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Extended Affine Surface Area

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During the twentieth century many of the notions of classical Differential Geometry have been extended so as to be defined on the boundaries of arbitrary convex bodies (without smoothness requirements). For example, the projection measures (Quermassintegrals) are the natural extensions of the integrals of mean curvature (see Santaló [27, pp. 215–232] for a discussion). The surface area measures of Aleksandrov–Fenchel–Jessen are extensions of (the indefinite integrals of) the elementary symmetric functions of the principal radii of curvature, while Federer's curvature measures (when restricted to convex hypersurfaces) are extensions of (the indefinite integrals of) the elementary symmetric functions of the principal curvatures (see Schneider [30, pp. 27–33]).

One of the important concepts of classical Differential Geometry, which has been difficult to extend is affine surface area. This has often led to difficulties which when circumvented, have been done so only by considerable effort and ingenuity. For example, suppose we are given a measure on the unit sphere, which satisfies the hypothesis of the Minkowski Problem. The solution of the Minkowski Problem guarantees the existence of a convex body whose surface area measure is the given measure. From the given measure we can find the surface area of the body. The Minkowski (Mixed Volume) Inequality can be used to obtain various upper bounds for the volume of the body. However, we cannot even inquire about the affine surface area of the body, unless we can somehow convince ourselves that the convex body is sufficiently smooth.

Results regarding affine surface area have a surprising number of applications. For example, a number of geometric inequalities which appear to have no connection with affine surface area are in fact consequences of inequalities involving affine surface area (see, for example, Petty [25] and also [15]). Since polytopes have zero affine surface area, it seems quite surprising that the notion of affine surface area arises frequently in the area of polytope approximation (see the survey of Gruber [9]).

Recently, Leichtweiss [11, 12] presented a definition of extended affine

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surface area. The approach to this problem which is taken in this article is very different. The advantage of this approach is that the proofs of the basic properties of extended (and classical) affine surface area are greatly simplified. In addition, with this approach all of the powerful affine isoperimetric inequalities of Petty [22, 24, 25] which involve affine surface area, can now be established for arbitrary convex bodies. Another advantage is that a proof of the conjectured upper-semicontinuity of classical (as well as extended) affine surface area can now be given.

To define this extended affine surface area, generalized convex bodies are introduced. In this generalized setting, simple proofs can be given of the basic properties (such as affine invariance and Blaschke concavity) of classical as well as extended affine surface area. A great deal of freedom is available in this new setting. For example, taking the polars of star-shaped sets is perfectly permissible. In fact, the flexibility to be able to do such things is the reason that generalized convex bodies are introduced.

Since interest in affine surface area is not limited to specialists in geometric convexity, the author has attempted to write a reasonably self-contained article. Toward that end, and for quick reference, a number of elementary definitions (Minkowski and Blaschke addition, mixed volumes, surface area measures) and well-known results (the weak solution of the Minkowski Problem, the Blaschke–Santaló Inequality) are stated in Section 0. The survey of Schneider [30], as well as the texts of Busemann [6], Leichtweiss [10], and Santaló [27], are recommended as references. For reference regarding dual mixed volumes see [13, 17, 18] and Burago and Zalgaller [4, pp. 158–160]. It should be noted that nothing listed in Section 0 is new.

In Section 1 a list is given of various properties of classical affine surface area which extended affine surface area should have. Extended affine surface area as defined in this article satisfies all these requirements. In addition, all the well-known inequalities which involve affine surface area (with their equality condition) will be shown to hold, for arbitrary convex bodies. It should be noted that what is called "classical affine surface area" in this article, is in fact already an extension of classical affine surface area (to bodies with positive continuous curvature functions). The affine isoperimetric inequality (of Affine Differential Geometry) is due to Blaschke (see [2]) and Santaló [26]. The (extended) form of the affine isoperimetric inequality which is stated in Section 1 is due to Petty [25]. The properties of classical affine surface area which are listed in Section 1, can be found in (or derived from) the works of Blaschke [2], Santaló [26], and Petty [25] (see also [16]). It should be noted that nothing listed in Section 1 will be used, in any way, in this article. Rather, it will be shown that all the properties (of classical affine surface area) listed in Section 1, hold for arbitrary convex bodies.

Polar curvature images of star bodies are defined in Section 2. Various results which are required later, regarding these polar curvature images, are also obtained in this section.

Generalized convex bodies are introduced in Section 3. A number of the results obtained in this section will not be used in this article. They are presented because they might be of (independent) interest. Definitions are given for the volume (and mixed volume) of generalized convex bodies. Also defined are Minkowski sums and affine images of these bodies. The Brunn–Minkowski and Minkowski inequalities remain valid in this generalized setting. Also presented is an extension, to star-shaped bodies, of the Blaschke–Santaló inequality (with equality conditions).

Extended affine surface area is defined in Section 4. Here, the basic properties of extended affine surface area are established. It should be noted that these properties are obtained without assuming that they have already been established for classical affine surface area—in fact it is not assumed that classical affine surface area has even been defined. An extended version (valid for all convex bodies) of the affine isoperimetric inequality of Affine Differential Geometry (with equality conditions) is proven in Section 4. At the end of this section, it is shown that on convex bodies for which classical affine surface area is defined, extended and classical affine surface area agree.

In [16] the author introduced mixed affine surface areas. In Section 5 it is shown how (a simple extension of) one of the mixed affine surface areas can be used to define extended affine surface area. While such a definition avoids the use of generalized convex bodies, it has some undesirable qualities: A number of proofs become much less transparent. In addition, such a definition would assume that affine surface area has already been defined on bodies with continuous positive curvature functions. Finally, such a definition has the (computational) deficiency of being based on a class of bodies which are only defined implicitly.

Petty [24] has introduced the important concept of geominimal surface area. By exploiting the duality between centroids and Santaló points, it is possible to choose a definition of geominimal surface area which involves centroids rather than (the much less familiar concept of) Santaló points. This is the definition used in Section 6—It involves only a trivial modification of Petty's definition. In Section 6 it is shown that Petty's important inequality between affine surface area and geominimal surface area can now be established for all convex bodies. At the end of this section, a new representation of geominimal surface area is obtained (which involves mixed affine surface areas).

The definition of projection bodies (zonoids) is restated in Section 7. The excellent survey of Schneider and Weil [31] should be consulted for this topic. Petty [22] established a fundamental inequality between the affine surface area of a convex body (with positive continuous curvature function) and the volume of the projection body of the convex body. Petty's affine projection inequality is closely related to both the Petty projection inequality [23] and the Busemann–Petty centroid inequality [5, 21] (see [15] for a discussion). An extension (valid for all convex bodies) of Petty's affine projection inequality (with equality conditions) is given in Section 7.

In [24], Petty obtained a strong generalization of the monotonicity result of Winternitz (see [2, p. 200]) regarding affine surface area. It is shown in Section 8 that Petty's generalization will hold for arbitrary convex bodies. If the areas of the projections of a convex body do not exceed those of an ellipsoid, then the affine surface area of the body does not exceed that of the ellipsoid. For bodies with positive continuous curvature functions, this result, and a strong generalization, were proven in [18]. In Section 8, this result, and the generalization, are established for arbitrary convex bodies.

0. Background and Notation

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean n-space, $\mathbb{R}^n$.

Associated with a body $K \in \mathscr{K}^n$, is its support function, $h_K: \mathbb{R}^n \to \mathbb{R}$, defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \operatorname{Max}\{x \cdot y : y \in K\}, \tag{0.1}$$

where $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$.

For $\phi \in SL(n)$, write ϕ^{-1}, ϕ^t, and ϕ^{-t}, for the inverse, transpose, and transpose of the inverse of ϕ. From definition (0.1), it follows immediately that for $K \in \mathscr{K}^n$, $x \in \mathbb{R}^n$, and $\phi \in SL(n)$,

$$h_{\phi K}(x) = h_K(\phi^t x). \tag{0.2}$$

For $K, L \in \mathscr{K}^n$, and $\alpha, \gamma \geqslant 0$ (not both 0), the Minkowski linear combination $\alpha K + \gamma L \in \mathscr{K}^n$, can be defined as the body whose support function is given by

$$h_{\alpha K + \gamma L} = \alpha h_K + \gamma h_L. \tag{0.3}$$

From (0.2) and (0.3), it follows that for $\phi \in SL(n)$,

$$\phi(\alpha K + \gamma L) = \alpha \phi K + \gamma \phi L. \tag{0.4}$$

The Hausdorff metric, δ, on $\mathscr{K}^n$ can be defined as follows: For K, $L \in \mathscr{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ is the max-norm on the space $C(S^{n-1})$ of continuous functions on the unit sphere.

Lebesgue measure on $\mathbb{R}^n$ will be denoted by V, and for Lebesgue measure on the unit sphere S^{n-1} write S. The unit ball in $\mathbb{R}^n$ will be denoted by B, and let $\omega_n = V(B)$.

For $K, L \in \mathscr{K}^n$, the mixed volume $V_1(K, L)$ is defined by

$$nV_1(K, L) = \lim_{\varepsilon \downarrow 0} \frac{V(K+\varepsilon L) - V(K)}{\varepsilon}. \tag{0.5}$$

Obviously,

$$V_1(K, K) = V(K). \tag{0.6}$$

From definition (0.5), it follows easily that for $K, L, M \in \mathscr{K}^n$, $x, y \in \mathbb{R}^n$, and $\alpha, \gamma > 0$,

$$V_1(x+\alpha K, y+\gamma L) = \alpha^{n-1}\gamma V_1(K, L), \tag{0.7}$$

and

$$V_1(M, \alpha K+\gamma L) = \alpha V_1(M, K) + \gamma V_1(M, L). \tag{0.8}$$

From (0.4) and definition (0.5), it follows that, for $\phi \in SL(n)$, $V_1(\phi K, \phi L) = V_1(K, L)$, or equivalently, that

$$V_1(K, \phi L) = V_1(\phi^{-1}K, L). \tag{0.9}$$

A body $K \in \mathscr{K}^n$ is said to have a positive continuous curvature function, $f_K: S^{n-1} \to (0, \infty)$, provided the integral representation

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, f_K(u)\, dS(u), \tag{0.10}$$

holds for all $L \in \mathscr{K}^n$. As an aside, we note that if the boundary of K is of class C^2, and has positive Gaussian curvature, then the reciprocal Gaussian curvature (as a function of the outer normals) is the curvature function of K. Let $\mathscr{F}^n$ denote the class of convex bodies which have a positive continuous curvature function.

It will be convenient to extend the definition of the curvature function of a body $K \in \mathscr{F}^n$, so that f_K is defined on $\mathbb{R}^n \backslash \{0\}$, as a homogeneous function of degree $-(n+1)$: For $u \in S^{n-1}$, and $\lambda > 0$, let

$$f_K(\lambda u) = \lambda^{-(n+1)} f_K(u). \tag{0.11}$$

Associated with a convex body $K \in \mathscr{K}^n$, is a regular Borel measure S_K on S^{n-1}, called the surface area measure of K, with the property that the integral representation,

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, dS_K(u), \tag{0.12}$$

holds for all $L \in \mathscr{K}^n$. As an aside, we note that there is a simple geometric description of the measure S_K: If $A \subset S^{n-1}$ is a Borel set, then $S_K(A)$ is just the $(n-1)$-dimensional Hausdorff measure of the set of points of ∂K which have an outer unit normal vector in A.

From (0.10) and (0.12), it follows that, for $K \in \mathscr{F}^n$,

$$f_K = dS_K/dS, \tag{0.13}$$

where the derivative is the Radon–Nikodym derivative of the surface area measure of K, with respect to spherical Lebesgue measure.

From (0.7), and definitions (0.10) and (0.12), it follows immediately that for $K \in \mathscr{K}^n$, $x \in \mathbb{R}^n$, and $\lambda > 0$,

$$S_{x+\lambda K} = \lambda^{n-1} S_K, \tag{0.14a}$$

and, if $K \in \mathscr{F}^n$,

$$f_{x+\lambda K} = \lambda^{n-1} f_K. \tag{0.14b}$$

From the translation invariance of mixed volumes (0.7), it follows that the centroid of the surface area measure of a convex body is at the origin; i.e., for $K \in \mathscr{K}^n$,

$$\int_{S^{n-1}} u\, dS_K(u) = 0. \tag{0.15a}$$

In particular, if $K \in \mathscr{F}^n$,

$$\int_{S^{n-1}} u f_K(u)\, dS(u) = 0. \tag{0.15b}$$

Conversely, the (weak) solution of the Minkowski problem states that if a Borel measure μ on S^{n-1} is not concentrated on a great sphere and has the property that

$$\int_{S^{n-1}} u d\mu(u) = 0, \tag{0.16a}$$

then there exists a body $K \in \mathscr{K}^n$, unique up to translation, such that $S_K = \mu$. The body K is called the solution of the Minkowski problem for the measure μ. Thus, given a continuous function $f: S^{n-1} \to (0, \infty)$, such that

$$\int_{S^{n-1}} uf(u)\, dS(u) = 0, \tag{0.16b}$$

there exists a body $K \in \mathscr{F}^n$, such that $f_K = f$.

Suppose $K, L \in \mathscr{K}^n$, and $\alpha, \gamma \geqslant 0$ (not both 0). From (0.15a) it follows that the measure $\alpha S_K + \gamma S_L$ satisfies (0.16a). The Blaschke linear combination $\alpha \cdot K + \gamma \cdot L$ is the solution to the Minkowski problem for the measure $\lambda S_K + \mu S_L$; i.e.,

$$S_{\alpha \cdot K + \gamma \cdot L} = \alpha S_K + \gamma S_L. \tag{0.17a}$$

If $K, L \in \mathscr{F}^n$, then obviously

$$f_{\alpha \cdot K + \gamma \cdot L} = \alpha f_K + \gamma f_L. \tag{0.17b}$$

From (0.14a), and definition (0.17a), it is easy to see that the relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{0.18}$$

From (0.12) and (0.17a), it follows that for $K, L, M \in \mathscr{K}^n$, and $\alpha, \gamma \geqslant 0$,

$$V_1(\alpha \cdot K + \gamma \cdot L, M) = \alpha V_1(K, M) + \gamma V_1(L, M). \tag{0.19}$$

Let $\mathscr{K}_0^n$ denote the class of convex bodies which contain the origin in their interiors. For $K \in \mathscr{K}_0^n$, use K^* to denote the polar body of K (with respect to the unit sphere centered at the origin):

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, \text{ for all } y \in K\}. \tag{0.20}$$

It is easily verified that

$$K^{**} = K. \tag{0.21}$$

It is also easily verified (see, for example, [22] or [18]) that for $K \in \mathscr{K}_0^n$, and $\phi \in SL(n)$,

$$(\phi K)^* = \phi^{-t} K^*. \tag{0.22}$$

For $K \in \mathscr{K}^n$, let $\text{Cen}(K)$ denote the centroid of K. The Santaló point of K can be defined as the point $s \in \text{int}\, K$, for which

$$\text{Cen}((-s + K)^*) = 0. \tag{0.23}$$

Let $\mathscr{K}_c^n$ denote the class of convex bodies whose centroid is at the origin. Let $\mathscr{F}_c^n$ denote the class of bodies in $\mathscr{K}_c^n$ which have positive continuous curvature functions.

Since a Blaschke linear combination is defined only up to translation, we can fix the linear combination by requiring the Blaschke linear combination to have the origin as its centroid. This would make both $\mathscr{K}_c^n$ and $\mathscr{F}_c^n$ closed with respect to Blaschke linear combinations.

The Blaschke–Santaló inequality (see [1, 26, 25, 7]) states that for $K \in \mathscr{K}_c^n$,

$$V(K)\, V(K^*) \leqslant \omega_n^2, \tag{0.24}$$

with equality if and only if K (or, equivalently, K^*) is an ellipsoid.

A compact subset of $\mathbb{R}^n$ is said to be star-shaped about the origin if it contains the line segment joining any of its points to the origin. Associated with a compact set K, which is star shaped about the origin, is its radial function, $\rho_K: S^{n-1} \to \mathbb{R}$, defined, for $u \in S^{n-1}$, by

$$\rho_K(u) = \operatorname{Max}\{\lambda \geqslant 0 : \lambda u \in K\}. \tag{0.25}$$

If ρ_K is a positive continuous function, K is called a star body, and $\mathscr{S}_0^n$ will be used to denote the class of star bodies in $\mathbb{R}^n$. It will be convenient to extend ρ_K from a function on S^{n-1} to a function on $\mathbb{R}^n \backslash \{0\}$, by making it homogeneous of degree -1; i.e., for $u \in S^{n-1}$, and $\lambda > 0$, let

$$\rho_K(\lambda u) = \lambda^{-1} \rho_K(u). \tag{0.26}$$

From (0.25), it follows immediately that for $K \in \mathscr{S}_0^n$, $\phi \in SL(n)$,

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1} x), \tag{0.27}$$

for all $x \in \mathbb{R}^n \backslash \{0\}$.

From (0.1) and (0.25), it follows that for $K \in \mathscr{K}_0^n$,

$$\rho_{K^*} = 1/h_K \qquad \text{and} \qquad h_{K^*} = 1/\rho_K. \tag{0.28}$$

Use $\mathscr{S}_c^n$ to denote the class of star bodies which have their centroids at the origin. Hence, $K \in \mathscr{S}_c^n$, if and only if,

$$\int_K x\, dV(x) = 0,$$

or, equivalently (see, for example, [8, p. 250]), if and only if,

$$\int_{S^{n-1}} u \rho_K(u)^{n+1}\, dS(u) = 0. \tag{0.29}$$

For $K, L \in \mathscr{S}_0^n$, and $\alpha, \gamma \geqslant 0$ (not both 0), the harmonic linear combination, $\alpha \blacklozenge K \hat{+} \gamma \blacklozenge L \in \mathscr{S}_0^n$ is defined by

$$1/\rho_{\alpha \blacklozenge K \hat{+} \gamma \blacklozenge L} = \alpha/\rho_K + \gamma/\rho_L. \tag{0.30}$$

Note that from (0.3), (0.28), and (0.30) it follows that if $K, L \in \mathscr{K}_0^n$, then

$$\alpha \blacklozenge K \hat{+} \gamma \blacklozenge L = (\alpha K^* + \gamma L^*)^*.$$

From (0.27) and (0.30), it follows immediately that for $\phi \in SL(n)$,

$$\phi(\alpha \blacklozenge K \hat{+} \gamma \blacklozenge L) = \alpha \blacklozenge \phi K \hat{+} \gamma \blacklozenge \phi L. \tag{0.31}$$

For $K, L \in \mathscr{S}_0^n$, the dual mixed volume $\tilde{V}_{-1}(K, L)$ can be defined by

$$n\tilde{V}_{-1}(K, L) = \lim_{\varepsilon \downarrow 0} \frac{V(K) - V(K \hat{+} \varepsilon \blacklozenge L)}{\varepsilon}. \tag{0.32}$$

Obviously,

$$\tilde{V}_{-1}(K, K) = V(K). \tag{0.33}$$

From (0.31), and definition (0.32), it follows that the dual mixed volume $\tilde{V}_{-1}$ is invariant under (simultaneous) unimodular centro-affine transformation; i.e., if $K, L \in \mathscr{S}_0^n$, and $\phi \in SL(n)$, then $\tilde{V}_{-1}(\phi K, \phi L) = \tilde{V}_{-1}(K, L)$, or equivalently,

$$\tilde{V}_{-1}(K, \phi L) = \tilde{V}_{-1}(\phi^{-1} K, L). \tag{0.34}$$

From the polar coordinate formula for volume, and definitions (0.30) and (0.32), one obtains the following integral representation of the dual mixed volume $\tilde{V}_{-1}$: If $K, L \in \mathscr{S}_0^n$, then

$$\tilde{V}_{-1}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+1} \rho_L(u)^{-1} \, dS(u). \tag{0.35}$$

The Minkowski (Mixed Volume) Inequality states that for $K, L \in \mathscr{K}^n$,

$$V_1(K, L)^n \geqslant V(K)^{n-1} V(L), \tag{0.36}$$

with equality if and only if K and L are homothetic. From the Minkowski inequality, it follows easily (see [3]) that if $K, L \in \mathscr{K}_c^n$, and

$$V_1(K, M) = V_1(L, M), \qquad \text{for all } M \in \mathscr{K}_c^n, \tag{0.37}$$

then $K = L$.

This result has proven to be remarkably useful in answering a variety of uniqueness questions. To demonstrate its power, a proof is given of the following result (which will be needed in Section 3): If $K, L \in \mathscr{K}^n$, $\alpha, \gamma \geqslant 0$, and $\phi \in SL(n)$, then

$$\phi(\alpha \cdot K + \gamma \cdot L) = \alpha \cdot \phi K + \gamma \cdot \phi L. \tag{0.38}$$

Suppose $M \in \mathscr{K}_c^n$. From (0.9), (0.19), again (0.9) and (0.19),

$$\begin{aligned} V_1(\phi(\alpha \cdot K + \gamma \cdot L), M) &= V_1(\alpha \cdot K + \gamma \cdot L, \phi^{-1} M) \\ &= \alpha V_1(K, \phi^{-1} M) + \gamma V_1(L, \phi^{-1} M) \\ &= \alpha V_1(\phi K, M) + \gamma V_1(\phi L, M) \\ &= V_1(\alpha \cdot \phi K + \gamma \cdot \phi L, M). \end{aligned}$$

Note that (0.38) now follows immediately from (0.37).

A simple consequence of the Minkowski Inequality (0.36), and (0.8), is the Brunn–Minkowski Inequality: For $K, L \in \mathscr{K}^n$,

$$V(K+L)^{1/n} \geqslant V(K)^{1/n} + V(L)^{1/n}, \tag{0.39}$$

with equality if and only if K and L are homothetic.

The following analogue of the Minkowski Inequality for the dual mixed volume $\tilde{V}_{-1}$ will be needed: If $K, L \in \mathscr{S}_0^n$, then

$$\tilde{V}_{-1}(K, L)^n \geqslant V(K)^{n+1}\, V(L)^{-1}, \tag{0.40}$$

with equality if and only if K and L are dilates. This inequality is a simple consequence of the Hölder inequality [8, p. 88] and the integral representation (0.35).

1. Requirements for an Extended Affine Surface Area

The affine surface area, $\Omega(K)$, of a body $K \in \mathscr{F}^n$ is defined by

$$\Omega(K) = \int_{S^{n-1}} f_K(u)^{n/(n+1)}\, dS(u). \tag{1.1}$$

The functional $\Omega\colon \mathscr{F}^n \to (0, \infty)$ has the following basic properties:

I. For $K \in \mathscr{F}^n$, $x \in \mathbb{R}^n$, and $\lambda > 0$,

$$\Omega(x + \lambda K)^{(n+1)/n} = \lambda^{n-1} \Omega(K)^{(n+1)/n}.$$

II. For $K \in \mathscr{F}^n$, and $\phi \in SL(n)$,

$$\Omega(\phi K) = \Omega(K).$$

III. The functional $\Omega^{(n+1)/n}$ is concave with respect to Blaschke addition; i.e., for K, $L \in \mathscr{F}^n$,

$$\Omega(K + L)^{(n+1)/n} \geqslant \Omega(K)^{(n+1)/n} + \Omega(L)^{(n+1)/n}, \tag{1.2}$$

with equality if and only if K and L are homothetic.

IV. If $E \in \mathscr{F}^n$ is an ellipsoid and $K \in \mathscr{F}^n$, then

$$\text{if } K \subset E, \text{ it follows that } \Omega(K) \leqslant \Omega(E). \tag{1.3}$$

V. The affine isoperimetric inequality (of Affine Differential Geometry) is satisfied: For $K \in \mathscr{F}^n$,

$$\Omega(K)^{n+1} \leqslant \omega_n^2 n^{n+1} V(K)^{n-1}, \tag{1.4}$$

with equality if and only if K is an ellipsoid.

An extension $\Omega: \mathscr{K}^n \to [0, \infty)$, of $\Omega: \mathscr{F}^n \to (0, \infty)$, will allow us to replace $\mathscr{F}^n$ by $\mathscr{K}^n$ in properties (I)–(V), and should have the additional property:

VI. $\Omega(P) = 0$, for all polytopes $P \in \mathscr{K}^n$.

Note that the equality condition for inequality (1.2) will have to be sacrificed since it is incompatible with condition (VI)—take K and L to be nonhomothetic polytopes and both sides of the inequality in (1.2) are equal (to 0).

Ideally, an extension of classical affine surface area should also provide insight, which would suggest simplified proofs of the classical properties (I)–(V), even when extended to $\mathscr{K}^n$.

To properties (I)–(V) one might add other requirements an extended affine surface area should have. For example one might require it to satisfy a "stronger" inequality than (1.4)—specifically, Petty's inequality between affine surface area and geominimal surface area (to be defined later). One might also require it to satisfy a stronger requirement than Winternitz' monotonicity result (1.3)—specifically, the extension of it given by Petty [24]. Another very desirable property would be that Petty's affine projection inequality (with the conditions for equality) should hold.

Extended affine surface area as presented here meets all these requirements. It should be noted that none of the properties of classical affine surface area will be used to prove anything in this article.

In addition to properties (I)–(V), it had been conjectured that classical affine surface area possesses another important property:

VII. The functional $\Omega: \mathscr{F}^n \to (0, \infty)$ is upper-semicontinuous.

It will be proven that extended affine surface area $\Omega: \mathscr{K}^n \to [0, \infty)$ is upper-semicontinuous. From this the conjectured property (VII) of classical affine surface area obviously follows.

2. Polar Curvature Images

For $K, L \in \mathscr{S}_0^n$, and $\alpha, \gamma \geq 0$ (not both 0), the harmonic Blaschke linear combination $\alpha K \hat{+} \gamma L \in \mathscr{S}_0^n$ can be defined (see [18]) by

$$\frac{1}{V(\alpha K \hat{+} \gamma L)} \rho_{\alpha K \hat{+} \gamma L}^{n+1} = \frac{\alpha}{V(K)} \rho_K^{n+1} + \frac{\gamma}{V(L)} \rho_L^{n+1}. \tag{2.1}$$

Note that for bodies in $\mathscr{K}_0^n$, harmonic Blaschke scalar multiplication and Minkowski scalar multiplication agree (and the same notation is used).

From (0.29) it follows immediately that $\mathscr{S}_c^n$ is closed with respect to harmonic Blaschke linear combinations. A mapping is now introduced which transforms harmonic Blaschke linear combinations into Blaschke linear combinations.

Define the mapping,

$$\Lambda: \mathscr{S}_c^n \to \mathscr{F}_c^n, \tag{2.2}$$

as follows: Suppose $K \in \mathscr{S}_c^n$. From (0.29) it follows that the function $\omega_n \rho_K^{n+1}/V(K)$ satisfies the conditions of the Minkowski problem (0.16b). Hence, there exists a unique convex body $\Lambda K \in \mathscr{F}_c^n$, such that

$$f_{\Lambda K} = \frac{\omega_n}{V(K)} \rho_K^{n+1}. \tag{2.3}$$

Note that from (0.11) and (0.26), it follows that (2.3) holds not only on S^{n-1} but on $\mathbb{R}^n \setminus \{0\}$ as well.

Proposition (2.4). *The mapping $\Lambda: \mathscr{S}_c^n \to \mathscr{F}_c^n$, is bijective.*

Proof. The injectivity of Λ follows trivially from definition (2.3). To see that Λ is surjective, suppose $K \in \mathscr{F}_c^n$. From (0.15b),

$$\int_{S^{n-1}} u f_K(u)\, dS(u) = 0.$$

Define the star body L by

$$\rho_L = c f_K^{1/(n+1)},$$

where

$$c = \frac{1}{n\omega_n} \int_{S^{n-1}} f_K(u)^{n/(n+1)}\, dS(u).$$

From (0.29), it follows that $L \in \mathscr{S}_c^n$, and it is trivial to verify from (2.3), and the polar coordinate formula, that $\Lambda L = K$. ∎

From (2.3), (2.1), and (0.17b), one easily sees that Λ transforms harmonic Blaschke linear combinations into Blaschke linear combinations, i.e., if $K, L \in \mathscr{S}_c^n$, and $\alpha, \gamma \geqslant 0$, then

$$\Lambda(\alpha K \hat{+} \gamma L) = \alpha \cdot \Lambda K + \gamma \cdot \Lambda L. \tag{2.5}$$

From (0.10), (2.3), (0.28), and (0.35), it follows immediately that for $K \in \mathscr{S}_c^n$, and $L \in \mathscr{K}_0^n$,

$$V_1(\Lambda K, L^*) = \omega_n \tilde{V}_{-1}(K, L)/V(K), \tag{2.6}$$

or equivalently by (2.6) and (0.21), that

$$V_1(\Lambda K, L) = \omega_n \tilde{V}_{-1}(K, L^*)/V(K). \tag{2.7}$$

Write Λ^2 for the composite mapping $\Lambda\Lambda$; i.e., for $K \in \mathscr{S}_c^n$, $\Lambda^2 K = \Lambda(\Lambda K)$. The next result contains the observation that the mapping Λ^2 commutes with members of $SL(n)$.

PROPOSITION (2.8). *If $K \in \mathscr{S}_c^n$, and $\phi \in SL(n)$, then*

$$\Lambda \phi K = \phi^{-t} \Lambda K. \tag{2.8}$$

Proof. Suppose $L \in \mathscr{K}_0^n$. From (2.7), (0.34), (0.22), again (2.7), and (0.9), it follows that

$$\begin{aligned} V_1(\Lambda\phi K, L) &= \omega_n \tilde{V}_{-1}(\phi K, L^*)/V(\phi K) \\ &= \omega_n \tilde{V}_{-1}(K, \phi^{-1} L^*)/V(K) \\ &= \omega_n \tilde{V}_{-1}(K, (\phi^t L)^*)/V(K) \\ &= V_1(\Lambda K, \phi^t L) \\ &= V_1(\phi^{-t} \Lambda K, L). \end{aligned}$$

Since $V_1(\Lambda\phi K, L) = V_1(\phi^{-t}\Lambda K, L)$, for all $L \in \mathcal{K}_0^n$, the desired result follows from (0.37). ∎

The transformation rule for curvature functions is a simple consequence of Proposition (2.8).

PROPOSITION (2.9). *For $K \in \mathcal{F}^n$, and $\phi \in SL(n)$,*

$$f_{\phi K}(u) = f_K(\phi^t u), \tag{2.9}$$

for all $u \in S^{n-1}$.

Proof. From (0.14b) we see that $K \in \mathcal{F}_c^n$ can be assumed. Choose $L \in \mathcal{S}_c^n$, such that $\Lambda L = K$, and let $u \in S^{n-1}$. From (2.8), (2.3), (0.27), and again (2.3), it follows that

$$\begin{aligned} f_{\phi K}(u) &= f_{\phi\Lambda L}(u) \\ &= f_{\Lambda\phi^{-t}L}(u) \\ &= \omega_n \rho_{\phi^{-t}L}^{n+1}(u)/V(\phi^{-t}L) \\ &= \omega_n \rho_L^{n+1}(\phi^t u)/V(L) \\ &= f_{\Lambda L}(\phi^t u) \\ &= f_K(\phi^t u). \quad ∎ \end{aligned}$$

From definition (2.3), it follows that for the unit ball $B \in \mathcal{S}_c^n$, $\Lambda B = B$. It follows, from (2.8) and (0.22), that if E is a centered ellipsoid, such that $V(E) = \omega_n$, then

$$\Lambda E = E^*.$$

From the injectivity of Λ, it can be seen that for $K \in \mathcal{S}_c^n$, ΛK is an ellipsoid if and only if K is an ellipsoid.

THEOREM (2.10). *If $K \in \mathcal{S}_c^n$, $L \in \mathcal{K}^n$, then*

$$V_1(\Lambda K, L)^n \geqslant \omega_n^{n-2} V(K)\, V(L), \tag{2.10}$$

with equality if and only if L is an ellipsoid and K is a dilate of $(-c+L)^*$, where $c = \mathrm{Cen}(L)$.

Proof. By (0.7) and (2.7),

$$V_1(\Lambda K, L) = V_1(\Lambda K, -c+L) = \omega_n \tilde{V}_{-1}(K, (-c+L)^*)/V(K).$$

From this, the dual mixed volume inequality (0.40) yields

$$V_1(\Lambda K, L)^n \geqslant \omega_n^n V(K)\, V((-c+L)^*)^{-1},$$

with equality if and only if K and $(-c+L)^*$ are dilates. The Blaschke–Santaló inequality (0.24) now yields the desired result. ▮

In Theorem (2.10) take $L = \Lambda K$, use (0.6), and get:

COROLLARY (2.11). *If $K \in \mathscr{S}_c^n$, then*

$$V(\Lambda K)^{n-1} \geqslant \omega_n^{n-2} V(K), \tag{2.11}$$

with equality if and only if K is an ellipsoid.

3. GENERALIZED CONVEX BODIES

A functional on $\mathscr{K}_c^n$,

$$\Phi\colon \mathscr{K}_c^n \to (0, \infty),$$

is said to be Blaschke linear if for all $K, L \in \mathscr{K}_c^n$, $\alpha, \gamma \geqslant 0$ (not both 0),

$$\Phi(\alpha \cdot K + \gamma \cdot L) = \alpha\Phi(K) + \gamma\Phi(L).$$

Let $\mathscr{G}\mathscr{K}^n$ denote the set of continuous Blaschke linear functionals on $\mathscr{K}_c^n$. The definition of each $\Phi \in \mathscr{G}\mathscr{K}^n$ can be extended so that it is defined for all bodies in $\mathscr{K}^n$ (rather than just the bodies in $\mathscr{K}_c^n$). Do this by letting Φ assume the same value for convex bodies which are translates.

Associate with each $K \in \mathscr{K}^n$ the continuous Blaschke linear functional $\Phi_K = V_1(\cdot, K)$; i.e., for each $Q \in \mathscr{K}^n$,

$$\Phi_K(Q) = \frac{1}{n}\int_{S^{n-1}} h_K(u)\, dS_Q(u). \tag{3.1}$$

We shall identify K and Φ_K, and regard $\mathscr{K}^n$ as a subset of $\mathscr{G}\mathscr{K}^n$. With this identification, members of $\mathscr{K}^n$ will occasionally be referred to as proper convex bodies.

In fact, rather than identifying bodies in $\mathscr{K}^n$ with members of $\mathscr{G}\mathscr{K}^n$, if careful, one should identify translation equivalence classes of $\mathscr{K}^n$ (i.e., members of $\mathscr{K}^n/\mathbb{R}^n$) with members of $\mathscr{G}\mathscr{K}^n$. However, a less formal presentation should create no difficulties.

For $Q \in \mathscr{K}^n$, and $\Phi \in \mathscr{G}\mathscr{K}^n$, define the mixed volume $V_1(Q, \Phi)$ by

$$V_1(Q, \Phi) = \Phi(Q). \tag{3.2}$$

From (3.1) it can be seen that if Φ is a proper convex body, this definition agrees with the usual definition. Since the members of $\mathscr{G}\mathscr{K}^n$ are Blaschke linear, the following extension of (0.19) holds: For $\Phi \in \mathscr{G}\mathscr{K}^n$, $K, L \in \mathscr{K}^n$, $\alpha, \gamma \geqslant 0$,

$$V_1(\alpha \cdot K + \gamma \cdot L, \Phi) = \alpha V_1(K, \Phi) + \gamma V_1(L, \Phi). \tag{3.3}$$

Given Φ, $\Phi' \in \mathscr{G}\mathscr{K}^n$, and λ, $\lambda' \geqslant 0$ (not both 0), define the Minkowski linear combination $\lambda\Phi + \lambda'\Phi' \in \mathscr{G}\mathscr{K}^n$ in the obvious manner: For $Q \in \mathscr{K}^n$,

$$(\lambda\Phi + \lambda'\Phi')(Q) = \lambda\Phi(Q) + \lambda'\Phi'(Q). \tag{3.4}$$

From (0.3), and (3.1), it can be seen that this agrees with the usual definition of a Minkowski linear combination if Φ and Φ' happen to be proper convex bodies.

By (3.4), (0.18), and the Blaschke linearity of members of $\mathscr{G}\mathscr{K}^n$, the following extension of (0.7) holds: If $K \in \mathscr{K}^n$, $\Phi \in \mathscr{G}\mathscr{K}^n$, $\alpha, \gamma > 0$, and $x \in \mathbb{R}^n$, then

$$V_1(x + \alpha K, \gamma\Phi) = \alpha^{n-1}\gamma V_1(K, \Phi). \tag{3.5}$$

By (3.2), and the definition of a Minkowski sum (3.4), the following extension of (0.8) also holds: If $K \in \mathscr{K}^n$, Φ, $\Phi' \in \mathscr{G}\mathscr{K}^n$, λ, $\lambda' \geqslant 0$ (not both 0), then

$$V_1(K, \lambda\Phi + \lambda'\Phi') = \lambda V_1(K, \Phi) + \lambda' V_1(K, \Phi'). \tag{3.6}$$

For $\Phi \in \mathscr{G}\mathscr{K}^n$, define the volume, $V(\Phi)$, of the generalized convex body Φ, by

$$V(\Phi)^{1/n} = \operatorname{Inf}\{V_1(Q, \Phi)/V(Q)^{(n-1)/n} : Q \in \mathscr{K}^n\}. \tag{3.7}$$

From (3.1), (3.2), and the Minkowski inequality (0.36), it can be seen that if Φ is a proper convex body, then this definition agrees with the usual definition of volume. Note that in definition (3.7) the class $\mathscr{K}^n$ could have been (and will sometimes be) replaced by the class $\mathscr{K}^n_c$.

Observe, that built into the definition of volume is a generalized Minkowski inequality: If $K \in \mathscr{K}^n$, $\Phi \in \mathscr{G}\mathscr{K}^n$, then

$$V_1(K, \Phi)^n \geqslant V(K)^{n-1}\, V(\Phi). \tag{3.8}$$

Since $\mathscr{K}^n_c \subset \mathscr{G}\mathscr{K}^n$, an extension of (0.37) obviously holds: If $K, L \in \mathscr{K}^n_c$, and

$$V_1(K, \Phi) = V_1(L, \Phi), \qquad \text{for all } \Phi \in \mathscr{G}\mathscr{K}^n,$$

then $K = L$.

In definition (3.7), for each $Q \in \mathscr{K}^n$, the functional,

$$V_1(Q, \cdot)/V(Q)^{(n-1)/n}: \mathscr{G}\mathscr{K}^n \to (0, \infty),$$

is, by (3.6), a Minkowski linear functional. But the infimum of linear functionals is obviously concave. Hence there is the following extension of the Brunn–Minkowski Inequality (0.39): If $\Phi, \Phi' \in \mathscr{G}\mathscr{K}^n$, then

$$V(\Phi + \Phi')^{1/n} \geq V(\Phi)^{1/n} + V(\Phi')^{1/n}. \tag{3.9}$$

Suppose $K \in \mathscr{K}^n$, $\phi \in SL(n)$. From (3.1) and (0.9) it follows that for $Q \in \mathscr{K}^n$,

$$\begin{aligned} \Phi_{\phi K}(Q) &= V_1(Q, \phi K) \\ &= V_1(\phi^{-1}Q, K) \\ &= \Phi_K(\phi^{-1}Q) \\ &= (\Phi_K \circ \phi^{-1})(Q). \end{aligned}$$

This suggests the obvious definition: Given $\phi \in SL(n)$, and a generalized convex body $\Phi \in \mathscr{G}\mathscr{K}^n$, define the image, $\phi\Phi$, of the body Φ under ϕ, as the composite of the functions Φ and ϕ^{-1}:

$$\phi\Phi = \Phi \circ \phi^{-1}. \tag{3.10}$$

The Blaschke linearity of $\phi\Phi$ is an immediate consequence of (0.38) and the Blaschke linearity of Φ. The continuity of the generalized body $\phi\Phi$ obviously follows from that of Φ and ϕ^{-1}.

Definition (3.10) can be written as an extension of (0.9): For $K \in \mathscr{K}^n$, $\Phi \in \mathscr{G}\mathscr{K}^n$, and $\phi \in SL(n)$,

$$V_1(K, \phi\Phi) = V_1(\phi^{-1}K, \Phi). \tag{3.11}$$

Just as for $K \in \mathscr{K}^n$, and $\phi \in SL(n)$, $V(\phi K) = V(K)$, it follows from definitions (3.10) and (3.7) that for $\Phi \in \mathscr{G}\mathscr{K}^n$, and $\phi \in SL(n)$,

$$V(\phi\Phi) = V(\Phi).$$

The following extension of (0.4) is an immediate consequence of (3.4) and (3.10): If $\Phi, \Phi' \in \mathscr{G}\mathscr{K}^n$, $\lambda, \lambda' \geq 0$, and $\phi \in SL(n)$, then

$$\phi(\lambda\Phi + \lambda'\Phi') = \lambda\phi\Phi + \lambda'\phi\Phi'.$$

For $K \in \mathcal{S}_0^n$, define the generalized convex body $K^* \in \mathcal{G}\mathcal{K}^n$, by

$$V_1(Q, K^*) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{-1}\, dS_Q(u), \tag{3.12}$$

for $Q \in \mathcal{K}^n$. The Blaschke linearity of K^* follows immediately from (0.17a). The continuity of K^* follows from the weak continuity of the surface area measures. Note that from (0.12) and (0.28), it follows that for $K \in \mathcal{K}_0^n$, definition (3.12) of K^* agrees with the usual definition.

The following extension of (2.6) is an immediate consequence of (0.35), (0.13), and definitions (3.12) and (2.3).

PROPOSITION (3.13). *If $K \in \mathcal{S}_c^n$, $L \in \mathcal{S}_0^n$, then*

$$V_1(\Lambda K, L^*) = \omega_n \tilde{V}_{-1}(K, L)/V(K). \tag{3.13}$$

An extension of (0.22) holds as well.

PROPOSITION (3.14). *If $K \in \mathcal{S}_0^n$, $\phi \in SL(n)$, then*

$$(\phi K)^* = \phi^{-t} K^*. \tag{3.14}$$

Proof. Suppose $Q \in \mathcal{S}_c^n$. By (3.13), (0.34), again (3.13), (2.8), and (3.11),

$$\begin{aligned} V_1(\Lambda Q, (\phi K)^*) &= \omega_n \tilde{V}_{-1}(Q, \phi K)/V(Q) \\ &= \omega_n \tilde{V}_{-1}(\phi^{-1} Q, K)/V(\phi^{-1} Q) \\ &= V_1(\Lambda \phi^{-1} Q, K^*) \\ &= V_1(\phi^t \Lambda Q, K^*) \\ &= V_1(\Lambda Q, \phi^{-t} K^*). \end{aligned}$$

Since, $\Lambda(\mathcal{S}_c^n) = \mathcal{F}_c^n$, it follows from (3.5) that, for all $L \in \mathcal{F}^n$,

$$V_1(L, (\phi K)^*) = V_1(L, \phi^{-t} K^*).$$

But this must hold for all $L \in \mathcal{K}^n$, since $\mathcal{F}^n$ is dense in $\mathcal{K}^n$, and the members of $\mathcal{G}\mathcal{K}^n$ are continuous. Hence, $(\phi K)^* = \phi^{-t} K^*$. ∎

An extension of the Blaschke–Santaló inequality is contained in:

THEOREM (3.15). *If $K \in \mathcal{S}_c^n$, then*

$$V(K)\, V(K^*) \leqslant \omega_n^2, \tag{3.15}$$

with equality if and only if K is an ellipsoid.

Proof. By definition (3.7),

$$V(K^*) = \operatorname{Inf}\{V_1(Q, K^*)^n / V(Q)^{n-1} : Q \in \mathscr{K}_c^n\}.$$

Since $K \in \mathscr{S}_c^n$, $\Lambda K \in \mathscr{K}_c^n$. Hence, by (3.13) and (0.33),

$$\begin{aligned} V(K^*) &\leqslant V_1(\Lambda K, K^*)^n / V(\Lambda K)^{n-1} \\ &= \omega_n^n \tilde{V}_{-1}(K, K)^n / V(K)^n \, V(\Lambda K)^{n-1} \\ &= \omega_n^n / V(\Lambda K)^{n-1}. \end{aligned}$$

The desired inequality is now an immediate consequence of Corollary (2.11). ∎

A functional, $v: \mathscr{K}^n \to \mathbb{R}$, is said to be a valuation on $\mathscr{K}^n$, provided that whenever $K, L, K \cup L \in \mathscr{K}^n$,

$$v(K \cup L) + v(K \cap L) = v(K) + v(L).$$

Schneider [29] (see also [20]) observes that for $K, L, K \cup L \in \mathscr{K}^n$,

$$S_{K \cup L} + S_{K \cap L} = S_K + S_L.$$

From this and definition (0.17a), it follows immediately that

$$(K \cup L) + (K \cap L) = K + L.$$

Hence, a (translation invariant) functional, $v: \mathscr{K}^n \to \mathbb{R}$ which is Blaschke linear must be a valuation on $\mathscr{K}^n$. The problem of characterizing such continuous valuations is solved by McMullen [19].

4. Extended Affine Surface Area

For a convex body $K \in \mathscr{K}^n$, define the affine surface area of K, $\Omega(K)$, by

$$n^{-1/n} \Omega(K)^{(n+1)/n} = \operatorname{Inf}\{n V_1(K, Q^*) \, V(Q)^{1/n} : Q \in \mathscr{S}_c^n\}. \tag{4.1}$$

From (3.5) it follows that for $K \in \mathscr{K}^n$, $x \in \mathbb{R}^n$, $\lambda > 0$, and each $Q \in \mathscr{S}_c^n$,

$$V_1(x + \lambda K, Q^*) = \lambda^{n-1} V_1(K, Q^*).$$

Hence, from definition (4.1) there is an extension of Property (I) of Section 1:

PROPOSITION (4.2). *If* $K \in \mathscr{K}^n$, $x \in \mathbb{R}^n$, *and* $\lambda > 0$, *then*

$$\Omega(x+\lambda K)^{(n+1)/n} = \lambda^{n-1}\Omega(K)^{(n+1)/n}. \tag{4.2}$$

An extension of Property (II) is contained in:

PROPOSITION (4.3). *If* $K \in \mathscr{K}^n$, $\phi \in SL(n)$, *then*

$$\Omega(\phi K) = \Omega(K). \tag{4.3}$$

Proof. From definition (4.1), (3.11), and (3.14), it follows that

$$\begin{aligned} n^{-1/n}\Omega(\phi K)^{(n+1)/n} &= \operatorname{Inf}\{nV_1(\phi K, Q^*)\, V(Q)^{1/n} \colon Q \in \mathscr{S}_c^n\} \\ &= \operatorname{Inf}\{nV_1(K, \phi^{-1}Q^*)\, V(Q)^{1/n} \colon Q \in \mathscr{S}_c^n\} \\ &= \operatorname{Inf}\{nV_1(K, (\phi^t Q)^*)\, V(\phi^t Q)^{1/n} \colon Q \in \mathscr{S}_c^n\} \\ &= n^{-1/n}\Omega(K)^{(n+1)/n}. \end{aligned}$$

The last step is justified by the fact that $\phi^t(\mathscr{S}_c^n) = \mathscr{S}_c^n$. ∎

For each $Q \in \mathscr{S}_c^n$, the functional

$$V_1(\cdot, Q^*)\, V(Q)^{1/n} \colon \mathscr{K}^n \to (0, \infty),$$

is in $\mathscr{G}\mathscr{K}^n$, and hence is continuous and Blaschke linear. Thus, the infimum of such functionals is upper-semicontinuous, and concave, with respect to Blaschke addition. This proves the next two propositions and extends Properties (VII) and (III).

PROPOSITION (4.4). *The functional,* $\Omega \colon \mathscr{K}^n \to [0, \infty)$, *is upper-semicontinuous.*

PROPOSITION (4.5). *If,* $K, L \in \mathscr{K}^n$, *then*

$$\Omega(K \,\#\, L)^{(n+1)/n} \geqslant \Omega(K)^{(n+1)/n} + \Omega(L)^{(n+1)/n}. \tag{4.5}$$

It will now be shown that the affine isoperimetric inequality of Affine Differential Geometry (with the same equality conditions) holds for arbitrary convex bodies.

THEOREM (4.6). *If* $K \in \mathscr{K}^n$, *then*

$$\Omega(K)^{n+1} \leqslant n^{n+1}\omega_n^2\, V(K)^{n-1}, \tag{4.6}$$

with equality, if and only if, K *is an ellipsoid.*

Proof. By (0.23), $(-s+K)^* \in \mathscr{K}_c^n$, where $s \in \operatorname{int} K$ is the Santaló point of K. Hence, from definition (4.1),

$$\Omega(K)^{n+1} \leqslant n^{n+1} V_1(K, (-s+K)^{**})^n \, V((-s+K)^*).$$

By (0.21), (0.7), and (0.6), $V_1(K, (-s+K)^{**}) = V(K)$. But from the Blaschke–Santaló inequality (0.24), and (0.21), it follows that

$$V((-s+K)^*) \leqslant \omega_n^2 V(K)^{-1},$$

with equality if and only if $-s+K$ is an ellipsoid. ∎

Theorem (4.6) shows that Property (V) holds. That Property (VI) holds is trivial: Suppose P is a polytope in $\mathscr{K}^n$, and $Q \in \mathscr{S}_c^n$. By definition (3.12),

$$V_1(P, Q^*) = \frac{1}{n} \int_{S^{n-1}} \rho_Q(u)^{-1} \, dS_P(u).$$

Recall that the surface area measure of a polytope is concentrated on a finite set of points, glance at definition (4.1), and get:

Proposition (4.7). *If $P \in \mathscr{K}^n$ is a polytope, then $\Omega(P) = 0$.*

That extended affine surface area satisfies condition (IV) will be shown in Section 8. What remains to be shown is that extended affine surface area agrees with the "classical" definition for bodies in $\mathscr{F}^n$. This is contained in:

Theorem (4.8). *If $K \in \mathscr{F}^n$, then*

$$\operatorname{Inf}\{V_1(K, Q^*) \, V(Q)^{1/n} : Q \in \mathscr{S}_c^n\} = \left[\frac{1}{n} \int_{S^{n-1}} f_K(u)^{n/(n+1)} \, dS(u)\right]^{(n+1)/n}. \tag{4.8}$$

Proof. By (0.14b) and (3.5), it may be assumed that $K \in \mathscr{F}_c^n$. From Proposition (2.4) it follows that there exists a $K_0 \in \mathscr{S}_c^n$, such that $\Lambda K_0 = K$. Hence, from definition (2.3),

$$f_K = f_{\Lambda K_0} = \frac{\omega_n}{V(K_0)} \rho_{K_0}^{n+1}.$$

From this and the polar coordinate formula for volume, it follows that the quantity on the right in Eq. (4.8) is just $\omega_n V(K_0)^{1/n}$.

Now from Proposition (3.13), it follows that for $Q \in \mathscr{S}_c^n$,

$$V_1(K, Q^*) \, V(Q)^{1/n} = V_1(\Lambda K_0, Q^*) \, V(Q)^{1/n} = \omega_n \tilde{V}_{-1}(K_0, Q) \, V(Q)^{1/n} / V(K_0).$$

Thus, to show that (4.8) holds, it must be shown that

$$\operatorname{Inf}\{\tilde{V}_{-1}(K_0, Q)\, V(Q)^{1/n} \colon Q \in \mathscr{S}_c^n\} = V(K_0)^{(n+1)/n}.$$

That the quantity on the left is no less than the quantity on the right is an immediate consequence of the dual mixed volume inequality (0.40). To see that the quantity on the right is no less than the quantity on the left, note that $K_0 \in \mathscr{S}_c^n$, and hence, by (0.33),

$$\begin{aligned}&\operatorname{Inf}\{\tilde{V}_{-1}(K_0, Q)\, V(Q)^{1/n} \colon Q \in \mathscr{S}_c^n\}\\ &\quad \leqslant \tilde{V}_{-1}(K_0, K_0)\, V(K_0)^{1/n} = V(K_0)^{(n+1)/n}. \quad \blacksquare\end{aligned}$$

5. Alternate Definition of Extended Affine Surface Area

The mixed affine surface area, $\Omega_{-1}(K, L)$, of $K, L \in \mathscr{F}^n$, was defined in [16] by

$$\Omega_{-1}(K, L) = \int_{S^{n-1}} f_K(u)\, f_L(u)^{-1/(n+1)}\, dS(u). \tag{5.1}$$

From (1.1), it follows that for $K \in \mathscr{F}^n$,

$$\Omega_{-1}(K, K) = \Omega(K). \tag{5.2}$$

It is shown in [16] that a consequence of the Hölder inequality is that, for $K, L \in \mathscr{F}^n$,

$$\Omega(K)^{(n+1)/n} \leqslant \Omega_{-1}(K, L)\, \Omega(L)^{1/n}, \tag{5.3}$$

with equality if and only if K and L are homothetic.

There is a natural extension of definition (5.1) of the mixed affine surface area Ω_{-1} from $\mathscr{F}^n \times \mathscr{F}^n$ to $\mathscr{K}^n \times \mathscr{F}^n$. Specifically, for $K \in \mathscr{K}^n$, and $L \in \mathscr{F}^n$, let

$$\Omega_{-1}(K, L) = \int_{S^{n-1}} f_L(u)^{-1/(n+1)}\, dS_K(u). \tag{5.4}$$

From (0.14a) and (0.14b), note that for $x, y \in \mathbb{R}^n$, and $\lambda > 0$,

$$\Omega_{-1}(x + \lambda K, y + L) = \lambda^{n-1} \Omega_{-1}(K, L). \tag{5.5}$$

From definition (5.4), (2.3), and (3.12), it follows immediately, that for $K \in \mathscr{K}^n$, $L \in \mathscr{S}_c^n$,

$$\omega_n \Omega_{-1}(K, \Lambda L)^{n+1} = n^{n+1} V(L)\, V_1(K, L^*)^{n+1}. \tag{5.6}$$

Note that L^* is a generalized convex body.

Take ΛL for K in (5.6), use (5.2), and from (3.13) get

$$\omega_n \Omega(\Lambda L)^{n+1} = n^{n+1} V(L)\, V_1(\Lambda L, L^*)^{n+1}$$
$$= \omega_n^{n+1} n^{n+1} V(L)\, \tilde{V}_{-1}(L, L)^{n+1}/V(L)^{n+1}.$$

Thus, by (0.33),

$$\Omega(\Lambda L)^{n+1} = n^{n+1} \omega_n^n V(L). \tag{5.7}$$

Combine (5.6) and (5.7), and the result is:

PROPOSITION (5.8). *If $K \in \mathscr{K}^n$, and $L \in \mathscr{S}_c^n$, then*

$$n^{-1/n} \Omega_{-1}(K, \Lambda L)\, \Omega(\Lambda L)^{1/n} = n V_1(K, L^*)\, V(L)^{1/n}. \tag{5.8}$$

From (0.14a), Propositions (2.4) and (5.8), and definition (4.1), we immediately obtain:

THEOREM (5.9). *If $K \in \mathscr{K}^n$, then*

$$\Omega(K)^{(n+1)/n} = \operatorname{Inf}\{\Omega_{-1}(K, L)\, \Omega(L)^{1/n} \colon L \in \mathscr{F}^n\}. \tag{5.9}$$

An obvious consequence of Theorem (5.9) is:

COROLLARY (5.10). *If $K \in \mathscr{K}^n$, $L \in \mathscr{F}^n$, then*

$$\Omega(K)^{n+1}\, \Omega(L)^{-1} \leqslant \Omega_{-1}(K, L)^n. \tag{5.10}$$

Of course, Theorem (5.9) could have been used to define the affine surface area of arbitrary convex bodies.

6. GEOMINIMAL SURFACE AREA

Petty [24] introduced the important concept of geominimal surface area. For $K \in \mathscr{K}^n$, the geominimal surface area of K, $G(K)$, is defined by

$$\omega_n^{1/n} G(K) = \operatorname{Inf}\{n V_1(K, Q^*)\, V(Q)^{1/n} \colon Q \in \mathscr{K}_c^n\}. \tag{6.1}$$

By comparing definitions (4.1) and (6.1), the reader can see how Petty's definition of geominimal surface area motivated the definition of extended affine surface area given in this article.

Suppose $K \in \mathscr{K}^n$. By (0.23), $(-s+K)^* \in \mathscr{K}_c^n$, where $s \in \operatorname{int} K$ is the Santaló point of K. From definition (6.1),

$$\omega_n^{1/n} G(K) \leqslant n V_1(K, (-s+K)^{**})\, V((-s+K)^*)^{1/n}.$$

This together with (0.21), (0.7), (0.6), and the Blaschke–Santaló inequality (0.24), immediately yields Petty's geominimal surface area inequality

$$G(K)^n \leqslant n^n \omega_n V(K)^{n-1}, \tag{6.2}$$

with equality if and only if K is an ellipsoid.

Petty [24] obtained a "stronger" inequality than the affine isoperimetric inequality. He proved that for $K \in \mathscr{F}^n$,

$$\Omega(K)^{n+1} \leqslant n\omega_n G(K)^n. \tag{6.3}$$

That this inequality is "stronger" than the affine isoperimetric inequality can be seen from Petty's geominimal surface area inequality (6.2).

Since obviously $\mathscr{K}_c^n \subset \mathscr{S}_c^n$, from definitions (6.1) and (4.1), an extension of (6.3) is immediately obtained:

PROPOSITION (6.4). *If* $K \in \mathscr{K}^n$, *then*

$$\Omega(K)^{n+1} \leqslant n\omega_n G(K)^n. \tag{6.4}$$

There is a circuitous (but more interesting) road to Proposition (6.4).

Petty [23, 24] introduced the important class of bodies of constant relative curvature, $\mathscr{V}^n$. A body $K \in \mathscr{F}^n$ is said to have constant relative curvature if $1/f_K^{1/(n+1)}$ is the (restricted) support function of a convex body. (See Petty [24, pp. 86–88] for the interesting motivation for this definition.) Let $\mathscr{V}_c^n$ denote the class of bodies in $\mathscr{F}_c^n$ which have constant relative curvature.

PROPOSITION (6.5). $\Lambda(\mathscr{K}_c^n) = \mathscr{V}_c^n$.

Proof. If $K \in \mathscr{V}_c^n$, then by definition, $1/f_K = h_Q^{n+1}$, for some $Q \in \mathscr{K}^n$. From (0.28), it follows that

$$f_K = \rho_{Q^*}^{n+1}.$$

From (0.15b) and (0.29), it must be that $Q^* \in \mathscr{K}_c^n$. It is now obvious that K is the polar curvature image of a dilate of Q^*. This shows that $\mathscr{V}_c^n \subset \Lambda(\mathscr{K}_c^n)$. That $\Lambda(\mathscr{K}_c^n) \subset \mathscr{V}_c^n$ is an easy consequence of definition (2.3), and (0.28). ∎

From Proposition (6.5), definition (6.1), Proposition (5.8), and (5.5), comes the following new representation of geominimal surface area:

THEOREM (6.6). *If* $K \in \mathscr{K}^n$, *then*

$$n\omega_n G(K)^n = \operatorname{Inf}\{\Omega_{-1}(K, L)^n\, \Omega(L) : L \in \mathscr{V}^n\}. \tag{6.6}$$

By comparing the representation for affine surface area in Theorem (5.9) with the representation for geominimal surface area in Theorem (6.6), and noting that $\mathscr{V}^n \subset \mathscr{F}^n$, Proposition (6.4) is immediately obtained.

7. Extension of Petty's Affine Projection Inequality

For $u \in S^{n-1}$, let ξ_u denote the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that is orthogonal to u. For $K \in \mathscr{K}^n$, let $K|\xi_u$ denote the image of the orthogonal projection of K onto ξ_u, and let $v(K|\xi_u)$ denote the area ($(n-1)$-dimensional volume) of $K|\xi_u$. As is well known and easily shown,

$$v(K|\xi_u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot u'| \, dS_K(u'), \tag{7.1}$$

for all $u \in S^{n-1}$. For $K \in \mathscr{K}^n$, the projection body of K, ΠK, is the convex body whose support function is given by

$$h_{\Pi K}(u) = v(K|\xi_u), \tag{7.2}$$

for $u \in S^{n-1}$. Write $\Pi^* K$, rather than $(\Pi K)^*$, for the polar of ΠK. Since ΠK and $\Pi^* K$ are centered, they are members of $\mathscr{K}_c^n$. Write Π^n and Π^{*n} for the class of projection bodies (zonoids) and polars of projection bodies; i.e.,

$$\Pi^n = \{\Pi K \colon K \in \mathscr{K}^n\},$$

and

$$\Pi^{*n} = \{\Pi^* K \colon K \in \mathscr{K}^n\}.$$

From (7.1), (7.2), and (0.12), it follows that for $K, L \in \mathscr{K}^n$,

$$V_1(K, \Pi L) = V_1(L, \Pi K). \tag{7.3}$$

Petty's affine projection inequality states that for $K \in \mathscr{F}^n$,

$$\omega_{n-1}^n \Omega(K)^{n+1} \leqslant \omega_n^n n^{n+1} V(\Pi K), \tag{7.4}$$

with equality if and only if K is an ellipsoid. Inequality (7.4) is due to Petty [22]. It will be shown that this inequality (with the same equality conditions) holds for all convex bodies. In order to do so, the Petty projection inequality will be used. This inequality is also due to Petty [23] (see [14] for an alternate proof) and states that for $K \in \mathscr{K}^n$,

$$V(K)^{n-1} \, V(\Pi^* K) \leqslant (\omega_n/\omega_{n-1})^n, \tag{7.5}$$

with equality if and only if K is an ellipsoid. Also needed is the trivial fact that if E is a centered ellipsoid, then $V(E)\,V(E^*)=\omega_n^2$. This is easily verified by using (0.22).

A "stronger" inequality than (7.4) will be required. The inequality of the next theorem is due to Petty [24]. The equality conditions are easily obtained from the equality conditions of the Petty projection inequality.

THEOREM (7.6). *If $K\in\mathscr{K}^n$, then*

$$\omega_{n-1}^n G(K)^n \leqslant n^n \omega_n^{n-1} V(\Pi K), \tag{7.6}$$

with equality if and only if K is an ellipsoid.

Proof. From definition (6.1), it follows that for $Q\in\mathscr{K}_c^n$,

$$\omega_n G(K)^n \leqslant n^n V_1(K, Q^*)^n\, V(Q).$$

Suppose $L\in\mathscr{K}^n$. Take Π^*L for Q, use (0.21), and get

$$\omega_n G(K)^n \leqslant n^n V_1(K, \Pi L)^n\, V(\Pi^* L).$$

By (7.3), and the Petty projection inequality (7.5),

$$\omega_{n-1}^n G(K)^n \leqslant n^n \omega_n^{n-1} V_1(L, \Pi K)^n\, V(L)^{-(n-1)},$$

with equality implying that L is an ellipsoid. Now take ΠK for L, use (0.6), and the result is the desired inequality (7.6). Note that equality in (7.6) would imply that ΠK is an ellipsoid.

Suppose there is equality in (7.6):

$$\omega_{n-1}^n G(K)^n = n^n \omega_n^{n-1} V(\Pi K).$$

Hence ΠK is a centered ellipsoid, and $V(\Pi K)\,V(\Pi^* K)=\omega_n^2$. From definition (6.1), it follows that for all $Q\in\mathscr{K}_c^n$,

$$n^n(\omega_n/\omega_{n-1})^n\, V(\Pi K) = \omega_n G(K)^n \leqslant n^n V_1(K, Q^*)^n\, V(Q).$$

Take $(-s+K)^*\in\mathscr{K}_c^n$ for Q, use (0.21), (0.7), and (0.6), and get

$$(\omega_n/\omega_{n-1})^n\, V(\Pi K) \leqslant V(K)^n\, V((-s+K)^*).$$

The Blaschke–Santaló inequality (0.24), and (0.21), now shows that

$$(\omega_n/\omega_{n-1})^n\, V(\Pi K) \leqslant \omega_n^2 V(K)^{n-1}.$$

But, as noted previously, $V(\Pi K)=\omega_n^2 V(\Pi^* K)^{-1}$. Hence the last inequality is

$$(\omega_n/\omega_{n-1})^n \leqslant V(K)^{n-1}\, V(\Pi^* K).$$

The equality conditions of the Petty projection inequality (7.5) show that K must therefore be an ellipsoid. ▮

If the inequality of Theorem (7.6) is combined with that of Proposition (6.4), the result is the promised extension of inequality (7.4):

COROLLARY (7.7). *If* $K \in \mathscr{K}^n$, *then*

$$\omega_{n-1}^n \Omega(K)^{n+1} \leqslant \omega_n^n n^{n+1} V(\Pi K), \tag{7.7}$$

with equality if and only if K is an ellipsoid.

8. MONOTONICITY RESULTS

Winternitz (see [2, p. 200]) proved that if $K \in \mathscr{F}^n$ (actually a more restrictive condition) and E is an ellipsoid such that

$$K \subset E,$$

then it follows that

$$\Omega(K) \leqslant \Omega(E).$$

By Propositions (6.5) and (2.8), the class $\mathscr{V}^n$ is an affine invariant class. Since balls are obviously members of $\mathscr{V}^n$, it follows that all ellipsoids are members of $\mathscr{V}^n$. Petty [24] proved the following extension of Winternitz' monotonicity result. If $K \in \mathscr{F}^n$, $L \in \mathscr{V}^n$, and

$$K \subset L,$$

then

$$\Omega(K) \leqslant \Omega(L).$$

It will now be shown that this is also correct when K is an arbitrary convex body.

THEOREM (8.1). *If* $K \in \mathscr{K}^n$, $L \in \mathscr{V}^n$, *and*

$$K \subset L, \tag{8.1a}$$

then

$$\Omega(K) \leqslant \Omega(L). \tag{8.1b}$$

Proof. Since $L \in \mathscr{V}^n$, by Proposition (6.5) there exists a $Q \in \mathscr{K}_c^n$, such that $L = \Lambda Q$, up to translation. Since $Q^* \in \mathscr{K}^n$, from the monotonicity of mixed volumes, and (8.1a),

$$V_1(K, Q^*) \leqslant V_1(L, Q^*).$$

But from (5.6), (5.5), and (5.2),

$$n^{n+1} V(Q)\, V_1(K, Q^*)^{n+1} = \omega_n \Omega_{-1}(K, \Lambda Q)^{n+1} = \omega_n \Omega_{-1}(K, L)^{n+1},$$

and

$$\begin{aligned} n^{n+1} V(Q)\, V_1(L, Q^*)^{n+1} &= \omega_n \Omega_{-1}(L, \Lambda Q)^{n+1} \\ &= \omega_n \Omega_{-1}(L, L)^{n+1} = \omega_n \Omega(L)^{n+1}. \end{aligned}$$

Hence, $\Omega_{-1}(K, L) \leqslant \Omega(L)$, and the desired result is now seen to be a direct consequence of inequality (5.10). ∎

Since all ellipsoids are members of $\mathscr{V}^n$, Theorem (8.1) provides the promised extension of Property (IV) of Section 1.

Recall, that a body $K \in \mathscr{F}^n$ is said to belong to $\mathscr{V}^n$ if $1/f_K^{1/(n+1)}$ is the support function of a convex body. A body $K \in \mathscr{F}^n$ is defined to be in $\mathscr{W}^n$ if $1/f_K^{1/(n+1)}$ is the support function of a zonoid. Both $\mathscr{V}^n$ and $\mathscr{W}^n$ are affine invariant classes (see [18]). Obviously, all balls belong to $\mathscr{W}^n$, and hence all ellipsoids belong to $\mathscr{W}^n$ as well. Let $\mathscr{W}_c^n$ denote the members of $\mathscr{W}^n$ whose centroids are at the origin.

In the same way that Proposition (6.5) is established, one easily shows that

$$\Lambda(\Pi^{*n}) = \mathscr{W}_c^n. \tag{8.2}$$

Winternitz' monotonicity result can be extended in a direction different from Theorem (8.1): If $K \in \mathscr{K}^n$, and $E \in \mathscr{K}^n$ is an ellipsoid, then if the areas of the projections of K do not exceed those of E, it follows that $\Omega(K) \leqslant \Omega(E)$. This is a special case of the next theorem.

THEOREM (8.3). *If $K \in \mathscr{K}^n$, $L \in \mathscr{W}^n$, and for all $u \in S^{n-1}$,*

$$v(K \mid \xi_u) \leqslant v(L \mid \xi_u), \tag{8.3a}$$

then

$$\Omega(K) \leqslant \Omega(L). \tag{8.3b}$$

Proof. By Proposition (4.2), it may be assumed that $L \in \mathscr{W}_c^n$. By (8.2) there exists a $Q \in \mathscr{K}^n$, such that $L = \Lambda\Pi^*Q$. From (8.3a) and (7.2),

$$h_{\Pi K} \leqslant h_{\Pi L}.$$

Since S_Q is a positive measure, it follows from (0.12), that

$$V_1(Q, \Pi K) \leqslant V_1(Q, \Pi L).$$

Hence, by (7.3),

$$V_1(K, \Pi Q) \leqslant V_1(L, \Pi Q).$$

As in the proof of Theorem (8.1), rewrite this, by using (5.6), as

$$\Omega_{-1}(K, \Lambda\Pi^*Q) \leqslant \Omega_{-1}(L, \Lambda\Pi^*Q).$$

Recall that $L = \Lambda\Pi^*Q$, use (5.2), and the last inequality becomes

$$\Omega_{-1}(K, L) \leqslant \Omega(L).$$

Inequality (5.10) immediately now gives (8.3b). ∎

For bodies with positive continuous curvature functions, Theorem (8.3) was proven in [18]. It should be noted that the special case of Theorem (8.3), where L is an ellipsoid, can also be obtained from the Petty–Schneider theorem [22, 28] together with the extended affine isoperimetric inequality (4.6).

Reference added in proof. K. Leichtweiss, Bemerkungen zur Definition einer erweiterten Affinoberfläche von E. Lutwak, *Manuscripta Math.* **65** (1989), 181–197.

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