))$ and $((2, 1), (2, 2), (3, 2))$ are all possible taquin paths of $T$; the first one is vertical, and the second one is zigzag.

Finally we need the notion of a hook. Let $T$ be a skew tableau, $(a, b) \in |T|$. Then

$$H_{ab}(T) := \{(c, d) \in |T|: \ (c = a \text{ and } d \geq b) \text{ or } (d = b \text{ and } c \geq a)\}$$

is the *hook* of $T$ with *corner* $(a, b)$.

The element $(a, h) \in H_{ab}(T)$ (resp. $(f, b) \in H_{ab}(T)$) is called the *hand* (resp. *foot*) of this hook, iff $(a, h+1) \notin H_{ab}(T)$ (resp. $(f+1, b) \notin H_{ab}(T)$).

## 3. Proof of the Theorem

In order to prove the Theorem, we distinguish several cases. In each case we can find hooks which guarantee the connectedness of some $J_{ik}(T)$.

*Case* 1. There is an $(i, k)$-taquin path of $T$ which is zigzag.

*Claim.* $J_{ik}(T)$ is connected.

To prove this, let us consider the hook $H_{ik}(J_{ik}(T))$. The crucial property of this hook is the fact that its foot coincides with the foot of $H_{i+1,k}(T)$ and its hand coincides with the hand of $H_{i,k+1}(T)$; hence $J_{ik}(T)$ is connected.

*Case* 2. No taquin path of $T$ is zigzag.

Here we distinguish three subcases. First of all, it is convenient to label all upper corner squares of $T$ lexicographically:

$$(i^{(1)}, k^{(1)}) < \cdots < (i^{(s)}, k^{(s)}).$$

*Case* 2.1. The taquin path of $T$ with respect to $(i, k) := (i^{(1)}, k^{(1)})$ is horizontal.

*Claim.* $J_{ik}(T)$ is connected.

If $i = 1$, our claim holds. Now suppose $i > 1$. Since $T$ is connected, $\{(i, k+1), (i+1, k), (i+1, k+1)\}$ is a subset of $|T|$. Furthermore, since $J_{ik}(T)$ is a skew tableau and the $(i, k)$-taquin path of $T$ is horizontal, even $(i, k+2) \in |T|$. Hence $H_{1,k+1}(T) \subseteq |J_{ik}(T)|$. Our claim follows.

*Case* 2.2. The taquin path of $T$ with respect to $(i, k) := (i^{(s)}, k^{(s)})$ is vertical.

*Claim.* $J_{ik}(T)$ is connected.

*Proof.* Similar to Case 2.1.

*Case* 2.3. The $(i^{(1)}, k^{(1)})$-taquin path of $T$ is vertical and the $(i^{(s)}, k^{(s)})$-taquin path of $T$ is horizontal.

In this case there exists a $p$, $1 \leq p < s$, such that the taquin path of $T$ with respect to $(i^{(p)}, k^{(p)}) =: (i, k)$ is vertical and the taquin path of $T$ with respect to $(i^{(p+1)}, j^{(p+1)}) =: (i', j')$ is horizontal.

*Claim.* $J_{ik}(T)$ and $J_{i'k'}(T)$ are connected.

Similarly to Case 2.1 one shows that both $\{(i', k'+1), (i'+1, k'), (i'+1, k'+1), (i', k'+2)\}$ and $\{(i, k+1), (i+1, k), (i+1, k+1), (i+2, k)\}$ are subsets of $|T|$. Hence the hooks of $T$, $J_{ik}(T)$ and $J_{i'k'}(T)$ with common corner $(i+1, k'+1)$ are equal; this implies the connectedness of $J_{ik}(T)$ and $J_{i'k'}(T)$.

This completes the proof of the Theorem.

## References

[1] A. Lascoux and M.P. Schützenberger, Le monoïde plaxique, Quaderni de la Ricera Scientifica 109 (1981) 129–156.

[2] G.P. Thomas, On a construction of Schützenberger; Discrete Math. 17 (1977) 107–118.