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Stable Cycles for Attractors of Strongly Monotone Discrete-Time Dynamical Systems

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INTRODUCTION

At the present time, there is considerable interest in the study of asymptotic behavior of strongly monotone dynamical systems. The path-breaking work of M. W. Hirsch [1] and later improvements by Smith and Thieme [5, 6] established that most positive orbits of a strongly monotone continuous-time local semiflow on a strongly ordered space X tend to the set E of equilibria. Not long ago, there was an attempt to show similar convergence properties (that is, most orbits converge to the set of fixed points) for strongly monotone discrete-time dynamical systems. However, examples of stable k-cycles, $k \geq 2$, for strongly monotone discrete-time dynamical systems have been constructed by Takáč [8–11] and Dancer and Hess [12, 13]. By imposing suitable conditions and using some ideas from Takáč [9], Poláčik and Tereščák [14, 15] have proved that most positive orbits of a strongly monotone discrete-time dynamical system converge to a cycle. These convergent results show that a strongly monotone dynamical system cannot be very chaotic. The results on attractors obtained by

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Hirsch [2, 3] further indicate this fact. In Chapter III of [2], he studied the structure of attractors for strongly monotone continuous-time flows. He showed that every attractor K contains an order-stable equilibrium (see [2, Theorem 4.1]) and that if the number of equilibria in K is finite then K contains an asymptotically order-stable equilibrium (see [2, Theorem 5.6]). In the same paper, he still obtained the following conclusion (see [2, Theorem 4.3]):

THEOREM A. *Let K be an attractor for the strongly monotone continuous-time flow ϕ. Suppose z is attracted to K but is not quasiconvergent. Then K contains two order-stable equilibria p, q such that $p \ll \omega(z) \ll q$.*

In earlier work [16], it is verified that the result of Theorem A still holds if the condition "nonquasiconvergent" is replaced by "nonconvergent" (see [16, Theorem 2]).

For attractors of strongly monotone discrete-time dynamical systems, replacing the term "equilibrium" by "cycle," Hirsch [3] proved that every one of the above-mentioned results, except Theorem A and its generalization [16, Theorem 2], holds. Observing the processes of proofs for Theorem A and its generalization, we find that they strongly depend on the following ω-limit set dichotomy theorem for the continuous-time case: if $x < y$, then either $\omega(x) \ll \omega(y)$ or else $\omega(x) = \omega(y) \subset E$, the set of equilibria. The existence of stable cycles for strongly monotone discrete-time dynamical systems (see [8–13]) shows that the limit set dichotomy of Hirsch [1] for strongly monotone semiflows does not carry over to strongly monotone discrete-time dynamical systems. Therefore, the methods used in the proofs of Theorem A and its generalization are not valid for strongly monotone discrete-time dynamical systems. In our opinion, this is the reason Theorem A has not been generalized to strongly monotone discrete-time dynamical systems.

In this paper, using the decomposition of the omega limit set and monotonicity, we shall prove that the result similar to Theorem A and Theorem 2 in [16] holds. Somewhat more precisely, we shall show that if K is an attractor for the strongly monotone map T and z is attracted to K but $\omega(z)$ is not a cycle, then K contains two order-stable cycles. Moreover, we shall give various conditions under which one obtains order-stable cycles, asymptotically order-stable cycles, and a globally asymptotically order-stable cycle.

This paper is organized as follows. In Section 1 we agree on some notation, give important definitions, and state some known results which will be essential to our proofs. In Section 2 we state our main results. The proofs are contained in Section 3.

1. DEFINITIONS AND PRELIMINARY RESULTS

We start with some notation and a few definitions.

The space X is called ordered if it is a topological space together with a closed partial order relation $R \subset X \times X$. We write

$$x \leq y \qquad \text{if } (x, y) \in R,$$
$$x < y \qquad \text{if } x \leq y \text{ and } x \neq y,$$
$$x \ll y \qquad \text{if } (x, y) \in \operatorname{Int} R,$$

where Int indicates the interior of a set. Notations such as $y > x$ have the obvious meanings.

If $A, B \subset X$ are subsets then $A < B$ means $a < b$ for all $a \in A$, $b \in B$; and similarly for $A \leq B$, $A \ll B$, etc.

The ordered space X is called strongly ordered if every open subset U of X satisfies:

(SO1) If $x \in U$ then $a \ll x \ll b$ for some $a, b \in U$. It is easy to see this implies

(SO2) If $a, b \in U$ and $a \ll b$ then $a \ll x \ll b$ for some $x \in U$.

Suppose that V is a real Banach space and $V_+ \subset V$ is a closed convex cone satisfying $V_+ \cap (-V_+) = \{0\}$. We write $y \geq x$ if $y - x \in V_+$ and $y > x$ if $y \geq x$ but $\neq x$. If $\operatorname{Int} V_+ \neq \emptyset$, then V is strongly ordered.

Throughout the rest of this paper X denotes a strongly ordered space.

Any points a, b in X determine the closed order interval

$$[a, b] = \{x \in X: a \leq x \leq b\}$$

and the open order interval

$$[[a, b]] = \{x \in X: a \ll x \ll b\}.$$

If A and B are subsets of X then we define

$$[[A, B]] = \{x \in X: A \ll x \ll B\}$$

and similarly for $[A, B]$.

Let X be strongly ordered. A topological space X is defined by giving the set X the topology generated by all open order intervals $[[a, b]]$ with $a \ll b$. T: $X \to X$ is called order-compact if $T[a, b]$ has a compact closure in X for each $[a, b]$ in X.

The orbit of $x \in X$ is the set

$$O(x) = \{T^m x: m \in Z_+\},$$

where Z_+ denotes the set of nonnegative integers. The closure of $O(x)$, denoted by $\overline{O}(x)$, called the orbit closure of x. The ω-limit set of x is defined by $\omega(x) = \{y \in X\colon T^{n_k}x \to y (k \to \infty)$ for some sequence $n_k \to \infty$ in $Z_+\}$. Notice that if $\overline{O}(x)$ is compact in X, then $\omega(x) \neq \emptyset$ and is totally invariant, i.e., $T\omega(x) = \omega(x)$.

A point $p \in X$ is wandering if there exist a neighborhood U of p and $n_0 \in Z_+$ such that

$$U \cap T^n U = \emptyset \qquad (n > n_0).$$

The nonwandering set is

$$\Omega = \{p \in X\colon p \text{ is not wandering}\}.$$

Ω contains all limit points.

A set $K \subset K$ attracts a point $y \in X$ if $\overline{O}(y)$ is compact and $\omega(y) \subset K$. An attractor K is a compact nonempty invariant set (i.e., $TK \subset K$) which attracts some neighborhood of K. The basin of K is denoted by $B(K) = \{x \in X\colon \omega(x) \subset K\}$.

It follows easily from Zorn's lemma that every nonempty compact subset of X contains a maximal and a minimal element. Let K be an attractor. Then the set $\Omega \cap K$ is compact, invariant, and nonempty. For any $z \in \Omega \cap K$ there are minimal and maximal elements p, q of $\Omega \cap K$ such that $p \leq z \leq q$.

The point p is called a (k-) periodic point of T if $T^k p = p$. We call $O(p)$ a cycle, or a k-cycle. If $Tp = p$, then we say p is a fixed point. Let $\mathbb{P}$ denote the set of all periodic points.

A fixed point $p \in X$ is upper stable if for every $y \gg p$ there exists $z \in [[p, y]]$ such that

$$T^n[p, z] \subset [p, y], \qquad \text{for any } n \in Z_+.$$

If in addition z can always be chosen so that

$$\lim_{n\to\infty} T^n x = p \qquad (x \in [p, z])$$

then p is called asymptotically upper stable. We define lower stable and asymptotically lower stable analogously.

If p is either upper stable or lower stable, then p is said to be semistable. If p is both upper stable and lower stable we say p is order-stable. If p is both asymptotically upper stable and asymptotically lower stable we call p asymptotically order-stable.

Now let $q \in X$ have period $m > 1$. We say the m-cycle $O(q)$ is upper stable provided q is an upper stable fixed point for the map T^m. The other types of stability defined above are similarly extended to cycles.

It is easy to see that an m-cycle $O(p)$ is asymptotically upper stable if and only if there exists $x \gg p$ such that

$$\lim_{j \to \infty} T^{jm} x = p,$$

and analogously for asymptotically lower stable.

Finally, we state several known results.

THEOREM 1.1. *Let K be an attractor for the strongly monotone map T: $X \to X$. Then there exists an integer $m > 0$ such that K contains an order-stable m-cycle.*

THEOREM 1.2. *Let K be an attractor for the strongly monotone map T: $X \to X$ and $p \in K$ an m-periodic point which is not lower stable. Then there exists a unique m-periodic point $q \in K$ with the following property: $q \ll p$, $O(q)$ is asymptotically upper stable, and $\lim_{n \to \infty} T^{nm} x = q$ for all x such that $q < x < p$. A similar result holds when p is not upper stable.*

THEOREM 1.3 (Nonordering of Limit Sets). *An omega limit set of a strongly monotone map cannot contain two points related by $>$.*

THEOREM 1.4 (Krein-Rutman). *Let V be a Banach space with $\operatorname{Int} V_+ \neq \emptyset$ and T be a compact and strongly positive linear operator on V. Then the spectral radius $r(T) > 0$ is a simple eigenvalue of T with an eigenvector $v \in \operatorname{Int} V_+$ and $|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.*

Theorems 1.1 and 1.2 are due to Hirsch and can be found in [3]; Theorem 1.3 is contained in [9, p. 112]; Theorem 1.4 is adapted from Deimling [19, p. 228].

2. THE MAIN RESULTS

Assume that T: $X \to X$ is strongly monotone, $\overline{O}(z)$ is compact for the point $z \in X$, and that m is a positive integer. Let $\omega_m^0(z)$ denote the ω-limit set of z for the strongly monotone map T^m. Then we first give the decomposition of $\omega(z)$ as

$$\omega(z) = \bigcup_{j=0}^{m-1} \omega_m^j(z), \tag{2.1}$$

where $\omega_m^j(z) = T^j \omega_m^0(z)$ for $j = 1, 2, \ldots, m-1$.

By the definition of $\omega(z)$, it is obvious that $\bigcup_{j=0}^{m-1} \omega_m^j(z) \subset \omega(z)$.

Fix any point $y \in \omega(z)$. Then there exists a sequence $n_k \in Z_+$ such that $n_k \to \infty$ and $T^{n_k} z \to y$ as $k \to \infty$. Divide n_k by m and we get $h_k \in Z_+$

and j_k/m, that is,

$$n_k = mh_k + j_k,$$

where $0 \le j_k < m$. It is easy to see that $h_k \to \infty$ as $k \to \infty$ and there exist a sequence k_i in Z_+ and $j \in \{0, 1, 2, \dots, m-1\}$ such that $j_{k_i} \equiv j$ for $i = 1, 2, \dots$. Therefore, we can assume without loss of generality that $T^{mh_k} z \to x$ as $k \to \infty$ and $j_k \equiv j$ for $k = 1, 2, \dots$. By the continuity of T, $y = T^j x$. It follows from the definition of $\omega_m^0(z)$ and $\omega_m^j(z)$ that $x \in \omega_m^0(z)$ and $y \in \omega_m^j(z)$. Since y is an arbitrary point in $\omega(y)$, $\omega(z) \subset \bigcup_{j=0}^{m-1} \omega_m^j(z)$. This proves (2.1).

We are in position to state our main results.

THEOREM 1. *Let K be an attractor for strongly monotone map $T\colon X \to X$. Suppose that z is attracted to K such that either $\omega(z)$ is not a cycle or $\omega(z)$ is a cycle but is not semistable. Then there exist an order-stable n-cycle $O(p) \subset K$ and an order-stable m-cycle $O(q) \subset K$ such that*

$$p \lll \omega_{mn}^0(z) \lll q.$$

THEOREM 2. *Let $X \subset V$ be order-open where V is a real Banach space with* $\operatorname{Int} V_+ \neq \emptyset$. *Suppose that $T\colon X \to X$ is analytic and order-compact. If $DT(x)$ is a strongly positive operator for each $x \in X$, then every stable cycle in an attractor K is asymptotically stable. Moreover, if z is attracted to K but $\omega(z)$ is not a cycle, then K contains two asymptotically stable cycles.*

THEOREM 3. *Suppose every nonempty and compact subset of the strongly ordered space X has both a greatest lower bound and a least upper bound in X. If $T\colon X \to X$ is monotone, then T has a globally asymptotically order-stable fixed point if and only if*

(a) *$\overline{O}(z)$ is compact for any $z \in X$, and*

(b) *there is not more than one fixed point.*

Remark. Theorem 1 is a generalization of [3, Theorem 4.3] and [16, Theorem 2], where the order relation between $\omega(z)$ and p, q was given. Because the limit set dichotomy theorem for strongly monotone flows does not carry over to strongly monotone discrete-time dynamical systems, there is great possibility that the order relation $O(p) \lll \omega(z) \lll O(q)$ does not hold in our Theorem 1. But if the basin $B(K)$ of K has the property that any two elements of B have both a least upper bound and a greatest lower bound in B, then p and q in Theorem 1 are fixed points (see [3, Theorem 5.1]) and $p \lll \omega(z) \lll q$. Moreover, if $K \cap \mathbb{P}$ is finite, then K contains two asymptotically order-stable cycles. If T is a strongly monotone map, then Theorem 3 has been proved by Takáč in either [9, Theorem 2.4] or [9, Corollary 6.5]. Thus, Theorem 3 is in the spirit of without strong assumption and gives a proper credit to [9] for original results. Therefore, Theorem 3 here generalized [4, Theorem 9; 17, Theorem B; 18, Theorem B].

3. THE PROOF OF RESULTS

Before proceeding to the proof of our main results, we present two lemmas which are taken from [9].

LEMMA 3.1. *Suppose* $x \ll y$ *and* $\omega(x) \neq \omega(y)$. *Then*:

(i) *If* $\omega(x) = \{p, Tp, \dots, T^{m-1}p\}$ *is an* m-*cycle and* $\omega_m^0(x) = p$, *then*

$$T^j p \ll \omega_m^j(y), \qquad j = 0, 1, \dots, m-1. \tag{3.1}$$

(ii) *If* $\omega(y) = \{p, Tp, \dots, T^{m-1}p\}$ *is an* m-*cycle and* $\omega_m^0(y) = p$, *then*

$$\omega_m^j(x) \ll T^j p, \qquad j = 0, 1, 2, \dots, m-1. \tag{3.2}$$

Lemma 3.1 is a corollary of Theorem 3.10 in [9]. Applying it, we can give the proof of the following lemma. Since this proof has been presented in Takáč's proof of [9, Theorem 6.1], we omit it.

LEMMA 3.2. *Let* K *be an attractor for the strong monotone map* T. *Suppose* z *is attracted to* K *and either* $\omega(z)$ *is not a cycle or it is a cycle but is not semistable*. *Then we have*:

(i) *There exist an asymptotically lower stable* n-*cycle* $O(y) \subset K$ *and* $z_1 = T^N z \in O(z)$ *such that*

$$y \ll z_1,$$

and

$$T^j y \ll \omega_n^j(z_1), \qquad j = 0, 1, 2, \dots, n-1. \tag{3.3}$$

(ii) *There exists an asymptotically upper stable* m-*cycle* $O(w) \subset K$ *such that*

$$z_1 \ll w,$$

and

$$\omega_m^j(z_1) \ll T^j w, \qquad j = 0, 1, 2, \dots, m-1, \tag{3.4}$$

where z_1 *is the same point as* (i). *Moreover*, *we can choose* N *to be a multiple of* mn.

Proof of Theorem 1. Recall from (i) of Lemma 3.2 that there exist a point $z_1 \in O(z)$ and an asymptotically lower stable n-cycle $O(y) \subset K$ such that

$$y \lessdot z_1,$$

and

$$T^j y \lessdot \omega_n^j(z_1), \qquad j = 0, 1, 2, \dots, n-1.$$

Set

$$\mathbb{P}_0 = \left\{x \in K \cap \mathbb{P} : O(x) \text{ is a lower stable } n\text{-cycle and } x \lessdot \omega_n^0(z_1)\right\}.$$

Obviously, $y \in \mathbb{P}_0$.

Denote by $\mathbb{Y}$ the set of all simply ordered subsets Y of $\mathbb{P}_0$ such that they contain y. We shall show that the ordered set $\mathbb{Y}$ endowed with the $\subset$ ordering possesses a maximal element. Consider a nonempty, simply ordered subset $\mathbb{Y}'$ of $\mathbb{Y}$. Set

$$Z = \bigcup\{Y : Y \in \mathbb{Y}'\}.$$

It is easy to see that $y \in Z$ and $Z \subset \mathbb{P}_0$. For any $q, r \in Z$, there exist $Y_q, Y_r \subset Z$ such that $q \in Y_q$ and $r \in Y_r$. Since $\mathbb{Y}'$ is simply ordered under the ordering $\subset$, either $Y_q \subset Y_r$ or $Y_r \subset Y_q$, which implies that either $q, r \in Y_r$ or $q, r \in Y_q$. Because Y_q and Y_r are simply ordered subsets of $\mathbb{P}_0$, q and r are related by R; that is, Z is a simply ordered subset of $\mathbb{P}_0$. This proves $Z \in \mathbb{Y}$ is an upper bound of $\mathbb{Y}'$. Hence, we may apply Zorn's lemma to conclude that $\mathbb{Y}$ possesses a maximal element, say H. Since all points in H are n-periodic, for any $x, y \in H$ with $x \neq y$, either $x \lessdot y$ or $y \lessdot x$. We shall show that H has the following properties:

(i) H has an upper bound p in H; and

(ii) such a p is upper stable.

(i) Since $\operatorname{clos} H$ is a nonempty compact set, it follows from Zorn's lemma that $\operatorname{clos} H$ contains a maximal element p. If $p \in H$, then $h \leq p$ for every element h of H. Otherwise, there is $h \in H$ such that $h \leq p$ doesn't hold. Since H is simply ordered and $p, h \in H$, we have $p \lessdot h$, a contradiction to the maximality of p. So if $p \in H$ then it is an upper bound of H. In order to prove (i), it suffices to show that $p \in H$. Suppose not, then $p \in \operatorname{clos} H - H$. By definition, there is a sequence $\{p_i\} \subset H$ such that $p_i \to p$ as $i \to \infty$. We assert that for any $h \in H$, there exists i_h such that $h \lessdot p_i$ for $i > i_h$. In fact, if it is not true, there are a point $h \in H$ and a sequence $i_k \to \infty$ as $k \to \infty$ such that $h \gtrdot p_{i_k}$. Now letting $k \to \infty$, we have $h \geq p$. Since $h \neq p$, $h \gtrdot p$, contradicting that p is a maximal element of clos H. Hence, our assertion holds. Letting $i \to \infty$, we obtain $h \le p$ for any $h \in H$, that is, p is an upper bound of H. It is easy to see that we can choose the sequence $\{p_i\}$ such that it is monotone, that is, $p_i \ll p_{i+1}$ for $i = 1, 2, \dots$. Hence, p is also n-periodic and lower stable. Because $p_i \ll \omega_n^0(z_1)$ for all $i, p \le \omega_n^0(z_1)$. We claim that $p \,\overline{\in}\, \omega_n^0(z_1)$. If $\omega(z) = \omega(z_1)$ is not a periodic orbit, then from (2.1) we conclude that $p \ne \omega_n^0(z_1)$. Theorem 1.3 implies that $p \,\overline{\in}\, \omega_n^0(z_1)$. If $\omega(z)$ is a periodic orbit, by supposition, $\omega(z)$ is not semistable, which shows $\omega(z) \ne O(p)$. Therefore, the claim is also true, i.e., $p < \omega_n^0(z_1)$. By the strong monotonicity of $T^n, p \ll \omega_n^0(z_1)$. This proves $p \in \mathbb{P}_0$. The above proof shows that $H \cup \{p\}$ is simply ordered and $H \cup \{p\} - H = \{p\}$, contradicting the maximality of H. This proves (i).

(ii) We claim that the point p obtained in (i) is upper stable. In order to prove this claim, let us assume the contrary. Then it follows from Theorem 1.2 that there is a unique n-periodic point $q \in K$ with the following property: $p \ll q$, $O(q)$ is asymptotically lower stable, and $T^{mn}x \to q$ as $m \to \infty$ for all x such that $p < x < q$. We shall prove that $q \ll \omega_n^0(z_1)$. Since $p \ll \omega_n^0(z_1)$, we can choose $x \in [[p, q]]$ such that $x \ll \omega_n^0(z_1)$. By strong monotonicity, $T^{mn}x \ll T^{mn}\omega_n^0(z_1) = \omega_n^0(z_1)$ for $m = 1, 2, \dots$. Letting $m \to \infty$, we get that $q \le \omega_n^0(z_1)$. By supposition, $\omega_n^0(z_1) \ne q$. Theorem 1.3 implies that $q < \omega_n^0(z_1)$. Again, using the strong monotonicity of T^n, we obtain that $q \ll \omega_n^0(z_1)$. This shows $q \in \mathbb{P}_0$. Obviously, $H \cup \{q\}$ is simply ordered and $H \cup \{q\} - H = \{q\}$, contradicting the maximality of H. This proves (ii). So far, we have proved that K contains an order-stable n-periodic point p such that $p \ll \omega_n^0(z_1)$.

Similarly, we set

$$\mathbb{P}^0 = \left\{x \in K \cap \mathbb{P}\colon O(x) \text{ is a upper stable } m\text{-cycle and } x \gg \omega_m^0(z_1)\right\},$$

where m and z_1 are given in (ii) of Lemma 3.2. It follows from (ii) of Lemma 3.2, $w \in \mathbb{P}^0$. In quite the same manner, we can prove that $\mathbb{P}^0$ contains an order-stable m-periodic point q such that $\omega_m^0(z_1) \ll q$. Finally, from

$$\omega_{mn}^0(z_1) \subset \omega_m^0(z_1) \cap \omega_n^0(z_1),$$

we conclude that $p \ll \omega_{mn}^0(z_1) \ll q$. Since $z_1 = T^N z$ and N is a multiple of mn, $\omega_{mn}^0(z_1) = \omega_{mn}^0(z)$, therefore, $p \ll \omega_{mn}^0(z) \ll q$. The proof of Theorem 1 is complete.

LEMMA 3.3. *Let V be a real Banach space with* Int $V_+ \ne \emptyset$ *and $X \subset V$ be order-open. Suppose $T\colon X \to X$ is analytic and order-compact. If p is a fixed point of T and $DT(p)$ is strongly positive with $r(DT(p)) = 1$, then there exist a neighborhood Ω of p and a simply ordered arc $C \subset \Omega$ containing p such* *that either C consists of fixed points of T or p is a unique fixed point of T in Ω. In the latter case, if p is order-stable, then it is asymptotically order-stable.*

Proof. Since T is order-compact, $DT(p)$ is compact. By the Krein–Rutman theorem, $r(DT(p)) = 1$ is a simple eigenvalue of $DT(p)$ to which there exists a positive eigenvector $v \in \operatorname{Int} V_+$.

Let $L \subset V$ be a one-dimensional space spanned by v. By the Hahn–Banach Theorem, L has a complementary closed linear subspace $M \subset V$ so that $V = M \oplus L$. Thus, without loss of generality, we may assume that $p = 0$ and

$$T(x) = (y + f(y,z), Az + g(y,z)), \qquad y \in L \text{ and } z \in M,$$

where f and g are analytic near 0 with $Df(0) = 0$ and $Dg(0) = 0$. The Krein-Rutman Theorem implies that all eigenvalues of A have modulus less than 1. Now we restrict our attention to $U \times W$, where U is a neighborhood of the origin in L and W is a neighborhood of the origin in M. Since each eigenvalue of A has modulus less than 1, $A - I$ is invertible where I: $M \to M$ is the identity operator. It is easy to see that $(A - I)z + g(y,z)$ vanishes at the origin. By the Implicit Function Theorem (see [19, Theorem 15.3]), there is a neighborhood $\Omega = U_1 \times W_1 \subset U \times W$ of $(0,0)$ and an analytic function $\varphi: U_1 \to W_1$, $\varphi(0) = 0$ such that $(A - I)z + g(y,z) = 0$ for $(y,z) \in \Omega$ if and only if $z = \varphi(y)$. This proves that all fixed points of T in Ω lie on the arc C: $z = \varphi(y)$ for $y \in U_1$. Let $F(y) = f(y, \varphi(y))$, $y \in U_1$. Then the set of fixed points for T contained in Ω is

$$\{(y,z): F(y) = 0, z = \varphi(y), y \in U_1\}.$$

Since f and φ are analytic, so is F. Thus either

(a) there is an integer $q \geq 2$ and a nonzero constant β such that

$$F(y) = \beta y^q + O(|y|^{q+1}) \qquad \text{as } |y| \to 0;$$

or else

(b) $F(y) \equiv 0$ for $y \in U_1$.

If (a) occurs, then $(0,0)$ is a unique fixed point in Ω. If (b) occurs, then C is composed of fixed points for T.

Since $(A - I)\varphi(y) + g(y, \varphi(y)) \equiv 0$ for $y \in U_1$ and $Dg(0) = 0$, and $A - I$ is invertible, $D\varphi(0) = 0$. This implies that C is tangent to v at $p = 0$. Therefore, C can be regarded as a simply ordered curve. It is easy to see that in the case (a), $(0,0)$ is asymptotically order-stable if and only if q is odd and $\beta < 0$. Therefore, if $(0,0)$ is order-stable and isolated, then q is odd and $\beta < 0$. This shows that $p = (0,0)$ is asymptotically order-stable.

Proof of Theorem 2. We note that if $T: X \to X$ is strongly monotone and order-compact then the concepts of stability and order-stability for discrete-time dynamical systems in X are equivalent. Therefore, by Theorems 1.1 and 1, and replacing T by T^m for some $m > 0$, in order to prove Theorem 2, we only have to prove that every order-stable fixed point in K is asymptotically order-stable. By Lemma 3.3, it suffices to show that every order-stable fixed point in K is isolated.

Let $p \in K$ be an order-stable fixed point for T. Suppose that r is the spectral radius of the strongly positive operator $DT(p)$. Then it follows from the stability of p that $r \leq 1$. If $r < 1$, then p is asymptotically stable, and hence it is isolated. We only have to consider the case $r = 1$. Suppose $p \in K$ is not an isolated order-stable fixed point for T. Then, by Lemma 3.3, there exists a neighborhood Ω of p in which the set of all fixed points of T is simply ordered arc C. Let $D \subset K$ be a simply ordered arc of fixed points of T such that $p \in D$, and D is maximal with respect to set inclusion; such a D exists by Zorn's lemma. By maximality we know that D is compact, hence, D has a least upper bound $q \in D$. Since the fixed point q is not isolated and lower stable, the spectral radius of $DT(q)$ is 1. Applying Lemma 3.3, we conclude that the simply ordered arc D of fixed points can be property extended, in contradiction to its maximality. This proves Theorem 2.

Proof of Theorem 3. Suppose T is monotone and has a globally asymptotically order-stable fixed point, then it is obvious that (a) and (b) hold.

We shall prove the converse (the ‘‘if’’ half of Theorem 3). Fix any $x \in X$, by (a), $\omega(x) \subset X$ is compact. By assumption, $\omega(x)$ has both a greatest lower bound p and a least upper bound q. Therefore, $p \leq \omega(x) \leq q$. By the total invariance of $\omega(x)$ and the monotonicity of T, $Tp \leq \omega(x) \leq Tq$. It follows from the definitions of p and q that $Tp \leq p$ and $Tq \geq q$. Applying Lemma 3.1 in [3, p. 145], we know that $\omega(p)$ is a singleton, and so is $\omega(q)$. Since $T^n p \leq p$ for $n = 1, 2, \dots$, $\omega(p) \leq p$. Similarly, $q \leq \omega(q)$. So far we have proved that $\omega(p)$ and $\omega(q)$ are fixed points and $\omega(p) \leq p \leq \omega(x) \leq q \leq \omega(q)$. By (b), $\omega(p) = \omega(q) = r$, the unique fixed point of T, and $\omega(x) = r$, that is, T has the unique fixed point r such that $T^n x \to r$, as $n \to \infty$ for any $x \in X$. Since X is strongly ordered, (SO1) tells us that there exist $x, y \in X$ with $x \ll r \ll y$. $\omega(x) = \omega(y) = r$ implies that r is asymptotically order-stable, hence it is globally asymptotically order-stable.

Finally, we shall present two examples to show applications of our results.

EXAMPLE 1. Consider the system of ordinary differential equations

$$\dot{x} = f(t, x), \qquad x \in X, \tag{3.5}$$

where $X = \mathbb{R}^n$ or $\mathbb{R}^n_+$ and $f: \mathbb{R} \times X \to \mathbb{R}^n$ is analytic and 2π-periodic in t. If (1.1) is cooperative in x, $Df(t, x)$ has all off-diagonal terms nonnegative for each (t, x). Assume that $Df(t, x)$ is irreducible for every $(t, x) \in \mathbb{R} \times X$. It is know that the Poincaré map $T(x) = \phi(2\pi, x)$, i.e., the period map of this equation, has strongly positive derivatives and is analytic in X. Suppose that the system (3.5) is dissipative, i.e., there is a compact set $K \subset X$ such that all solutions of (3.5) will eventually enter the compact set K. Then from Theorems 1 and 2 we can conclude that either every solution of (3.5) is asymptotic to a semistable $2m\pi$-periodic solution or (3.5) has at least two asymptotically stable subharmonic solutions. This result can be applied to the single loop positive feedback system in $\mathbb{R}^n_+$ (see [7])

$$\begin{aligned} \frac{dx_1}{dt} &= f(t, x_n) - \alpha_1(t) x_1 \\ \frac{dx_i}{dt} &= x_{i-1} - \alpha_i(t) x_1; \qquad i = 2, 3, \ldots, n, \end{aligned} \tag{3.6}$$

where we assume that $\alpha_i(t)$ and $f(t, x_n)$ are analytic and 2π-periodic in $t \in \mathbb{R}$ and that

$$f(t, u) \le au + b, \qquad a, b > 0$$
$$\alpha_i(t) \ge \alpha_i > 0, \qquad 1 \le i \le n$$
$$a < \prod_{i=1}^{n} \alpha_i .$$

If $f(0, 0) > 0$ and $(\partial f / \partial u)(t, u) > 0$ for all $u \ge 0$, $t \in \mathbb{R}$, then (3.6) is cooperative and irreducible. Therefore, it follows from [7] that there exists a closed order interval $[x_0, y_0]$ such that all solutions of (3.6) will eventually enter it. Thus, either every solution of (3.6) is asymptotic to a semistable subharmonic solution or (3.6) has at least two asymptotically stable subharmonic solutions.

EXAMPLE 2. Consider a time-periodic reaction diffusion equation

$$\begin{aligned} u_t &= \Delta u + f(t, x, u), && \text{on } \Omega \times (0, \infty) \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) &= u_0(x) && \text{on } \overline{\Omega}, \end{aligned} \tag{3.7}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth compact n-dimensional submanifold with interior Ω and boundary $\partial\Omega$, $f(t, x, u): \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is analytic function 2π-periodic in t, and $\partial/\partial n$ is the Neumann boundary operator in the outward normal direction. This equation is well posed on the space $X := C(\overline{\Omega})$ (see [20]), that is, for every $u_0 \in C(\overline{\Omega})$, (3.7) has a unique solution defined by $\phi_t(u_0)x = u(x,t)$. The period map $T(u_0) = \phi_{2\pi}(u_0)$ is analytic and compact. The space X with the pointwise ordering is a strongly ordered Banach space and the Parabolic Strong Maximum Principle implies that the derivative $DT(u_0)$ is a strongly positive operator for every $u_0 \in X$. If there exist constants $c_1, c_2 > c_1$ such that

$$f(t,x,c_1) > 0 > f(t,x,c_2)$$

for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$, then it can be shown that the order interval $[c_1, c_2]$ is positively invariant under the flow that (3.7) induces (see [21]). Let $K = \overline{T[c_1, c_2]}$. Then $K \subset [c_1, c_2]$ is compact. So K is an attractor. By Theorems 1 and 2, we obtain that either every orbit for T starting from $[c_1, c_2]$ converges to a semistable cycle or K contains at least two asymptotically stable cycles. Equivalently, either every solution $\phi_t(u_0)$ of (3.7) for $u_0 \in [c_1, c_2]$ is asymptotic to a semistable subharmonic solution or $[c_1, c_2]$ contains at least two asymptotically stable subharmonic solutions.

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