Existence and Uniqueness Theorems for Fourth-Order Boundary Value Problems

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1. INTRODUCTION

One of the most used elements in structures such as aircraft, buildings, ships, and bridges is the elastic beam. The deformations of the beam are described by a fourth-order boundary problem [3]. R. A. Usmani [4] has studied the existence of a unique solution of the real two-point linear problem

\[ L \psi \equiv \left[ \frac{d^4}{dx^4} + f(x) \right] \psi = g(x), \quad 0 < x < 1, \]

\[ y(0) = A_1, \quad y(1) = A_2, \quad y''(0) = B_1, \quad y''(1) = B_2. \]

In this paper we shall consider a more general problem, namely,

\[ y^{(4)} = f(x, y, y'') \quad \text{(1.1)} \]

where \( f \in C[0, 1] \times \mathbb{R} \times \mathbb{R} \), under the following types of boundary conditions:

\[ y(0) = y_0, \quad y(1) = y_1, \quad y''(0) = \bar{y}_0, \quad y''(1) = \bar{y}_1; \quad \text{(1.2)} \]
\[ y(0) = y_0, \quad y(1) = y_1, \quad y''(0) = \bar{y}_0, \quad y''(1) = \bar{y}_1; \quad \text{(1.3)} \]
\[ y(0) = y(1) = y''(0) = y''(1) = 0; \quad \text{(1.4)} \]
\[ y(0) = y(1) = 0, \quad y'''(0) - hy''(0) = 0, \quad y'''(1) + ky''(1) = 0, \quad \text{(1.5)} \]

\[ h, k \geq 0 \text{ and } h + k > 0. \]

We shall prove that under adequate conditions imposed on \( f \), Eq. (1.1) together with one of the above boundary conditions has a solution, and we shall discuss the uniqueness of such solutions for (1.1). To do this, we transform Eq. (1.1) into a second-order integro-differential equation. Then we apply known results for second-order boundary value problems and

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Schauder's fixed point theorem to obtain existence and uniqueness results for fourth-order boundary value problems of the form (1.1). This method leads us to obtain results for more general cases considered and we believe this technique can be applied to differential equations of other orders besides two and three.

2. PRELIMINARY RESULTS

We present some results which help simplify the proofs of our main results.

The homogeneous boundary value problem

\[ u'' = 0 \]

together with

\[ u(a) = u(b) = 0, \]

or

\[ u(a) = u'(b) = 0, \]

has Green's function \( G(x, t) \), \( H(x, t) \), respectively, defined by

\[ G(x, t) = \begin{cases} 
- (b - x)(t - a)/(b - a), & a \leq t \leq x \leq b, \\
- (b - t)(x - a)/(b - a), & a \leq x \leq t \leq b,
\end{cases} \]

and

\[ H(x, t) = \begin{cases} 
t - a, & a \leq t \leq x \leq b, \\
x - a, & a \leq x \leq t \leq b.
\end{cases} \]

We need some estimates related to \( G(x, t) \) and \( H(x, t) \) which are collected in the following lemma.

**LEMMA 2.1.** The following hold true:

\[ \int_a^b |G(x, t)| \, dt = (x - a)(b - x)/2 \leq (b - a)^2/8, \]

\[ \int_a^b |G_x(x, t)| \, dt = ((x - a)^2 + (b - x)^2)/2(b - a) \leq (b - a)/2, \]

\[ \int_a^b |H(x, t)| \, dt = (x - a)(2b - x - a)/2 \leq (b - a)^2/2, \]

\[ \int_a^b |H_x(x, t)| \, dt = b - x \leq b - a. \]
**Lemma 2.2** (see [2]). Consider the second-order linear differential equation

\[ y'' = a(x) y + b(x), \]  
(2.1)

with the boundary conditions

\[ y'(0) - hy(0) = 0, \quad y'(1) + ky(1) = 0, \quad h, k \geq 0, \quad h + k > 0, \]  
(2.2)

where \( a(x), b(x) \in C[[0, 1], \mathbb{R}] \), and \( a(x) \geq a_0 > 0 \). Then problem (2.1), (2.2) has a unique solution satisfying

\[ \sup_{0 \leq x \leq 1} |y(x)| \leq \max \frac{\max |b(x)|}{a_0}, \quad 0 \leq x \leq 1. \]  
(2.3)

**Lemma 2.3.** (see [4]). If \( y(0) = y(1) = 0 \) and \( y(x) \in C^1[0, 1] \), then

\[ \int_0^1 y^2(x) \, dx \leq \frac{1}{\pi^2} \int_0^1 (y'(x))^2 \, dx. \]  
(2.4)

**Lemma 2.4.** (see [4]). If \( y(0) = y(1) = 0 \) and \( y(x) \in C^1[0, 1] \), then

\[ \sup_{0 \leq x \leq 1} |y(x)| \leq \frac{1}{2} \left[ \int_0^1 [y'(x)]^2 \, dx \right]^{1/2}. \]  
(2.5)

**Lemma 2.5.** If \( y(0) = y(1) = 0 \), and \( a(x) > -m > -\pi^2 \), then any solution of the linear differential equation

\[ y'' = a(x) y + b(x) \]  
(2.6)

satisfies

\[ \sup_{0 \leq x \leq 1} |y(x)| \leq \frac{\pi}{2(\pi^2 - m)} \sup_{0 \leq x \leq 1} |b(x)|, \quad 0 \leq x \leq 1. \]  
(2.7)

**Proof.** On multiplying (2.6) by \( y(x) \), and integrating the result from 0 to 1, we find

\[ -\int_0^1 (y'(x))^2 \, dx = \int_0^1 a(x) y^2(x) \, dx + \int_0^1 b(x) y(x) \, dx. \]

Using the Cauchy–Schwarz inequality and the fact that \( a(x) > -m \), we obtain

\[ \int_0^1 (y'(x))^2 \, dx \leq m \int_0^1 y^2(x) \, dx + \sup_{0 \leq x \leq 1} |b(x)| \left[ \int_0^1 y^2(x) \, dx \right]^{1/2}. \]
From Lemma 2.3 and the above inequality we have
\[
\left( \int_0^1 (y'(x))^2 dx \right)^{1/2} \leq \frac{\pi}{\pi^2 - m} \sup |b(x)|. \tag{2.8}
\]
Therefore, (2.8) and (2.5) imply (2.7).

**Lemma 2.6.** (see [1]). If \( f(x, y, z) \) is continuous and has continuous first partials with respect to \( y \) and \( z \) on \([a, b] \times K\), where \( K \) is an open convex set, then for \((x, y_1, z_1), (x, y_2, z_2) \in [a, b] \times K\),
\[
f(x, y_1, z_1) - f(x, y_2, z_2) = f_2(x, r(x), s(x))(y_1 - y_2)
+ f_3(x, \tilde{r}(x), \tilde{s}(x))(z_1 - z_2),
\]
where
\[
f_2(x, r(x), s(x)) \equiv \int_0^1 f_2(x, ty_1 + (1 - t)y_2, tz_1 + (1 - t)z_2) dt
\]
and
\[
f_3(x, \tilde{r}(x), \tilde{s}(x)) \equiv \int_0^1 f_3(x, ty_1 + (1 - t)y_2, tz_1 + (1 - t)z_2) dt
\]
are continuous functions on \([a, b] \times K\) with \( s(x), \tilde{s}(x) \) between \( z_1 \) and \( z_2 \), \( r(x), \tilde{r}(x) \) between \( y_1 \) and \( y_2 \).

3. **Existence Results**

**Theorem 3.1.** Suppose \( f \) is bounded on \([0, 1] \times R \times R\). Then problem (1.1) together with either (1.2) or (1.3) has a solution.

**Proof.** First, we consider the boundary conditions (1.2). Let \( u = y'' \). Since \( y(0) = y_0 \) and \( y(1) = y_1 \), one has
\[
y(x) = y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) u(t) dt. \tag{3.1}
\]
Using the above transformation and (3.1), problem (1.1), (1.2) becomes
\[
u'' = f \left( x, y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) u(t) dt, u \right) \tag{3.2}
\]
with
\[
u(0) = \bar{y}_0, \quad u(1) = \bar{y}_1. \tag{3.3}
\]
From (3.2) and (3.3) we have
\[
    u(x) = y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) f \left( t, y_0 + t(y_1 - y_0) \right) dt.
\]
(3.4)

Now, define an operator \( T \) on \( E = C[[0, 1], \mathbb{R}] \) by
\[
    Tu(x) = y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) f \left( t, y_0 + t(y_1 - y_0) \right) dt.
\]
(3.5)

If \( u \in E \), the norm is defined by
\[
    |u|_E = \max_{0 \leq x \leq 1} |u(x)|.
\]

Let \( M \) be the bound of \( f \) on \([0, 1] \times \mathbb{R} \times \mathbb{R} \). Then from (3.5) and Lemma 2.1, it follows that
\[
    |(Tu)(x)| \leq |y_0| + |y_1 - y_0| + M/8, \quad (3.6)
\]
and
\[
    |(Tu)'(x)| \leq |y_1 - y_0| + M/2. \quad (3.7)
\]

Hence \( T \) maps the closed, bounded, and convex set
\[
    B = \{ u \in E : |u(x)| \leq |y_0| + |y_1 - y_0| + M/8 \}
\]
into itself. Moreover, since \((Tu)'(x)\) verifies (3.7), \( T \) is completely continuous on \( C[[0, 1], \mathbb{R}] \) by Ascoli's theorem. The Schauder's fixed point theorem then yields the fixed point of \( T \), which is a solution of (3.2) and (3.3). Since \( u = y'' \), then (3.1) is a solution of (1.1) and (1.2), thus completing the proof of the theorem. The other part, i.e., for (1.3), can be proved analogously, using the function \( H(x, t) \) and the estimates in Lemma 2.1.

**Theorem 3.2.** Assume that there exist positive numbers \( m \) and \( r \) such that
\[
    \begin{align*}
    & (i) \quad \sup |f(x, y, 0)| \leq mr \text{ for } 0 \leq x \leq 1, \ |y| \leq r/8; \\
    & (ii) \quad f(x, y, z) \text{ has a continuous partial derivative with respect to } z \text{ on } [0, 1] \times \mathbb{R} \times \mathbb{R} \text{ and} \\
    & \quad f_3(x, y, z) \geq m > 0 \quad \text{ for } 0 \leq x \leq 1, \ |y| \leq r/8, \ z \in \mathbb{R}.
    \end{align*}
\]

Then the boundary value problem (1.1) and (1.5) has a solution.
Proof. Let \( u = y'' \), then from \( y(0) = y(1) = 0 \), we have

\[
y(x) = \int_0^1 G(x, t) \, u(t) \, dt.
\] (3.8)

Using (3.8) we get, from (1.1) and (1.5),

\[
u'' = f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, u\right),
\] (3.9)

and

\[
u'(0) - hu(0) = 0, \quad u'(1) + ku(1) = 0.
\] (3.10)

Let

\[
B_r = \{ u \in C[0, 1]; |u| \leq r \}.
\]

For \( u \in B_r \) and \( 0 \leq x \leq 1 \), define a mapping \( T: C[0, 1] \to C[0, 1] \) by \((Tu)(x) = v(x)\), where

\[
v'' = f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, v\right)
\] (3.11)

and

\[
v'(0) - hv(0) = 0, \quad v'(1) + kv(1) = 0.
\] (3.12)

Equation (3.11) is equivalent to

\[
v'' = f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, v\right) - f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, 0\right) + f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, 0\right).
\] (3.13)

Using Lemma 2.6, (3.13) becomes

\[
v'' = \left(\int_0^1 f_3\left(x, \int_0^1 G(x, t) \, u(t) \, dt, Tv\right) \, dT\right) v + f\left(x, \int_0^1 G(x, t) \, u(t) \, dt, 0\right).
\] (3.14)

Equation (3.14) is a form of (2.1), and since for \( u \in B_r \),

\[
\int_0^1 |G(x, t)| \, u \, dt \leq r/8,
\]
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the using Lemma 2.2, and conditions (i), (ii) we obtain

$$\sup_{0 \leq x \leq 1} |v(x)| \leq \frac{1}{m} \sup_{0 \leq x \leq 1} \left| f\left(x, \int_0^1 G(x, t) \, u \, dt, 0\right)\right| \leq r. \quad (3.15)$$

Hence, $T$ maps the closed, bounded, and convex set $B$, into itself, and clearly $T$ is continuous. Moreover,

$$v'(x) = v'(0) + \int_0^x f\left(t, \int_0^t G(t, s) \, u(s) \, ds, v(t)\right) \, dt$$

or

$$|v'(x)| \leq |v'(0)| + \int_0^x \left| f\left(t, \int_0^t G(t, s) \, u(s) \, ds, v(t)\right)\right| \, dt. \quad (3.16)$$

From $v'(0) = hv(0)$, and using (3.15), we have

$$|v'(0)| \leq hr. \quad (3.17)$$

Also, $|f(x, y, z)| \leq k$ for $0 \leq x \leq 1$, $|y| \leq r/8$, and $|z| \leq r$, therefore from (3.16) and (3.17) we get

$$|v'(x)| \leq hr + k.$$

All of these considerations imply that $T$ is completely continuous by Ascoli's theorem. Schauder's fixed point theorem then yields the fixed point of $T$, which is a solution of (3.9) and (3.10). But $u = y''$ and (3.8) implies that (1.1), (1.5) has a solution, completing the proof of the theorem.

**Corollary 3.1.** In Theorem 3.2, if all conditions hold true, except condition (i) which is replaced by $f(x, y, 0)$ bounded on $[0, 1] \times R$, then the conclusion of the theorem remains valid.

**Theorem 3.3.** Suppose there exist positive numbers $r$ and $m < \pi^2$ such that

(i) $\sup |f(x, y, 0)| \leq (2(\pi^2 - m)/\pi) \, r$, for $0 \leq x \leq 1$, $|y| \leq r/8$;

(ii) $f(x, y, z)$ has a continuous partial derivative with respect to $z$ on $[0, 1] \times R \times R$ and

$$f_3(x, y, z) \geq -m > -\pi^2, \quad \text{for } 0 \leq x \leq 1, \ |y| \leq r/8, \ z \in R.$$

Then problem (1.1), (1.4) has a solution.
Proof. Assuming \( u = y'' \), we have, from \( y(0) = y(1) = 0 \),

\[
y(x) = \int_0^1 G(x, t) u(t) \, dt.
\]  

(3.18)

Problem (1.1), (1.4) is equivalent, using (3.18) and \( u = y'' \), to

\[
u'' = f \left( x, \int_0^1 G(x, t) u(t) \, dt, u \right),
\]

(3.19)

and

\[
u(0) = u(1) = 0.
\]

(3.20)

Let

\[
B_r = \{ u \in C[0, 1]: |u| \leq r \}.
\]

For \( u \in B_r \) and \( 0 \leq x \leq 1 \), define a mapping \( T: C[0, 1] \to C[0, 1] \) by \( (Tu)(x) = v(x) \), where

\[
v'' = f \left( x, \int_0^1 G(x, t) u(t) \, dt, v \right),
\]

(3.21)

and

\[
v(0) = v(1) = 0.
\]

(3.22)

Equation (3.21) is equivalent, using Lemma 2.6, to

\[
v'' = \left( \int_0^1 f_3 \left( x, \int_0^1 G(x, t) u(t) \, dt, Tv \right) \, dT \right) v + f \left( x, \int_0^1 G(x, t) u(t) \, dt, 0 \right).
\]

(3.23)

Now, using Lemma 2.5, and condition (ii), we have

\[
\sup_{0 \leq x \leq 1} |v(x)| \leq \frac{\pi}{2(\pi^2 - m)} \sup_{0 \leq x \leq 1} \left| f \left( x, \int_0^1 G(x, t) u(t) \, dt, 0 \right) \right|,
\]

or from (i),

\[
\sup_{0 \leq x \leq 1} |v(x)| \leq r.
\]

(3.24)

This shows that \( T \) maps the closed, bounded, and convex set \( B_r \) into itself. Also, from (3.21) and (3.22),

\[
v(x) = \int_0^1 G(x, t) f \left( t, \int_0^1 G(t, s) u(s) \, ds, v(t) \right) \, dt,
\]
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and

\[ v'(x) = \int_0^1 G_x(x, t) f \left( t, \int_0^1 G(t, s) u(s) \, ds, v(t) \right) dt. \]

For \( 0 \leq x \leq 1 \), \( |y| \leq r/8 \), and \( |z| \leq r \), we have \( |f(x, y, z)| \leq k \). Therefore

\[ |v'(x)| \leq k/2. \]

All of these considerations imply that \( T \) is completely continuous by Ascoli's theorem, and Schauder's fixed point theorem then yields the fixed point of \( T \) which is a solution of (3.19) and (3.20). Thus \( u = y'' \) implies that (3.18) is a solution of (1.1) and (1.4). The proof is complete.

4. UNIQUENESS RESULTS

Consider problem (1.1) with either boundary conditions (1.2), (1.3), (1.4), or (1.5). Let

\[ y'' = u. \tag{4.1} \]

Problem (4.1) has the unique solution

\[ y(x) = y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) u(t) \, dt, \tag{4.2} \]

where \( y(0) = y_0 \) and \( y(1) = y_1 \). From (4.1) and (4.2) we have, using (1.1)–(1.5),

\[ u'' = f \left( x, y_0 + x(y_1 - y_0) + \int_0^1 G(x, t) u(t) \, dt, u \right), \tag{4.3} \]

with

\[ u(0) = \bar{y}_0, \quad u(1) = \bar{y}_1; \tag{4.4} \]

or

\[ u(0) = \bar{y}_0, \quad u'(1) = \bar{y}_1; \tag{4.5} \]

or

\[ u(0) = u(1) = 0; \tag{4.6} \]

or

\[ u'(0) - hu(0) = 0, \quad u'(1) + ku(1) = 0. \tag{4.7} \]
**Lemma 4.1.** If problem (4.3), (4.4) (or (4.5), (4.6), or (4.7)) has a unique solution, then problem (1.1), (1.2) (or (1.3), (1.4), or (1.5)) has a unique solution.

**Lemma 4.2.** Consider the fourth-order linear differential equation

\[ y^{(4)} = a(x) y'' + b(x) y + c(x) \]  

(4.8)

with boundary conditions

\[ y(0) = y(1) = y''(0) = y''(1) = 0, \]  

(4.9)

where \( a(x), b(x), c(x) \in C[0, 1] \), and \( a(x) \geq -a_o > -\pi^2 \), \( |b(x)| \leq b_o < \pi^4 \). If \( b_0 + a_o \pi^2 < \pi^4 \), then any solution \( y(x) \) of (4.8), (4.9) satisfies

\[ \sup |y(x)| \leq \frac{\pi}{2(\pi^4 - a_o \pi^2 - b_0)} \max |c(x)|, \quad 0 \leq x \leq 1. \]  

(4.10)

**Proof.** We use the same method given by [4, Lemma 4] to obtain the estimate (4.10). Problem (4.8), (4.9) is equivalent to

\[ y''(x) = z(x), \quad y(0) = y(1) = 0 \]  

(4.11)

and

\[ z''(x) = a(x) z(x) + b(x) y + c(x), \quad z(0) = z(1) = 0. \]  

(4.12)

On multiplying (4.11) by \( y(x) \) and integrating from 0 to 1, we find

\[ -\int_0^1 (y'(x))^2 dx = \int_0^1 y z \, dx. \]

Now, using the Cauchy–Schwartz inequality we obtain from the preceding equation

\[ \int_0^1 (y'(x))^2 dx \leq \left[ \int_0^1 y^2(x) \, dx \right]^{1/2} \left[ \int_0^1 z^2(x) \, dx \right]^{1/2}. \]  

(4.13)

Using Lemma 2.3, we derive from (4.13)

\[ \left[ \int_0^1 (y'(x))^2 dx \right]^{1/2} \leq \frac{1}{\pi^2} \left[ \int_0^1 (z'(x))^2 dx \right]^{1/2}. \]  

(4.14)
In a similar manner, from (4.12), we have
\[
\left( \int_0^1 (z'(x))^2 \, dx \right)^{1/2} \leq \frac{\pi^3}{\pi^4 - a_0 \pi^4 - b_0} \max |c(x)|, \quad 0 \leq x \leq 1. \tag{4.15}
\]
Inequalities (4.14), (4.15) and Lemma 2.4 imply the estimate (4.10). Having Lemma 4.2 at our disposal, it is easy to prove the following.

**Theorem 4.1.** Suppose there exist positive numbers \( m \) and \( r \) such that \( m < \pi^2 \) and \( r < \pi^4 \), and

(i) \( \sup |f(x, y, 0)| \leq (2(\pi^2 - m)/\pi) r \), for \( 0 \leq x \leq 1 \), \( |y| \leq r/8 \);

(ii) \( f(x, y, z) \) has a continuous partial derivative with respect to \( z \) on \([0, 1] \times R \times R\) and
\[
f_3(x, y, z) \geq -m \quad \text{on} \quad [0, 1] \times R \times R;
\]

(iii) \( f(x, y, z) \) has a continuous partial derivative with respect to \( y \) on \([0, 1] \times R \times R\) and
\[
|f_2(x, y, z)| \leq r \quad \text{on} \quad [0, 1] \times R \times R.
\]

Then problem (1.1), (1.4) has a unique solution.

**Proof.** Existence of a solution of (1.1), (1.4) follows from Theorem 3.3. Now, suppose that (1.1), (1.4) has two solutions \( y_1(x) \) and \( y_2(x) \). Let \( u(x) = y_1(x) - y_2(x) \), then
\[
u^{(4)}(x) = f(x, y_1, y''_1) - f(x, y_2, y''_2).
\]
From Lemma 2.6, we have
\[
u^{(4)}(x) = \left( \int_0^1 f_3(x, Ty_1 + (1 - T)y_2, Ty''_1 + (1 - T)y''_2) \, dT \right) u''
\]
\[
+ \left( \int_0^1 f_2(x, Ty_1 + (1 - T)y_2, Ty''_1 + (1 - T)y''_2) \, dT \right) u.
\]
Since \( u(0) = u(1) = u''(0) = u''(1) = 0 \), we have, by Lemma 4.2,
\[
\sup |u(x)| \leq 0, \quad 0 \leq x \leq 1.
\]
This implies that \( u(x) = 0 \), or \( y_1(x) = y_2(x) \). Hence problem (1.1), (1.4) has a unique solution.
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