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Ljapunov Approach to Multiple Hopf Bifurcation

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1. INTRODUCTION

During the past decade a considerable number of systems of ordinary autonomous differential equations have been investigated which exhibit the phenomenon of Hopf bifurcation, i.e. the appearance of periodic solutions (limit cycles) branching off from a stationary state of the system when certain changes of the parameters occur. Most of the examples known to us are models of chemical and biological systems. In spite of the fact that Hopf's original work on this subject appeared in 1942 [1], investigators had to rely on numerical methods until recently. Through the work of Andronov *et al.* [2], Cohen and Keener [3], Poore [4], Marsden and McCracken [5], Hsü and Kazarinoff [6], Brušlinskaya [7], and Hassard and Wan [8], alternative formulations of the theory have been established which facilitate its application.

Since the work of Andronov and his co-workers [2] it is known that the bifurcation of several limit cycles from a focus is directly related with the stability of the focus. They assign to a focus a set of numbers (α_1, α_3, α_5,...) which they call focal values. α_1 is simply related to the real part of the eigenvalues of the Jacobian of the system, α_3 corresponds to μ_2 commonly used in bifurcation theory. The sign of the first nonvanishing focal value determines the stability of the focus. Furthermore, the number of the leading α_i's ($i = 1, 3, 5,...$) which vanish simultaneously is the number of limit cycles which may bifurcate from a focus. This is the reason why the investigation of the bifurcation of limit cycles deals with the computations of focal values.

As far as we know, an explicit expression for the second focal value α_3 was given for the first time by Andronov et al. [2]. They restricted themselves to systems with two variables. Equivalent expressions for systems with three variables were derived in [3] and for systems of higher dimension in [4, 5, 6], and under special assumptions in [7]. Hassard and Wan [8] gave an expression for the third focal value without noting the relation to the bifurcation of several limit cycles. This problem was recently investigated by Chafee [9] for systems of higher dimension.

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In this paper we apply Ljapunov's direct method to the bifurcation problem. Ljapunov was the first who did this (cited by Malkin [11]). His approach is somewhat more appealing than ours, but his result, a general expression for all focal values, is not readily applicable. We restrict ourselves to systems with two variables. The construction of an appropriate Ljapunov function reduces the stability problem of a focus to the solution of a sequence of systems of linear equations. The derivatives which are necessary in other approaches (see [5, p. 125]) are very complicated because of the use of complex numbers and/or of powers of trigonometric functions. Both do not occur in our approach.

We derive explicit expressions for focal values of two-dimensional systems up to the fourth, which is a new result of this paper. Our second and third focal values agree with those given by Hassard and Wan [8].

In the following section we discuss a two-dimensional system of differential equations which was recently investigated by Troy [10], Hsü and Kazarinoff [6], and Hadeler *et al.* [12]. This system exhibits the bifurcation of two limit cycles. In Section 3 we sketch how Ljapunov's direct method is applied and how the focal values can be derived. In Section 4 we consider the behavior of a system close to bifurcation and show where limit cycles may appear, where a pair of limit cycles may co-exist, and where limit cycles disappear by coalescence. In Section 5 we establish the relation of our results to those of Andronov *et al.* In the Appendix we describe our calculations in greater detail and list our results.

2. A Simple Example

The system of two equations

$$\begin{aligned} \dot{x} &= x + y + x - x^3/3 \\ \dot{y} &= \rho(a - x - by) \end{aligned} \tag{2.1}$$

has been proposed by Fitzhugh as a rather simplified model of the nerve impulse. It was recently investigated by Troy [10], Hadeler *et al.* [12], and Hsü and Kazarinoff [6]. We adopt essentially the representation of Hsü and Kazarinoff. Two of the parameters b and ρ are restricted to the interval (0, 1). By introducing x_0 as the x-value of the unique stationary state of the system, the stability of the stationary state and that of possible limit cycles may be stated as follows:

1. The stationary state (x_0, y_0) is stable (unstable) if the quatity α_1 is negative (positive) with

$$\alpha_1 = 1 - x_0^2 - b\rho. \tag{2.2}$$

2. A bifurcating limit cycle is stable (unstable) for negative (positive) α_3

$$\alpha_3 = 2b - 1 - \rho b^2. \tag{2.3}$$

The bifurcation takes place at a point $(x_0, b, \rho)^0$ in the parameter space with $\alpha_1 = 0$ on that side where α_1 assumes the sign different from that of α_3 (stable focus with unstable limit cycle or unstable focus with stable limit cycle).

Inspection of Equations (2.2) and (2.3) shows that for a fixed value of x_0 with $x_0^2 < 1$ the conditions $\alpha_1 = 0$ and $\alpha_3 = 0$ can be fulfilled inside the square $b, \rho \in (0, 1)$.

For

$$\begin{aligned} b_0 &= 1/(1 + x_0^2) \\ \rho_0 &= 1 - x_0^4 \end{aligned} \tag{2.4}$$

both α_1 and α_3 vanish simultaneously.

According to a theorem of Andronov *et al.* [2, p. 254, Theorem 40], any neighborhood of (b_0, ρ_0) should contain points (b, ρ) giving rise to a pair of limit cycles as solutions to system (2.1). These were found by Hadeler *et al.* [12] by numerical methods. Our own numerical results are shown in Figs. 1 and 2. The subject of this paper is the prediction of this kind of behavior by analytical methods.

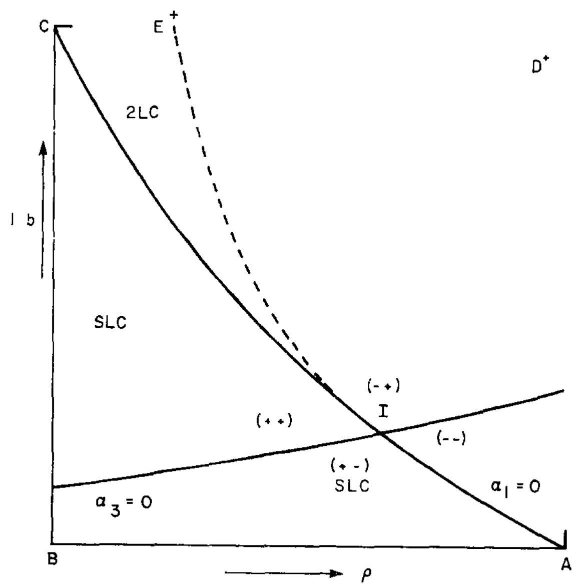

FIG. 1. Section of the parameter space of system (2.1) with $x_0 = 0.8$ and $\rho \in (0.36, 0.72)$ and $b \in (0.5, 1.0)$. The two solid curves show where α_1 and α_3, respectively, vanish. These curves separate four regions of different sign patterns of α_1 and α_3, given in brackets. A limit cycle bifurcates at the curve $\alpha_1 = 0$ on that side of the curve where the signs become different. The stability of the limit cycle is determined by the sign of α_3 ($\alpha_3 < 0$ means a stable limit cycle). The stable limit cycle (SLC) bifurcates between A and the intersection point I. The SLC persists in the $(++)$-region and between C and I an unstable limit cycle bifurcates inside of the stable one (2LC). The two coalesce and disappear at the dashed curve between E and I. This kind of behavior can be predicted qualitatively if it is known that the quantity P_6 derived in this paper is negative at I.

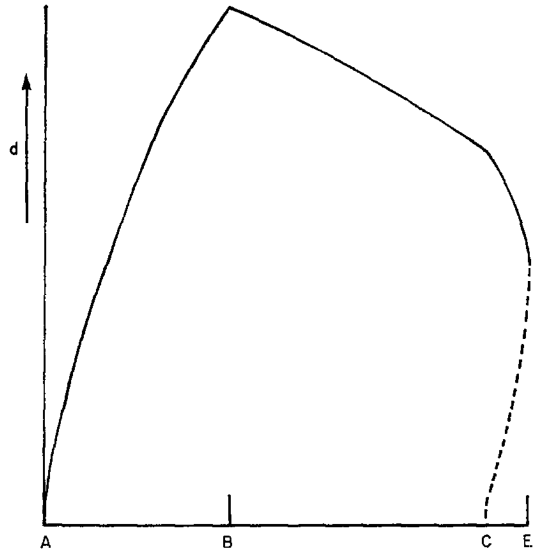

FIG. 2. Diameters d of the limit cycles of system (2.1) in arbitrary units. The parameters are varied along the margin of the rectangle of Fig. 1 starting at point A. The stable limit cycle is represented by the solid curve, the unstable one by the dashed curve.

3. The Stability of a Focus

From the work of Hopf and of Andronov *et al.* it is known that bifurcation of limit cycles may occur if a focus changes stability. Here we shall show how the stability of a focus may be determined in the marginal case where the Jacobian of a system has purely imaginary eigenvalues and therefore the stability cannot be deduced by linear stability analysis. We assume that the system under investigation has been transformed into a convenient form which is called the canonical form by Andronov *et al.*:

$$\begin{aligned} \dot{x} &= F(x, y) = -\omega y + \sum_{k=2}^{K} \sum_{l=0}^{k} F_{k-l,l} x^{k-l} y^{l}, \\ \dot{y} &= G(x, y) = \omega x + \sum_{k=2}^{K} \sum_{l=0}^{k} G_{k-l,l} x^{k-l} y^{l}. \end{aligned} \tag{3.1}$$

This canonical form looks rather special, but it can be derived from a given system under fairly general conditions. Consider a system of differential equations with two variables u and v:

$$\begin{aligned} \dot{u} &= U(u, v; \mathbf{a}), \\ \dot{v} &= V(u, v; \mathbf{a}). \end{aligned}$$

The vector $\mathbf{a}$ denotes the parameters of the system. The functions U and V are assumed to be C^K-functions of u and v and C^1-functions of the parameters. Suppose that $\mathbf{a}_0$ is a bifurcation point of the system, i.e. one of the critical points (u_0, v_0) of the system is a focus and the Jacobian of the system at this point has purely imaginary eigenvalues $\pm i\omega$ (we assume $\omega \neq 0$ throughout) at this critical point.

Then one has to shift the origin into the point (u_0, v_0) and to apply a non-singular linear transformation (see [2], p. 253) which yields an antisymmetric Jacobian of the transformed system. The original system is expressed in the new coordinates and the right hand sides are expanded into Taylor series about the origin with terms up to the order K, which results in a system of the canonical form (3.1). Terms of order higher than K are neglected.

For an investigation of the system for points close to $\mathbf{a}_0$ it suffices to consider deviations from $\mathbf{a}_0$ of the form $\mathbf{a} = \mathbf{a}_0 + \epsilon\mathbf{b}$. $\mathbf{b}$ is an arbitrary but fixed vector in parameter space and ϵ is a sufficiently small factor. In this case the critical point is a function of ϵ, say $(\xi(\epsilon), \eta(\epsilon))$ with $\xi(0) = u_0$ and $\eta(0) = v_0$, which is unique for small values of ϵ because we assume a non-singular Jacobian of the original system at (u_0, v_0). By shifting the origin into $(\xi(\epsilon), \eta(\epsilon))$ and applying the same linear transformation as mentioned above, one obtains a system of the form (4.1) by a Taylor expansion of the right hand sides including first order terms in ϵ.

We try to find a function $V(x, y)$ with

(i) $V(0, 0) = 0$,

(ii) $V(x, y) > 0$,

(iii) $\dot{V}(x, y) = FV_x + GV_y$ definite

for some sufficiently small neighborhood of the origin. Then V is a Ljapunov function and the sign of $\dot{V}$ determines whether or not the origin is a stable focus of system (3.1).

The first two conditions are satisfied by the function

$$V(x, y) = \sum_{k=2}^{K-1} \sum_{l=0}^{k} V_{k-l,l} x^{k-l} y^l \tag{3.2}$$

with $V_{20} = V_{02} = \frac{1}{2}$ and $V_{11} = 0$. In this case the second order terms of V are $(x^2 + y^2)/2 = r^2/2$ and V is approximately equal to $r^2/2$ for small values of r.

The function $\dot{V}$ is the directional derivative of V on the vectors $(\dot{x}, \dot{y})'$. By the form of V and that of F and G, it is easily seen that $\dot{V}$ also starts with second order terms, but these are zero. We assume that $\dot{V}$ has been arranged by powers of x and y, and by renaming the coefficients we obtain:

$$\dot{V} = \sum_{\mu=3}^{2K+1} \sum_{\nu=0}^{\mu} P_{\mu-\nu,\nu} x^{\mu-\nu} y^{\nu}. \tag{3.3}$$

In the Appendix we shall describe the calculations in detail; therefore, we give here only the results and sketch how they are obtained. The $P_{\mu-\nu,\nu}$ are linear functions of the $V_{k-l,l}$ with $k \leq \mu$. V's with $k > \mu$ do not occur. This is the reason why the P's and then the V's can be determined step by step starting with $\mu = 3$. The P's are determined by the following rules:

(a) $P_{\mu-\nu,\nu} = 0$ if μ or ν or both are odd numbers,

(b) $P_{\mu-\nu,\nu} = c(\mu/2, \nu/2)\, P_\mu$ if both μ and ν are even numbers.

By $c(\mu/2, \nu/2)$ we mean the $\nu/2$th coefficient of the binomial formula with the power $\mu/2$. P_μ is a new quantity which is uniquely determined by these equations. After this we arrive at

$$\dot{V} = P_4 r^4 + P_6 r^6 + \cdots = \sum_{\mu=4,6,\ldots} P_\mu r^\mu. \tag{3.4}$$

The expressions P_4, which was first published by Andronov *et al.*, P_6 and P_8 are given in the Appendix. After these coefficients are known, the stability of the origin is easily determined by Ljapunov's stability theory: in some neighborhood of the origin, with the exception of the origin itself, $\dot{V}$ is a definite function with the sign of the first non-vanishing coefficient P_4, P_6,.... Thus, the focus is asymptotically stable if the first non-vanishing coefficient is negative and unstable in the opposite case.

4. Bifurcation of Limit Cycles from a Focus

From the result of the preceding section we are able to decide whether or not a focus of a system in the form of (3.1) is stable. Here we shall investigate how small perturbations of such a system may give rise to the creation of limit cycles. The perturbed system may be written in the form

$$\begin{aligned} \dot{x} &= F(x, y) + \epsilon f(x, y), \\ \dot{y} &= G(x, y) + \epsilon g(x, y), \\ (f(0, 0) &= g(0, 0) = 0). \end{aligned} \tag{4.1}$$

The functions f and g are assumed to be given as polynomials in x and y like the functions F and G of Eq. (3.1). But their linear terms are completely determined by the process described below Eq. (3.1). Therefore, no special assumption on these terms can be made. As becomes apparent below, it is just the fact that the Jacobian of system (4.1) may have eigenvalues apart from the imaginary axis which gives rise to the bifurcation of limit cycles.

As a Ljapunov function we try some function W, which deviates by a small amount from the Ljapunov function V of system (3.1):

$$W(x, y; \epsilon) = V(x, y) + \epsilon v(x, y). \tag{4.2}$$

Then the directional derivative $\dot{W}$ of W on vectors $(\dot{x}, \dot{y})'$ is

$$\dot{W} = \dot{V} + \epsilon(Fv_x + fV_x + Gv_y + gV_y) + \epsilon^2(\cdots). \tag{4.3}$$

The partial derivatives v_x and v_y are multiplied in the same way with F and G as V_x and V_y in the previous section. The new terms fV_x and gV_y consist of quantities which have been determined before or are given by (4.1). Thus, if $v(x, y)$ is written in the form of (3.2), then the coefficients $v_{k-l,l}$ are determined by the same systems of linear equations as the $V_{k-l,l}$ but with different right hand sides, which contain the new terms. It is therefore possible to arrive at

$$\dot{W} = \epsilon p_2 r^2 + (P_4 + \epsilon p_4)\, r^4 + (P_6 + \epsilon p_6)\, r^6 + \cdots + \epsilon^2(\cdots). \tag{4.4}$$

Apart from the terms with ϵ^2, which we neglect because we are only interested in small values of ϵ, $\dot{W}$ is a polynomial in the variable r^2. A positive zero of this polynomial means that for a circle with the corresponding r, $\dot{W}$ changes its sign. If some curve can be found, which is defined by $W(x, y; \epsilon) = W_0$ and lies entirely in a region of one sign (say negative) of $\dot{W}$, all vectors $(\dot{x}, \dot{y})'$ point inwards on this curve. If this curve encloses a circle with $\dot{W} = 0$, another curve with $W(x, y; \epsilon) = W_1$ may be found inside of this circle with $\dot{W} > 0$. Then, according to the Poincaré–Bendixson theorem, between the two curves with constant $W = W_0$ and $W = W_1$ lies a limit cycle or a critical point of system (4.1). Here we only consider zeroes of $\dot{W}$ which are continuations of the multiple root $r^2 = 0$ of $\dot{V}$ and therefore they are arbitrarily small if small values of ϵ are considered. For small values of r and ϵ the curves of constant W are close to circles and a second critical point of system (4.1) is excluded because the origin is an isolated critical point of this system if $\omega \neq 0$. Under these assumptions, a zero of $\dot{W}$ is associated with a limit cycle nearby, which is stable if $\dot{W}$ changes from positive to negative values with increasing r.

Case 1. $P_4 \neq 0$. The smallest root of $\dot{W}$ apart from the trivial one ($r^2 = 0$) is near to

$$r_1^2 = -(\epsilon p_2)/P_4\,. \tag{4.5}$$

It is positive for different signs of (ϵp_2) and P_4. As shown in the appendix, ϵp_2 is the real part of the eigenvalues associated with the origin of system (4.1). Thus, we have the result:

If the focus of system (3.1) is stable because of negative P_4, a stable limit cycle bifurcates for that sign of ϵ where the origin of system (4.1) becomes an unstable focus because of the linear terms. For positive P_4 an unstable limit cycle bifurcates.

We do not investigate the case $p_2 = 0$. The radius of the limit cycle increases proportionally to $|\epsilon|^{1/2}$, which has an infinite derivative at $\epsilon = 0$.

Case 2. $P_4 = 0$, $P_6 \neq 0$. Suppose we have found a point $\mathbf{a}^0$ in the parameter space of system (4.1) which gives rise to purely imaginary eigenvalues of the system and to $P_4 = 0$. In order to see what happens for points $\mathbf{a}$ close to $\mathbf{a}^0$, we have to consider a two parameter bifurcation. The first parameter δ measures deviations from $\mathbf{a}^0$ along a direction where the eigenvalues remain imaginary. For a point with fixed δ the deviation along a different direction measured by ϵ is considered with sufficiently small ϵ. Then the essential terms of $\dot{W}$ with respect to the small roots are:

$$\dot{W} = r^2[\epsilon p_2 + \delta p_4 r^2 - P_6 r^4] + \cdots \tag{4.6}$$

The first nontrivial roots of $\dot{W}$ are approximately

$$r_{1,2}^2 = -(\delta p_4/2P_6) \pm [(\delta p_4/2P_6)^2 - (\epsilon p_2/P_6)]^{1/2}. \tag{4.7}$$

It is easily seen that a sufficient condition for one positive solution of (4.7) is that (ϵp_2) and P_6 have opposite signs. The stability of the corresponding limit cycle is determined by the sign of P_6. It is stable for negative P_6. This result is essentially the same as in Case 1 if P_4 is replaced by P_6.

For $\epsilon = 0$, (4.7) has a positive root for different signs of (δp_4) and P_6. In this case a second positive root appears according to Case 1 if small values are chosen which yield a sign of (ϵp_2) different from that of (δp_4). The two positive roots coincide for

$$(\delta p_4)^2 - 4(\epsilon_0 p_2) P_6 = 0 \tag{4.8}$$

and disappear as complex roots if ϵ exceeds this bound (ϵ may be negative). The interval $(0, \epsilon_0)$, where ϵ_0 is only an approximation of the true value, is the interval of coexistence of two limit cycles for some given δ. Because ϵ_0 is proportional to δ^2, the region of coexistence of two limit cycles becomes an arbitrarily small fraction of a neighbourhood of the point $\mathbf{a}^0$ in parameter space if the neighbourhood chosen is small enough. We state our results for negative P_6 by the pattern of signs of $(\epsilon p_2, \delta p_4, P_6)$ which designate different regions of a neighbourhood of $\mathbf{a}^0$:

$(- - -)$ no limit cycle,

$(+ - -)$ a stable limit cycle bifurcates at $\epsilon = 0$,

$(+ + -)$ a stable limit cycle persists. Its amplitude depends mainly on δ,

$(- + -)$ an unstable limit cycle inside of the stable one bifurcates at $\epsilon = 0$. The two coalesce and disappear if ϵ exceeds a bound near to ϵ_0 of (4.8).

For positive values of P_6 the plus and minus signs and the words "stable" and "unstable" have to be exchanged. The results given here are in good agreement with the results from numerical investigations, which are depicted in the Figures 1 and 2. For that example P_6 is negative.

5. Relation of our Results to Those of Andronov

In their investigation of the stability of foci of systems of the form (3.1), Andronov *et al.* [2] gave an expression which determines the stability of a focus if the associated eigenvalues are purely imaginary. This expression is identical with our P_4 of Section 3, apart from a positive factor. In this section we shall summarize their results and establish a relation between their focal values and our coefficients P_μ of the time-derivative of the Ljapunov function.

For this discussion we transform system (3.1) into polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$):

$$\begin{aligned}\dot{r} &= \sum_{k=2}^{K} r^k \sum_{l=0}^{k} [G_{k-l,l}\sin\theta + F_{k-l,l}\cos\theta]\cos^{(k-l)}\theta\sin^l\theta\\ \dot{\theta} &= \omega + \sum_{k=2}^{K} r^{k-1}\sum_{l=0}^{k} [G_{k-l,l}\cos\theta - F_{k-l,l}\sin\theta]\cos^{(k-l)}\theta\sin\theta.\end{aligned} \tag{5.1}$$

For sufficiently small values of r, a succession function $d_\theta(r)$ is defined as follows: Consider a solution of (5.1) starting on some ray with some fixed value of $\theta = \theta_0$. We assume $\omega > 0$ and therefore θ is a strictly increasing function of time as long as r is small enough. Thus, a solution starting at (r_0, θ_0) will intersect the ray at (r_1, θ_0), (r_2, θ_0),.... Then

$$d_{\theta_0}(r_0) = r_1 - r_0 \tag{5.2}$$

is called the succession function of system (5.1) on the ray $\theta = \theta_0$. Zeroes of d_{θ_0} occur if (r_0, θ_0) lies on a periodic solution or if $r_0 = 0$.

The mth derivative of d_{θ_0} with respect to r at $r = 0$ is called the mth focal value of the focus (the origin). Andronov *et al.* show that the focal values of even order m all vanish. The stability of the focus is determined by the first non-vanishing focal value (called the Ljapunov value of the focus); a negative Ljapunov value means that d_{θ_0} is a decreasing function of r for small values of r and that the focus is stable. If the inspection of the focal values of a focus yields:

$$\begin{gathered} d'(0) = d'''(0) = \cdots = d^{(2n-1)}(0) = 0,\\ d^{(2n+1)}(0) \neq 0, \qquad n = 1, 2,..., \end{gathered} \tag{5.3}$$

the focus is said to be of multiplicity n. Because $d'(0) = 0$ for a focus with purely imaginary eigenvalues, we are concerned here with foci of multiplicity $n \geq 1$.

In the framework of this paper the succession function is more conveniently defined in terms of

$$\rho = (2V)^{1/2} \tag{5.4}$$

with V expressed in polar coordinates:

$$V = r^2/2 + r^3(V_{30} \cos^3 \theta + \cdots + V_{03} \sin^3 \theta) + r^4(\cdots) + \cdots . \tag{5.5}$$

Thus, for small values of r the difference between r and ρ is of second and higher order in r. If distances from the origin are measured by ρ instead of r, the results are identical if the limit $r \to 0$ is considered as we do here.

The new succession function in terms of ρ is given by

$$d_{\theta_0}(r_0) = \rho(r_1, \theta_0) - \rho(r_0, \theta_0) = \int_0^T \dot{\rho}(r(t), \theta(t))\, dt. \tag{5.7}$$

T is the time elapsing between the start of a solution of system (5.1) at (r_0, θ_0) until the first intersection with the ray $\theta = \theta_0$ at (r_1, θ_0). The interval of time $[0, T]$ is equivalent to an interval of length 2π of the angular coordinate θ. For small values of r and positive ω the function $\theta(t)$ monotonously increases with t and in this case the succession function may be expressed by an integral over θ:

$$d_{\theta_0}(r_0) = \int_{\theta_0}^{\theta_0+2\pi} \dot{\rho}/\dot{\theta}\, d\theta = \int_{\theta_0}^{\theta_0+2\pi} \dot{V}/(\dot{\theta}(2V)^{1/2})\, d\theta. \tag{5.8}$$

Inserting $\dot{V}$ from Eq. (3.4) and expanding the functions in the denominator into powers of r yields

$$d_{\theta_0}(r_0)$$
$$= \int_{\theta_0}^{\theta_0+2\pi} [P_4 r^4 + P_6 r^6 + \cdots] \frac{1}{r} [1 - r(\cdots) + \cdots] \frac{1}{\omega} [1 - \omega r(\cdots) + \cdots]\, d\theta. \tag{5.9}$$

The lowest non-vanishing derivative of the succession function with respect to r does not depend on θ_0 at $r = 0$ and it is given by

$$d^{(2n+1)}(0) = 2\pi(2n+1)!\, P_{2n+2}/\omega. \tag{5.10}$$

P_{2n+2} is the first non-vanishing coefficient of $\dot{V}$ from Eq. (3.4). Andronov *et al.* define their focal values by

$$\alpha_{2n+1} = d^{(2n+1)}(0)/(2n+1)! \tag{5.11}$$

Thus, the first non-vanishing P and the first non-vanishing focal value (the Ljapunov value) of a focus are related by

$$\alpha_{2n+1} = 2\pi P_{2n+2}/\omega. \tag{5.12}$$

Because the stability of a focus depends on the sign of the first non-vanishing α or P respectively, this relation seems to give contradictory results for negative values of ω. But Andronov *et al.* restrict themselves to the case of positive ω and their expression for α_3 is not correct for negative values of ω.

As mentioned above, the multiplicity of a focus is equal to n if α_{2n+1} (or P_{2n+2} in our notation) is the first non-vanishing focal value of the focus. The reason why the number n is called multiplicity becomes apparent by a theorem of Andronov *et al.* [2, p. 254, Theorem 40]. This theorem has the consequence that if a system as given by Eq. (4.1) has for $\epsilon = 0$ a focus of multiplicity n, then there exist at most n limit cycles close to the focus for small values of ϵ. All of these n limit cycles exist if the functions f and g in (4.1) are chosen in an appropriate manner.

Appendix

We have left the derivations of the results cited in Section 3 to this Appendix because of their considerable length. We rewrite Eq. (3.1) and (3.2) in a somewhat different but equivalent form, which allows an easier manipulation of the subscripts:

$$\begin{aligned} \dot{x} &= -\omega y + \sum_{k=0} \sum_{l=0} F_{kl} x^k y^l, \\ \dot{y} &= \omega X + \sum_{k=0} \sum_{l=0} G_{kl} x^k y^l, \end{aligned} \tag{A.1}$$

$$V = \sum_{i=0} \sum_{j=0} V_{ij} x^i y^j. \tag{A2}$$

In the sequel we shall use the term "order" for the sum of subscripts belonging to a coefficient. The upper limits of the sums are specified below. Coefficients of order lower than two are set equal to zero because they do not occur in the corresponding formulae in Section 3. The same applies to negative subscripts which occur in some formulae below. The partial derivatives of V are

$$\begin{aligned} V_x &= \sum_{i=0} \sum_{j=0} (i+1)\, V_{i+1,j} x^i y^j, \\ V_y &= \sum_{i=0} \sum_{j=0} (j+1)\, V_{i,j+1} x^i y^j, \end{aligned} \tag{A.3}$$

where i has been replaced by $i+1$ in V_x and j by $j+1$ in V_y, respectively. The derivative of V with respect to time along a solution of (A.1) is given by:

$$\begin{aligned} \dot V &= \dot x V_x + \dot y V_y \\ &= \sum_{i=1} \sum_{j=0} \Big\{ [-\omega(i+1)\, V_{i+1,j}\, y + \omega(j+1)\, V_{i,j+1}\, x]\, x^i y^j \\ &\quad + \sum_{k=0} \sum_{l=0} [F_{kl}(i+1)\, V_{i-1,j} + G_{kl}(j+1)\, V_{i,j+1}]\, x^{i+k} y^{j+l} \Big\}. \end{aligned} \tag{A.4}$$

We wish to know if (A.4) may be transformed into Eq. (3.3) and then into (3.4):

$$\dot V = \sum_{\mu} \sum_{\nu=0}^{\mu} P_{\mu-\nu,\nu} x^{\mu-\nu} y^{\nu}, \tag{A.5–3.3}$$

$$\dot V = \sum_{\mu=4,6,\ldots} P_\mu (x^2+y^2)^{\mu/2}. \tag{A.6–3.4}$$

The first step is achieved with

$$\begin{aligned} P_{\mu-\nu,\nu} &= \omega[-(\mu-\nu+1)\, V_{\mu-\nu+1,\nu-1} + (\nu+1)\, V_{\mu-\nu-1,\nu+1}] \\ &\quad + \sum_{i=0} \sum_{j=0} [F_{\mu-\nu-i,\nu-j}(i+1)\, V_{i+1,j} + G_{\mu-\nu-i,\nu-j}(j+1)\, V_{i,j+1}]. \end{aligned} \tag{A.7}$$

The requirement that the order of any coefficient must be greater than one leads to the following inequalities:

$$\mu \geqslant 2, \qquad 1 \leqslant i+j \leqslant \mu-2.$$

For the twofold sums of (A.7) these inequalities have the consequence that they contain only V's of order lower than μ and that they are zero for $\mu \leqslant 2$. This is the reason why the V's can be determined for any order μ if the V's of lower order are known. Eq. (A.7) relates the V's of the order μ with the same number of unknowns P by a square matrix of dimension $\mu + 1$, which is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ -\mu & 0 & 2 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & (1-\mu) & 0 & 3 & 0 & \cdot & \cdot & 0 \\ \cdot & & (2-\mu) & \cdot & \cdot & & & \\ \cdot & & & \cdot & \cdot & \cdot & & \\ \cdot & & & & \cdot & \cdot & (\mu-1) & \\ 0 & \cdot & \cdot & \cdot & 0 & -2 & 0 & \mu \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{A.8}$$

By operations which do not change the determinant of this matrix with the rows with odd numbers starting with the first row, it is easy to see that this matrix is singular if μ is an even number and nonsingular for odd μ. Thus, the set of equations given by (A.7) can be solved in a simple manner only if μ is odd. Because $\dot{V}$ as expressed by (A.6) does not contain odd exponents of x or y, the first rule in determining the P's is:

$$P_{\mu-\nu,\nu} = 0 \qquad \text{if} \qquad \mu \quad \text{or} \quad \nu \quad \text{or} \quad \text{both are odd numbers.}$$

With this rule, the V's of odd order can be computed from (A.7). For even numbers μ the system of equations (A.7) has to be decomposed into two parts. Since (A.7) related the P's with even ν with V's with odd ν and vice versa, the numbers of unknowns in both parts are different. Those $P_{\mu-\nu,\nu}$ with odd ν are zero and the V's related with them must be restricted by an arbitrary equation (for example, $V_{\mu-\nu,\nu} = 0$ for one ν) in order to obtain a unique solution.

If both subscripts μ and ν are even, the identity of the right hand sides of (A.5) and (A.6) yields

$$P_{\mu-\nu,\nu} = c(\mu/2, \nu/2)\, P_\mu\,,$$

where $c(i, j)$ is the jth binomial coefficient of order i. Because there are $\mu/2 + 1$ coefficients $P_{\mu-\nu,\nu}$ with μ and ν even, which are related by (A.7) with the $\mu/2$ coefficients V of order μ with odd subscripts, the above relations complete the system of equations (A.7) to the necessary number of equations by introducing the one new unknown P_μ. Because we are looking essentially for the quantities P_4, P_6,... it is important to note that any P_μ can be computed before the V's of the same order are known.

The case $\mu = 2$ has to be considered separately. From Eq. (A.7) follows $P_{2,0} = -P_{0,2}$ and $V_{2,0} = V_{0,2}$. Therefore, a definite $\dot{V}$ can only be obtained with $P_{2,0} = P_{0,2} = V_{11} = 0$. By the choice $V_{20} = V_{02} = \frac{1}{2}$, we obtain a Ljapunov function V, which is positive definite in a neighbourhood of the origin and one factor in the partial derivatives of V becomes unity.

In Section 4 we investigated how the system (3.1) is affected by small changes of the parameters which are expressed by small values of ϵ in Eq. (4.1). If the time derivative $\dot{W}$ of the Ljapunov function W, associated with the perturbed system, shows a qualitative behavior which is different from the behavior of $\dot{V}$ in a neighbourhood of the origin, this is due to the second order term of $\dot{W}$ with the coefficient ϵp_2 (see Eq. (4.4)). This quantity can be computed from Eq. (4.3). Using the same notation as before with small letters, the second order terms of the first bracket of (4.3) are:

$$
\begin{aligned}
p_{20} &= \omega v_{11} + f_{10}, \\
p_{02} &= -\omega v_{11} + g_{01}, \\
p_{11} &= 2\omega(v_{02} - v_{20}) + f_{01} + g_{10}.
\end{aligned} \tag{A.9}
$$

The right hand sides of (4.3) and (4.4) become identical with respect to the terms of second order in x and y and of first order in ϵ if $p_{20} = p_{02} = p_2$ and $p_{11} = 0$. For a unique solution of these equations we need an additional equation (for example, $v_{20} = 0$). The first two equations yield

$$p_2 = (f_{10} + g_{01})/2. \tag{A.10}$$

It is easily seen that ϵp_2 is just one half of the trace of the Jacobian of system (4.1) at the origin and therefore it is identical with the real part of the eigenvalues of this matrix.

Below we give the results of our computation up to P_8:

$$
\begin{aligned}
P_4 = \frac{1}{8}\Big\{&3(F_{30} + G_{03}) + F_{12} + G_{21} + \frac{1}{\omega}[F_{11}(F_{20} + F_{02}) - G_{11}(G_{20} + G_{02}) \\
&+ 2(F_{02}G_{02} - F_{20}G_{20})]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
P_6 = \frac{1}{16}\{&(25F_{20} + 5F_{02})\,V_{50} + (25G_{02} + 5G_{20})\,V_{05} + (5G_{20} + G_{02} + 4F_{11})\,V_{41} \\
&+ (5F_{02} + F_{20} + 4G_{11})\,V_{14} + (3F_{20} + 3F_{02} + 2G_{11})\,V_{32} \\
&+ (3G_{02} + 3G_{20} + 2F_{11})\,V_{23} + (20F_{30} + 4F_{12})\,V_{40} + (20G_{03} + 4G_{21})\,V_{04} \\
&+ (5G_{30} + 3F_{03} + 3F_{21} + G_{12})\,V_{31} + (5F_{03} + 3G_{30} + 3G_{12} + F_{21})\,V_{13} \\
&+ (15F_{40} + 3F_{22} + 3F_{04})\,V_{30} + (15G_{04} + 3G_{22} + 3G_{40})\,V_{03} \\
&+ (5G_{40} + G_{22} + G_{04} + 2F_{31} + 2F_{13})\,V_{21} \\
&+ (5F_{04} + F_{22} + F_{40} + 2G_{13} + 2G_{31})\,V_{12} \\
&+ 5(F_{50} + G_{05}) + F_{32} + G_{23} + F_{14} + G_{41}\},
\end{aligned}
$$

$$P_8 = \frac{1}{128}\{35(C_1 + C_2) + 5(C_3 + C_4) + 3C_5\}$$

with

$$V_{20} = V_{02} = \tfrac{1}{2}, \qquad V_{11} = 0,$$

$$V_{30} = \frac{1}{3\omega}(F_{11} - G_{20} + 2G_{02}), \qquad V_{03} = \frac{-1}{3\omega}(G_{11} + F_{02} + 2F_{20}),$$

$$V_{21} = -\frac{1}{\omega}F_{20}, \qquad V_{12} = \frac{1}{\omega}G_{02},$$

$$\begin{aligned} V_{40} = \frac{1}{4\omega^2}[&F_{11}(F_{11} + G_{20} + 2G_{02}) - F_{20}(2F_{20} + G_{11}) \\ &+ 2G_{20}G_{02} + \omega(G_{30} + F_{21})], \end{aligned}$$

$$\begin{aligned} V_{04} = \frac{1}{4\omega^2}[&G_{11}(G_{11} + F_{02} + 2F_{20}) \\ &- G_{02}(2G_{02} + F_{11}) + 2F_{20}F_{02} - \omega(F_{03} + G_{12})], \end{aligned}$$

$$V_{22} = 0,$$

$$V_{31} = -\frac{1}{\omega^2}[F_{20}(F_{11} + 2G_{02}) + \omega(F_{20} - P_4)],$$

$$V_{13} = -\frac{1}{\omega^2}[G_{02}(G_{11} + 2F_{20}) + \omega(P_4 - G_{03})],$$

$$V_{41} = -\frac{1}{\omega}[4F_{20}V_{40} + G_{20}V_{31} + 3F_{30}V_{30} + G_{30}V_{21} + F_{40}],$$

$$V_{14} = \frac{1}{\omega}[4G_{02}V_{04} + F_{02}V_{13} - 3G_{03}V_{03} + F_{03}V_{12} + G_{04}],$$

$$\begin{aligned} V_{32} = \frac{1}{3\omega}[&3F_{02}V_{31} + (F_{20} + 3G_{11})V_{13} + 4G_{20}V_{04} + 3F_{03}V_{30} \\ &+ (2F_{12} + G_{03})V_{21} + (F_{21} + 2G_{12})V_{12} + 3G_{21}V_{03} + F_{13} + G_{22}] \\ &- 4V_{14}/3, \end{aligned}$$

$$\begin{aligned} V_{23} = -\frac{1}{3\omega}[&3G_{20}V_{13} + (G_{02} + 3F_{11})V_{31} + 4F_{02}V_{40} + 3G_{30}V_{03} \\ &+ (2G_{21} + F_{30})V_{12} + (G_{12} + 2F_{21})V_{21} + 3F_{12}V_{30} + G_{31} + F_{22}] \\ &+ 4V_{41}/3, \end{aligned}$$

$$\begin{aligned} V_{50} = \frac{1}{5\omega}[&4F_{11}V_{40} + (3F_{20} + G_{11})V_{31} + 3F_{21}V_{30} + (2F_{30} + G_{21})V_{21} \\ &+ 2G_{30}V_{12} + F_{31} + G_{40}] + 2V_{32}/5, \end{aligned}$$

$$\begin{aligned} V_{05} = \frac{-1}{5\omega}[&4G_{11}V_{04} + (3G_{02} + F_{11})V_{13} + 3G_{12}V_{03} + (2G_{03} + F_{12})V_{12} \\ &+ 2F_{03}V_{21} + G_{13} + F_{04}] + 2V_{23}/5, \end{aligned}$$

$$V_{60} = \frac{1}{6\omega}A_1,$$

$$V_{06} = -\frac{1}{6\omega}\left(A_2 + \frac{1}{2\omega}A_5\right),$$

$$V_{42} = 0,$$

$$V_{24} = -\frac{1}{4\omega}A_5,$$

$$V_{33} = \tfrac{1}{2}\{5\omega(V_{15} - V_{51}) + A_3 + A_4\},$$

$$V_{51} = -\frac{1}{\omega}A_6,$$

$$V_{15} = \frac{1}{\omega}A_7,$$

$$V_{61} = -\frac{1}{\omega}B_1,$$

$$V_{16} = \frac{1}{\omega}B_2,$$

$$V_{70} = \frac{1}{7\omega}(B_3 + 2\omega V_{52}),$$

$$V_{07} = -\frac{1}{7\omega}(B_4 - 2\omega V_{25}),$$

$$V_{43} = -\frac{1}{3\omega}(B_5 - 6\omega V_{61}),$$

$$V_{34} = \frac{1}{3\omega}(B_6 + 6\omega V_{16}),$$

$$V_{52} = \frac{1}{5\omega}(B_7 + 4\omega V_{34}),$$

$$V_{25} = -\frac{1}{5\omega}(B_8 - 4\omega V_{43}).$$

The expressions for $A_1, \ldots, A_7$, $B_1, \ldots, B_8$, $C_1, \ldots, C_5$ are given by

$$\begin{aligned}
A_1 = {} & F_{41} + 2F_{40}V_{21} + 3(F_{31}V_{30} + F_{30}V_{31}) + 4(F_{21}V_{40} + F_{20}V_{41}) + 5F_{11}V_{50} \\
& + G_{31}V_{21} + G_{21}V_{31} + G_{11}V_{41} + 2(G_{40}V_{12} + G_{20}V_{32}) + G_{50}, \\
A_3 = {} & F_{40}V_{12} + F_{32} + 2F_{31}V_{21} + 3(F_{22}V_{30} + F_{21}V_{31} + F_{20}V_{32}) \\
& + 4(F_{12}V_{40} + F_{11}V_{41}) + 5F_{02}V_{50} + G_{22}V_{21} + G_{12}V_{31} + G_{02}V_{41} + G_{41} \\
& + 2(G_{31}V_{12} + G_{11}V_{31}) + 3(G_{40}V_{03} + G_{30}V_{13} + G_{20}V_{23}), \\
A_5 = {} & F_{31}V_{12} + F_{30}V_{13} + F_{23} + 2(F_{22}V_{21} + F_{20}V_{23}) \\
& + 3(F_{13}V_{30} + F_{12}V_{31} + F_{11}V_{32}) + 4(F_{03}V_{40} + F_{02}V_{41}) + G_{13}V_{21} + G_{03}V_{31} \\
& + G_{32} - 2(G_{22}V_{12} + G_{02}V_{32}) + 3(G_{31}V_{03} + G_{21}V_{13} + G_{11}V_{23}) \\
& + 4(G_{30}V_{04} + G_{20}V_{14}), \\
A_6 = {} & F_{50} + 3F_{40}V_{30} + 4F_{30}V_{40} + 5F_{20}V_{50} + G_{40}V_{21} + G_{30}V_{31} + G_{20}V_{41}.
\end{aligned}$$

To get the expressions for A_2, A_4, and A_7, A_1, A_3, and A_6 have to be changed as follows: For $F_{ij}V_{kl}$ set $G_{ji}V_{lk}$ and for $G_{mn}V_{pq}$ set $F_{nm}V_{qp}$. We shall denote this operation by the sign $\mapsto$. Thus, $A_1 \mapsto A_2$, $A_3 \to A_4$, $A_6 \to A_7$.

$$\begin{aligned}
B_1 &= F_{60} + 3F_{50}V_{30} + 4F_{40}V_{40} + 5F_{30}V_{50} + 6F_{20}V_{60} + G_{50}V_{21} + G_{40}V_{31} \\
&\quad + G_{30}V_{41} + G_{20}V_{51}, \\
B_3 &= F_{51} + 2F_{50}V_{21} + 3(F_{41}V_{30} + F_{40}V_{31}) + 4(F_{31}V_{40} + F_{30}V_{41}) \\
&\quad + 5(F_{21}V_{50} + F_{20}V_{51}) + 6F_{11}V_{60} + G_{41}V_{21} + G_{31}V_{31} + G_{21}V_{41} + G_{11}V_{51} \\
&\quad + G_{60} + 2(G_{50}V_{12} + G_{30}V_{32} + G_{20}V_{42}), \\
B_5 &= F_{50}V_{12} + F_{42} + 2F_{41}V_{21} + 3(F_{32}V_{30} + F_{31}V_{31} + F_{30}V_{32}) \\
&\quad + 4(F_{22}V_{40} + F_{21}V_{41}) + 5(F_{12}V_{50} + F_{11}V_{51}) + 6F_{02}V_{60} + G_{32}V_{21} \\
&\quad + G_{22}V_{31} + G_{12}V_{41} + G_{02}V_{51} + G_{51} + 2(G_{41}V_{12} + G_{21}V_{32}) \\
&\quad + 3(G_{50}V_{03} + G_{40}V_{13} + G_{30}V_{23} + G_{20}V_{33}), \\
B_7 &= F_{41}V_{13} + F_{40}V_{14} + F_{33} + 2(F_{32}V_{21} + F_{30}V_{23}) + 3(F_{23}V_{30} + F_{22}V_{31} \\
&\quad + F_{21}V_{32} + F_{20}V_{33}) + 4(F_{13}V_{40} + F_{12}V_{41}) + 5(F_{03}V_{50} + F_{02}V_{51}) + G_{23}V_{21} \\
&\quad + G_{13}V_{31} + G_{03}V_{41} + G_{42} + 2(G_{32}V_{12} + G_{12}V_{32}) - 3(G_{41}V_{03} + G_{31}V_{13} \\
&\quad + G_{21}V_{23} + G_{11}V_{33}) + 4(G_{40}V_{04} + G_{30}V_{14} - G_{20}V_{24}),
\end{aligned}$$

$B_1 \mapsto B_2$, $B_3 \mapsto B_4$, $B_5 \mapsto B_6$, $B_7 \mapsto B_8$,

$$\begin{aligned}
C_1 &= F_{70} + 3F_{60}V_{30} + 4F_{50}V_{40} + 5F_{40}V_{50} + 6F_{30}V_{60} + 7F_{20}V_{70} + G_{60}V_{21} \\
&\quad + G_{50}V_{31} + G_{40}V_{41} + G_{30}V_{51} + G_{20}V_{61}, \\
C_3 &= F_{60}V_{12} + F_{52} + 2F_{51}V_{21} + 3(F_{42}V_{30} + F_{41}V_{31} + F_{40}V_{32}) \\
&\quad + 4(F_{32}V_{40} + F_{31}V_{41}) + 5(F_{22}V_{50} + F_{21}V_{51} + F_{20}V_{52}) + 6(F_{12}V_{60} \\
&\quad + F_{11}V_{61}) + 7F_{02}V_{70} + G_{42}V_{21} + G_{32}V_{31} + G_{22}V_{41} + G_{12}V_{51} + G_{02}V_{61} \\
&\quad + G_{61} + 2(G_{51}V_{12} + G_{31}V_{32} + G_{11}V_{52}) + 3(G_{60}V_{03} + G_{50}V_{13} \\
&\quad + G_{40}V_{23} + G_{30}V_{33} + G_{20}V_{43}), \\
C_5 &= F_{42}V_{12} + F_{41}V_{13} + F_{40}V_{14} + F_{34} + 2(F_{33}V_{21} + F_{31}V_{23} + F_{30}V_{24}) \\
&\quad + 3(F_{24}V_{30} + F_{23}V_{31} + F_{22}V_{32} + F_{21}V_{33} + F_{20}V_{34}) + 4(F_{14}V_{40} + F_{13}V_{41} \\
&\quad + F_{11}V_{43}) + 5(F_{04}V_{50} + F_{03}V_{51} + F_{02}V_{52}) + G_{24}V_{21} + G_{14}V_{31} + G_{04}V_{41} \\
&\quad + G_{43} + 2(G_{33}V_{12} + G_{13}V_{32}) + 3(G_{42}V_{03} + G_{32}V_{13} + G_{22}V_{23} + G_{12}V_{33} \\
&\quad + G_{02}V_{43}) + 4(G_{41}V_{04} + G_{31}V_{14} + G_{21}V_{24} + G_{11}V_{34}) + 5(G_{40}V_{05} \\
&\quad + G_{30}V_{15} + G_{20}V_{25}),
\end{aligned}$$

$C_1 \mapsto C_2$, $C_3 \mapsto C_4$.

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