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On Poincaré's Isoperimetric Problem for Simple Closed Geodesics*

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We show in the context of integral currents that Poincaré's isoperimetric variational problem for simple closed geodesics on ovaloids has a smooth extremal C without self-intersection. Provided the smooth Riemannian metric on the ovaloid M in question does not deviate too far from constant curvature, we further show that (i) this extremal C is connected and so is the desired non-trivial geodesic of shortest length on M and (ii) C is close (in the sense of Hausdorff distance) to a great circle.

1. Introduction

In a well-known paper [1] published in the year 1905, Poincaré suggested that a simple closed geodesic on an ovaloid M could be found by minimizing the arclength among all simple closed curves that bisect the integral curvature of M. Utilizing physical reasoning, Poincaré showed that it was plausible to conjecture that: (i) this minimum is attained by a closed geodesic C on M, (ii) this geodesic C has no double points, i.e., C is simple and (iii) under certain circumstances this geodesic possesses certain stability properties analogous to periodic solutions of Hamiltonian systems. Moreover Poincaré observed that his reasoning might be firmly established by utilizing techniques of the calculus of variations. However, subsequent studies of simple closed geodesics [2–5] on ovaloids have concentrated on variational methods connected with Hilbert manifolds of parametrized closed curves, utilizing the topology of these infinite-dimensional manifolds.

In this paper we study once again Poincaré's variational principle by means of the so-called direct methods in the calculus of variations, restricting most attention to ovaloids sufficiently close (in the C^3 sense) to the standard sphere. Although we do not claim to have proved new results on geodesics

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on a convex sphere, we believe that our approach is significantly different from what may be found in the current literature.

The present paper is organized as follows. In Section 2, we discuss Poincaré's isoperimetric variational principle [V] for simple closed geodesics and compute the first variation associated with [V] assuming the associated minimum is attained by a smooth (although not necessarily connected) curve C. In Section 4 we show that the infimum in [V] is attained by an integral current $\tilde{C}$ that is smooth everywhere. To this end we prove a new isoperimetric inequality associated with [V], since the known isoperimetric inequalities fail to yield the necessary positive lower bound. In Section 5, we prove the extremal curve $\tilde{C}$ is connected by computing the second variation of [V], and thus that $\tilde{C}$ is the desired simple closed geodesic. In Section 6 we investigate a weak notion of stability of the curve $\tilde{C}$ under a C^3 perturbation of the metric defining the ovaloid M. This notion of stability is much weaker than that considered by Poincaré and has nothing in common with the results of [4, 5].

For simplicity we have restricted attention to small C^3 deformations of the standard sphere and have used integral currents, for which a good notion of boundary is available. We expect that for general ovaloids our approach should still yield the existence of simple closed geodesics at the cost of complicating the proofs. The main difficulty to overcome is the connectedness of the extremal curve which cannot be proved in the general case in the context of integral currents, as simple examples show. One then should use the theory of varifolds to force the connectedness of the extremal; the proof of the regularity then becomes more difficult.

2. Poincaré's Isoperimetric Problem for Simple Closed Geodesics on Ovaloids

A natural approach to find nontrivial closed geodesics on a compact manifold M is to minimize the arclength of a homotopy class of closed curves on M. If M is simple connected, so that $\pi_1(M) = \{0\}$, this approach fails and other properties of the set of closed curves on M must be used to find closed geodesics. The recent monograph of Klingenberg [3] obtains many deep and interesting results in this direction by studying the topological invariants of the space of unparametrized closed curves on S^N. However, the question of whether or not a given closed geodesic (obtained by these topological methods) is simple (i.e., non-self-intersecting) requires special and often ingenious geometric constructions.

In 1905, Poincaré formulated an isoperimetric variational problem for simple closed geodesics on the two-dimensional ovaloids M^2, in which the Gauss–Bonnet theorem is used as a natural constraint. Thus differential geometric restrictions are used in place of topological ones in finding closed geodesics. Moreover Poincaré's principle has the advantage that, when the associated critical point is shown to exist, the associated closed curve C automatically has no self-intersections, i.e., C is simple.

The Isoperimetric Variational Problem

As originally stated by Poincaré in [1], the isoperimetric problem for simple closed geodesics on ovaloids can be stated as follows:

[V]. *The shortest nontrivial closed geodesic of an ovaloid $M \subset R^3$ can be characterized as the curve with shortest length among all simple smooth closed curves C that satisfy the constraint*

$$\int_{\Sigma(C)} K\,dV = 2\pi, \tag{1}$$

where K is the Gaussian curvature of M and $\Sigma(C)$ denotes a simply connected subset of M bounded by C.

We shall denote by $\mathscr{C}$ the class of smooth simple curves on M.

Now we shall show that the constraint (1) is "natural" for the problem of finding nontrivial closed simple smooth geodesic on M in the following senses

(A) every nontrivial closed simple smooth geodesic on M satisfies constraint (1),

(B) every critical point in $\mathscr{C}$ of the arclength functional on M with constraint (1) is automatically a critical point of the unrestricted arclength functional.

LEMMA 1. *Constraint* (1) *is a natural one for the arclength functional on M, in the sense of* (A) *and* (B) *described above.*

Proof (Poincaré [1]). Let C be a simple closed connected smooth curve on M. Then the Gauss–Bonnet theorem implies

$$\int_{\Sigma(C)} K\,dV + \int_C K_g\,ds = 2\pi,$$

where $\Sigma(C)$ denotes a simply connected subset of M bounded by C and K_g denotes the geodesic curvature.

Thus if C is a geodesic (i.e., a critical point of the arclength functional on M) $K_g = 0$ so the displayed formula above reduces to constraint (1). Thus condition (A) is satisfied.

Now we show that constraint (1) satisfies condition (B). To this end we suppose C is critical point in $\mathscr{C}$ of the arclength functional restricted to the constraint (1). Then C satisfies the Euler–Lagrange equation

$$K_g = \lambda K, \qquad \text{where } \lambda \text{ is a constant.} \tag{2}$$

This result follows from standard results of the calculus of variations and the first variation calculations

$$\delta \int ds = \int_C K_g \cdot n \, ds,$$

$$\delta \int K \, dV = \int_C K \cdot n \, ds.$$

We integrate (2) over C obtaining

$$\int_C K_g \, ds = \lambda \int_C K \, ds; \tag{3}$$

also the Gauss–Bonnet theorem applied to C and the fact that C satisfies constraint (1) yields

$$\int_C K_g \, ds = 2\pi - \int_{\Sigma(C)} K \, dV = 0.$$

Thus (3) reduces to

$$\lambda \int_C K \, ds = 0.$$

Since M is an ovaloid we have $K > 0$, which implies $\lambda = 0$. Consequently Eq. (2) reduces to $K_g = 0$ and C is a geodesic on M.

3. The Isoperimetric Inequality

The Poincaré variational problem just discussed requires us to find a positive absolute constant α such that

$$\inf_{\Gamma \in \Sigma} L(\Gamma) \geqslant \alpha > 0,$$

where $L(\Gamma)$ denotes the length of a simple closed curve Γ bounding a simply connected domain S of the manifold M subject to the constraint

$$\int_S K\,dV = 2\pi.$$

We have not found among the standard isoperimetric inequalities one that can be applied to this purpose. For example, we note the isoperimetric inequality of Huber [6] is

$$L^2(\Gamma) \geqslant 2\pi A(S)\left\{1 - \frac{1}{2\pi}\int_S K\,dV\right\},$$

where $A(S)$ is the area of S. Consequently, if we require the constraint displayed above, we note that no strictly positive lower bound is attained.

It is at this point that we make our first application of geometric measure theory to the problem of Poincaré. To this end, we use the techniques of the theory of integral currents of Federer and Fleming. Our notation is consistent with the standard one and we take Federer's book "Geometric Measure Theory" [7] as a reference.

Our notation is as follows:

V	isometric embedding of S^2 in R^N, as a C^∞ oriented manifold
$H(x)$	mean curvature vector of V at x
$K(x)$	Gauss curvature vector of V at x
$\mathbf{V}$	the two-dimensional integral current $\mathbf{V} \in I_2(\mathbb{R}^N)$ associated with V
$\mathbf{V}(\zeta) = \int_V \zeta$	is the value of the current V on the differential 2-form ζ
$\lVert\mathbf{V}\rVert$	is the total variation of $\mathbf{V}$, hence
$d\lVert\mathbf{V}\rVert$	is the area element on V
$I_k(\mathbb{R}^N)$	the k-dimensional integral currents on $\mathbb{R}^N$
$I_k(V)$	the k-dimensional integral currents on $\mathbb{R}^N$ with support in V

For $T \in I_k(V)$ we use the following notation

∂T	boundary of the current T
$T \llcorner A$	restriction of T to the Borel set A
$M(T)$	the mass of T hence

$$M(T) = \int_V d\lVert T\rVert$$

$N(T) = M(T) + M(\partial T)$ the norm of T

$F(T)$ the flat norm of T, given by

$$F(T) = \inf_{S \in I_{k+1}(V)} \{M(T - \partial S) + M(S)\}.$$

Flat convergence means convergence with respect to the flat norm. Similarly mass convergence means convergence with respect to the mass norm.

Weak convergence of integral currents denoted ($T_n \to T$ weak) means

$T_n(\zeta) \to T(\zeta)$ for all ζ

$B(r, a)$ the closed ball of center a and radius r, i.e.,

$$B(r, a) = \{x \mid x \in \mathbb{R}^N, |x - a| \leqslant r\}$$

$U(r, a)$ the open ball $B(r, a) \backslash \partial B(r, a)$

α_k the volume of the unit k-ball

$\Theta^*(T, a)$ the upper density of T at a so

$$\Theta^*(T, a) = \overline{\lim_{r \to 0}} \frac{M[T \llcorner B(r, a)]}{\alpha_k r^k}$$

$\Theta_*(T, a)$ the lower density of T at a, defined analogously to the upper density

$\Theta(T, a)$ the density of T at a, defined if $\Theta^*(T, a) = \Theta_*(T, a)$.

Having these conventions and notations defined, we can state and prove the new isoperimetric inequality that we shall need.

First we mention a known result that explicitly involves the mean curvature of V.

PROPOSITION 1. *Let* $T \in I_2(V)$. *Then* $\{M(T)\}^{1/2} \leqslant c_1\{M(\partial T) + \sup_V |H(x)| \, M(T)\}$, *where* c_1 *is a constant depending only on* V.

Proof. This inequality is due to Almgren [8] and is proven in detail in Michael and Simon [9].

THEOREM 1. *Let A be a Borel subset of V, let $T = \mathbf{V} \llcorner A$ and suppose $T \in I_2(V)$. There is a constant $c_2 = c_2(V)$ depending only on V such that if $\delta > 0$ satisfies*

$$\delta \leqslant \int K \, d \|T\| \leqslant 4\pi - \delta$$

then

$$M(\partial T) \geqslant c_2 \delta^{1/2}.$$

Proof. By the Gauss–Bonnet theorem we have

$$\int K \, d \, \|\mathbf{V}\| = 4\pi$$

thus, replacing T by $\mathbf{V} - T$ if needed, we may suppose that

$$\delta \leqslant \int K \, d \, \|T\| \leqslant 2\pi.$$

This yields

$$M(T) \geqslant c_3 \delta$$

with $c_3 = c_3(V) = (\max_V K)^{-1}$.

We prove Theorem 1 by contradiction. If the theorem were not true, we could find a sequence T_n, δ_n such that $T_n \in I_2(V)$,

$$\delta_n \leqslant \int K \, d \, \|T_n\| \leqslant 2\pi,$$

and

$$M(\partial T_n) \leqslant (1/n) \, \delta_n^{1/2}.$$

We also have

$$M(T_n) \geqslant c_3 \delta_n$$

by our previous remark.

We apply Proposition 1 and deduce

$$M(T_n)^{1/2} \leqslant c_4 M(\partial T_n) + c_4 M(T_n)$$

with some $c_4 = c_4(V)$, hence

either $$c_4 M(\partial T_n) \geqslant \tfrac{1}{2} M(T_n)^{1/2}$$

or $$M(T_n) \geqslant (2c_4)^{-2}.$$

The first alternative is untenable for large n, because $M(\partial T_n) \leqslant (1/n) \, \delta_n^{1/2}$ and $M(T_n) \geqslant c_3 \delta_n$. Hence we must have

$$M(T_n) \geqslant c_5 > 0$$

for some constant $c_5 = c_5(V)$. Also

$$N(T_n) = M(T_n) + M(\partial T_n)$$
$$\leqslant M(V) + (1/n)\,\delta_n^{1/2}$$

is bounded independently of n, hence by the fundamental compactness theorem for integral currents [7, 4.2.17(2), p. 414] there is a subsequence of $\{T_n\}$, again denoted by $\{T_n\}$, which is flat convergent to a current T. However since V is two-dimensional, the sequence of integral currents $\{T_n\}$ has maximal dimension and the flat norm and the mass norm coincide. We conclude that

$$M(\partial T) = M(\partial T_n) = 0,$$
$$M(T) = \lim M(T_n) \geqslant c_5 > 0.$$

Now since $\partial T = 0$ and T has maximal dimension we have $T = k\mathbf{V}$ for some integer k. In fact if $Z_2(V)$ denotes the group of integral 2-cycles in $I_2(V)$ we have

$$Z_2(V) \cong Z_2(V)/B_2(V) \cong H_2(V, \mathbb{Z}) \cong \mathbb{Z}$$

[7, 4.4.1, pp. 464, 465]. Hence $Z_2(V)$ is infinite cyclic generated by V, and we have just shown that $T \in Z_2(V)$.

Finally $\int K\,d\,\|T\| = \lim \int K\,d\,\|T_n\|$ by flat convergence, hence

$$0 \leqslant \int K\,d\,\|T\| \leqslant 2\pi.$$

On the other hand

$$\int K\,d\,\|T\| = \int K\,d\,\|k\mathbf{V}\| = 4\pi\,|k|$$

(by the Gauss–Bonnet theorem).

Comparison with the previous inequality yields $k = 0$ and so $T = 0$. This contradicts $M(T) \geqslant c_5 > 0$ and proves Theorem 1.

Remark. The proof shows the theorem remains true for a compact oriented 2-manifold without boundary, provided we replace the inequality

$$\delta \leqslant \int K\,d\,\|T\| \leqslant 4\pi - \delta$$

by

$$\inf_{n \in \mathbb{Z}} \left| \int K \, d \, \|T\| - 2\pi\chi(V) n \right| \geqslant \delta,$$

where $\chi(V)$ is the Euler–Poincaré characteristic of V.

4. Solution of the Isoperimetric Variational Problem and Its Regularity

In this section we begin by proving the existence of a solution of an integral current $\tilde{T} \in I_2(V)$ that attains the infimum in the isoperimetric variational problem

$$\inf_{T \in \Sigma} M(\partial T) \tag{P}$$

in the class

$$\sum = \left\{ T \mid T \in I_2(V), \int K \, d \, \|T\| = 2\pi \right\}$$

and

$$\Theta(\|T\|, a) = 1, \qquad \|T\| \text{ a.e. in } V.$$

Next we prove that the extremal integral current is actually a smooth oriented 1-manifold of class $C^{1,1/2-\epsilon}$ for some $\varepsilon > 0$ and in fact can be represented as a *finite* collection $C_1, C_2, \ldots, C_n$ of closed simple connected curves. In Section 5 we shall show $n = 1$ by requiring that V is a small deformation of the standard sphere.

Here we prove

Theorem 2. *The infimum in the variational problem* (P) *is attained by an integral current*

$$\tilde{T} \in \Sigma.$$

Proof. Let $\alpha = \inf_{T \in \Sigma} M(\partial T)$. Then Theorem 1 with $\delta = 2\pi$ shows that $\alpha > 0$. To show that α is actually attained within the class Σ, we suppose $T_i \in \Sigma$ is a minimizing sequence. Then $M(\partial T_i) \to \alpha$ while

$$2\pi = \int K \, d \, \|T_i\| \geqslant (\min K) \int d \, \|T_i\|$$

hence

$$M(T_i) \leqslant \frac{2\pi}{\min K}$$

is uniformly bounded.

By the compactness theorem [7, 4.2.17, p. 414] there is a subsequence, still denoted by $\{T_i\}$ and an integral current $T \in I_2(V)$ such that $T_i \to T$ in the flat norm and hence in the mass norm because we are in the maximal dimension. Now convergence in the mass norm implies convergence of the measures $\|T_i\| \to \|T\|$ in the sense that for every continuous function f we have

$$\int f\, d\|T_i\| \to \int f\, d\|T\|.$$

Taking $f = K$ we see that the limiting current T has the property

$$\int K\, d\|T\| = 2\pi.$$

Now $M(T \llcorner B(a,r)) \leqslant \liminf M(T_i \llcorner B(a,r)) \leqslant M(\mathbf{V} \llcorner B(a,r))$ because T_i has density 1, $\|T_i\|$ a.e. and it follows that $\Theta^*(\|T\|, a) \leqslant \Theta(\|V\|, a) = 1$ everywhere. Since $\Theta(\|T\|, a)$ exists and is a positive integer $\|T\|$ a.e. [7, 4.1.28(s), p. 385] we see that $\Theta(\|T\|, a) = 1$, $\|T\|$ a.e. and we conclude that $T \in \Sigma$.

Finally $\partial T_i \to \partial T$ in the flat topology, hence

$$M(\partial T) \leqslant \lim M(\partial T_i) = \alpha$$

by lower semi-continuity of mass [7, 5.1.5, p. 519]. Clearly $M(\partial T) = \alpha$, and so the integral current $T \in \Sigma$ and the infimum is attained by T in the class Σ.

We now proceed to study the properties of the extremal T whose existence was just established above. We begin by proving that the extremal T can be assumed to have the same orientation $\|T\|$ a.e. (Lemma 2 below). Then we establish the smoothness properties of ∂T by using the known regularity results on almost minimal integral currents.

LEMMA 2. *There exists an extremal $T \in \Sigma$ of the variational problem* (P) *of type $T = \mathbf{V} \llcorner A$ for some subset $A \subset V$.*

Proof. Let T denote the extremal of (P) shown to exist in Theorem 2; we show that if T does not have the required type, then there is another integral current $\tilde{T} \in \Sigma$, that is an extremal for (P) with the properties as stated. To this end, since

$$\Theta(\|T\|, a) = 1 \qquad \|T\| \text{ a.e.}$$

we have $T = \mathbf{V} \llcorner f$, where f is an integer-valued function taking only the values $+1, 0, -1$. Let $M_i = \{x \mid x \in V, f(x) \geqslant i,\ i \text{ an integer}\}$ and note that by [7, 4.5.17, p. 512][1] we have

$$T = \mathbf{V} \llcorner M_1 + \mathbf{V} \llcorner M_0 - \mathbf{V}$$

and

$$\|\partial T\| = \|\partial(\mathbf{V} \llcorner M_1)\| + \|\partial(\mathbf{V} \llcorner M_0)\|,$$

where we have used the fact that $\mathbf{V} = \mathbf{V} \llcorner M_{-1}$ and that V has no boundary.

Let us define

$$\tilde{T} = \mathbf{V} \llcorner M_1 + \mathbf{V} - \mathbf{V} \llcorner M_0. \tag{$\dagger$}$$

Then we note, first of all, that the two sets M_1 and $V \backslash M_0$ are disjoint, therefore

$$\begin{aligned} \|\tilde{T}\| &= \|\mathbf{V} \llcorner M_1\| + \|\mathbf{V} - \mathbf{V} \llcorner M_0\| \\ &= \|T\| \qquad \text{by } (\dagger). \end{aligned}$$

Consequently

$$\int K\, d\|\tilde{T}\| = \int K\, d\|T\| = 2\pi$$

and so $\tilde{T} \in \Sigma$. On the other hand

$$\begin{aligned} M(\partial\tilde{T}) &\leqslant M(\partial(\mathbf{V} \llcorner M_1)) + M(\partial(\mathbf{V} - \mathbf{V} \llcorner M_0)) \\ &\leqslant M(\partial(\mathbf{V} \llcorner M_1)) + M(\partial(\mathbf{V} \llcorner M_0)). \end{aligned}$$

It follows that $M(\partial\tilde{T}) \leqslant M(\partial T)$, and so $\tilde{T}$ is also extremal. Now $\tilde{T} = \mathbf{V} \llcorner h$, where $h(x) = 0$ or 1. Hence we have shown there is at least one extremal $\tilde{T} = \mathbf{V} \llcorner A$ for some subset $A \subset \mathbf{V}$ (in other words T has the same orientation, $\|T\|$ a.e.).

We are now in a position to apply the contemporary regularity theory for integral currents to our extremal T. First we state

[1] The theorem quoted is in Euclidean space, but it remains true on any oriented compact manifold without boundary, with identical proof.

DEFINITION. A point $a \in \operatorname{spt} \partial T$ is a regular point if $\operatorname{spt} \partial T$ is a manifold in a neighborhood of a.

LEMMA 3. *The set of regular points of the extremal T (of Lemma 2) is dense in* $\operatorname{spt} \partial T$. *Moreover, if $\Theta(\partial T, a) = 1$ and if there exists a tangent cone C to ∂T at a which is a linear subspace, then a is a regular point.*

Proof. We first show that given any $X \in I_2(V)$ with support in a sufficiently small neighborhood of a point $a \in V$ such that $M(X)$ and $M(\partial X) \leqslant 1$, we have

$$M(\partial T) \leqslant M(\partial(T + X)) + c_4 M(X), \qquad (*)$$

where $c_4 = c_4(V)$ depending only on V. Therefore by a standard isoperimetric inequality on V

$$M(\partial T) \leqslant M(\partial(T + X)) + c_5(\operatorname{diam} \operatorname{spt} \partial X)\, M(\partial X).$$

This last inequality implies that ∂T is almost minimal in the sense of Almgren [10] and the regularity theory of Almgren [10] and Bombieri [11] in the context of currents applies.

We prove $(*)$ in two steps.

Step 1. We may replace X by X', so that

(i) $\operatorname{spt} X' \subset \operatorname{spt} X$,

(ii) $M(X') \leqslant M(X)$,

(iii) $M(\partial(T + X')) \leqslant M(\partial(T + X))$,

(iv) $\Theta(\|T + X'\|, a) = 1$, $\|T + X'\|$ a.e.

Proof. Let $T = \mathbf{V} \llcorner f$ and $X = \mathbf{V} \llcorner g$ and define

$$M_i = \{x \mid x \in V, f(x) + g(x) \geqslant i \text{ for } i \in \mathbb{Z}\}.$$

Then we have as noted in the proof of Lemma 2

$$\partial(T + X) = \sum_{i \in \mathbb{Z}} \partial(\mathbf{V} \llcorner M_i)$$

with

$$\|\partial(T + X)\| = \sum_{i \in \mathbb{Z}} \|\partial(\mathbf{V} \llcorner M_i)\|.$$

Let

$$\begin{aligned} h(x) &= 1 \quad \text{if } f(x) = 0 \quad g(x) \neq 0 \\ &= -1 \quad \text{if } f(x) = 1 \quad g(x) = -1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and let $X' = \mathbf{V} \llcorner h$. Now

$$\begin{aligned} M(X') &\leqslant \|\mathbf{V}\| \llcorner \{g \neq 0\} \\ &\leqslant \sum_{i \in \mathbb{Z}} |i| \, \|\mathbf{V}\|\{g = i\} \\ &\leqslant M(X), \end{aligned}$$

which proves (i) and (ii). Also

$$\begin{aligned} f(x) + h(x) &= 1 \qquad \text{if} \quad f(x) + g(x) \neq 0 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Thus $T + X' = \mathbf{V} - \mathbf{V} \llcorner M_0 + \mathbf{V} \llcorner M_1$, and $\|\partial(T + X')\| = \|\partial(\mathbf{V} \llcorner M_0)\| + \|\partial(\mathbf{V} \llcorner M_1)\| \leqslant \|\partial(T + X)\|$. This proves (iii) and (iv).

Step 2. We may replace X' by $X' + Y$ so that

(i) $T + X' + Y \in \Sigma$,

(ii) $M(\partial(T + X' + Y)) \leqslant M(\partial(T + X')) + c_5 M(X)$.

This makes (*) obvious because then

$$M(\partial T) \leqslant M(\partial(T + X' + Y)).$$

Proof. For $S \in I_2(V)$ we let

$$\Phi(S) = \int K \, d \, \|S\|.$$

In order to define deformations of T within the class Σ let $\zeta(x)$ be a smooth tangent vector field on V and ζ_t: $V \to V$ be a diffeomorphism of class C^3 in t such that

$$\frac{d}{dt} \zeta_t \bigg|_{t=0} = \zeta(x).$$

Now we have the first variation $\delta(\Phi, S, \zeta)$ of Φ at S (see, for instance, [7, 5.1.7, p. 524]) defined by

$$\delta(\Phi, S, \zeta) = \frac{d}{dt} \Phi(\zeta_{t\#} S) \bigg|_{t=0}.$$

Note that we cannot have $\delta(\Phi, S, \zeta) = 0$ for all ζ with support in A unless $\partial S \llcorner A = 0$. In fact, if $S = \mathbf{V} \llcorner f$ then

$$\delta(\Phi, S, \zeta) = \int (DK \cdot \zeta) \, f d \, \|\mathbf{V}\| + \int K(Df \cdot \zeta) \, \|\mathbf{V}\|,$$

where Df is the vector-valued measure gradient of f. Since we assume $\delta = 0$ for every ζ with $\operatorname{spt} \zeta \subset A$ we obtain

$$(DK)f + K\, Df = D(Kf) = 0 \qquad \text{in } A.$$

Hence Kf is locally constant in A. Now f is integer-valued, thus f itself is locally constant and so $Df = 0$. Thus the Green–Gauss theorem [7, 4.5.6, p. 478] shows that $\partial S = 0$ in A as claimed.

We now choose a vector field $\zeta(x)$ with support disjoint from $\operatorname{spt} X$ such that the first variation $\delta_0 = \delta(\Phi, T, \zeta) \neq 0$ and we assume that ζ_t is the identity on $\operatorname{spt} X$. We shall show that we may take

$$Y = (\zeta_t)_\# T - T \qquad \text{for a suitable } t, \quad |t| \leqslant c_6 M(X').$$

First of all $\operatorname{spt} X'$ and $\operatorname{spt} Y$ are disjoint hence

$$T + X' + Y = \zeta_{t\#}(T + X')$$

satisfies the density condition (iv) of Step 1 for every t. Moreover by our previous hypothesis on ζ_t, $\zeta_{t\#} X' = X'$. Next

$$\begin{aligned} M(\partial(T + X' + Y)) - M(\partial(T + X')) \\ &= M(\partial\zeta_{t\#}(T + X')) - M(\partial(T + X')) \\ &= M(\zeta_{t\#}\partial(T + X')) - M(\partial(T + X')) \\ &\leqslant c_6 t \end{aligned} \qquad (**)$$

(by a version of [7, 4.1.14, p. 370 last line). We want to choose t so that

$$\Phi(T + X' + Y) = 2\pi.$$

We already have

$$\Phi(T + X' + Y) - \Phi(T + X') = \delta_0 t + O(t^2),$$

where the constant involved in the term $O(t^2)$ is bounded by $c_7 M(T + X') \leqslant c_8$, i.e., the estimate is uniform with respect to X'. Also

$$\Phi(T + X') - \Phi(T) = O(M(X')).$$

Hence

$$\begin{aligned} \Phi(T + X' + Y) &= \Phi(T) + O(M(X')) + \delta_0 t + O(t^2) \\ &= 2\pi + \delta_0 t + O(M(X')) + O(t^2), \end{aligned}$$

where $\delta_0 \neq 0$ does not depend on t or X' (δ_0 does not depend on X', because we have made sure that $\operatorname{spt} \zeta \cap \operatorname{spt} X' = \emptyset$).

Now it is obvious that if $M(X') \leqslant c_9$, where c_9 is a small constant, then we may choose t with

$$\Phi(T + X' + Y) = 2\pi.$$

This fact and (**) prove Step 2 and the Lemma.

Now we are ready to prove that the extremal T described in Lemmas 2 and 3 is smooth everywhere, and hence ∂T may be regarded as a finite collection of closed curves with no self-intersections (although T need not be connected).

THEOREM 3. *$\operatorname{spt}(\partial T)$ is a smooth 1-manifold.*

Proof. Let $a \in \operatorname{spt}(\partial T)$ and suppose λ is a real number with

$$\Theta_*(\|\partial T\|, a) \leqslant \lambda \leqslant \Theta^*(\|\partial T\|, a)$$

and λ is a finite limit point of

$$(1/2r)\, \|\partial T\|\, B(a, r) \qquad \text{as} \quad r \to 0.$$

There is a sequence $r_j \to 0$ with $(1/2r_j)\, \|\partial T\|\, B(a, r_j) \to \lambda$. Furthermore by an elementary slicing theorem [7, 4.2.1, pp. 395, 396] we may suppose $\|\partial T\| \llcorner \partial B(a, r_j) = 0$. Now let $\mu_r(x) = a + (1/r)(x - a)$ be the dilatation by the factor $1/r$ with center a and let $S_j = \mu_{r_j \#} T \llcorner U(a, 1)$ so $\partial S_j = \mu_{r_j \#}(\partial T) \llcorner U(a, 1)$. We claim that, by going to a subsequence if needed, we have

$$S_j \to S \qquad \text{and} \qquad \partial S_j \to \partial S$$

locally in the flat norm, with $S \in I_2^{\text{loc}}(U(a, 1))$, $\partial S \in I_1^{\text{loc}}(U(a, 1))$ and $M(\partial S) = 2\lambda$. In fact

$$\begin{aligned} M(\partial S_j) &= M(\mu_{r_j \#}(\partial T \llcorner U(a, r_j))) \\ &= r_j^{-1} M(\partial T \llcorner U(a, r_j)) \qquad \text{by [7, 4.2.8, p. 405]} \\ &\to 2\lambda. \end{aligned}$$

Similarly $M(S_j)$ is uniformly bounded. Hence by the standard compactness theorem [7, 4.2.14, p. 414], S_j and ∂S_j (after passing to a suitable subsequence) tend (in flat norm) to S and ∂S, respectively.

Let $D = \operatorname{Tan}(V, a) \cap U(a, 1)$ be the unit disk in the tangent space of V at a and let $\exp_a$ be the exponential map on V near a, normalized so that its gradient at a is the identity. Let Y be a two-dimensional integral current with compact support in D and let

$$Y_j = (\exp_a^{-1} \circ \mu_{r_j^{-1}})_{\#} Y.$$

Then spt $Y_j \subset V \cap U(a, r_j)$ for r_j sufficiently small. By the almost minimality of ∂T, expressed by the inequality $(*)$, we have

$$M(\partial T \llcorner U(a, r_j)) \leqslant M(\partial T \llcorner U(a, r_j) + \partial Y_j) + c_4 M(Y_j).$$

Now $(\partial T) \llcorner U(a, r_j) = \mu_{r_j^{-1}\#} S_j$. Hence dividing by r_j we find

$$M(\partial S_j) \leqslant M((\partial S_j) + \partial Y_j) + c r_j M(Y_j)$$

which shows that ∂S_j is again almost minimal, even in a stronger sense than before because of the factor r_j in the second term of the right hand side. By a simple modification of [7, 5.4.2, p. 620] now we can deduce that $\|\partial S_j\| \to \|\partial S\|$ weakly and letting $j \to \infty$ we find

$$M(\partial S) = \lim_{j \to \infty} M(\partial S_j) = 2\lambda$$

and

$$M(\partial S) \leqslant M(\partial(S + Y)).$$

This proves that ∂S is a minimal one-dimensional current in D and also that $S = \mathbf{D} \llcorner A$ for some set $A \subset D$. Also $O \in \operatorname{spt} \partial S$.

It follows that ∂S is smooth and S is a half-circle by known results, say, [7, 5.4.15, p. 644]. Moreover by a modified form of [7, 5.4.3(6), (7), pp. 621, 622] there exists a tangent cone to ∂T at a. Every such tangent cone restricted to $U(a, 1)$ is of the type described above and since minimal cones in $\mathbb{R}^2$ are straight lines (Fleming [13]) it follows that $M(\partial S) = 2$.

Hence by [7, 5.4.3(7)] mentioned above we obtain $\lambda = 1$. Thus we have shown that the density $\Theta(\|\partial T\|, a) = 1$ at every point of spt T, and any tangent cone to ∂T at a is a straight line. Now the result follows from Lemma 3.

5. The Connectedness of the Extremal for the Problem (P)

Our previous results have shown that there is an extremal C of the isoperimetric problem (P) consisting of a finite number of smooth non-self-intersecting closed curves $C_0, C_1, C_2, \ldots, C_n$, on each of which

$$K_g = \lambda K,$$

where λ is a constant independent of the components C_i.

In order to show C is actually a geodesic and so solve Poincaré's isoperimetric variational problem [V] we shall prove that C consists of a single closed curve C_0, provided the metric g of the manifold is sufficiently restricted; now the result follows from the discussion of Section 2.

To establish the connectedness of C we shall calculate the second variation for the variational problem (P). Then we show that for an admissible variation, if C had more than one connected component and the metric g is sufficiently restricted, then this second variation can be made negative.

LEMMA 4. *The second variation $\delta^2 J$ of the isoperimetric variational problem* (P) *is given by the formula*

$$\delta^2 J(n) = \int_C \left\{ \left(\frac{dn}{ds}\right)^2 - \left[\lambda \left(\frac{\partial K}{\partial v}\right) + K + (\lambda K)^2\right] n^2 \right\} ds$$

for all vector fields n normal to C such that $\int_C Kn\, ds = 0$, where $\partial K/\partial v$ is the derivative of K with respect to the inner normal to C.

Proof. Choose a geodesic parallel system of coordinates about the connected component L of C (see, e.g., Klingenberg [12, p. 80]). Thus the arclength element is given by the formula

$$ds^2 = A^2\, du^2 + dv^2.$$

Here the curve L is identified with the curve $v = 0$ and intersecting L orthogonally are the geodesics $u = \text{const}$. Orthogonal trajectories to these geodesics are the curves $v = \text{const}$. The parameter v measures the arclength along the geodesics $u = \text{const.}$, starting with $v = 0$ on L. The parameter u is the arclength along L, beginning with an arbitrary point. Here $A(u, 0) = 1$ since the line element along L is

$$ds = A(u, 0)\, du.$$

In this coordinate system the intrinsic quantities on L, measuring the geodesic curvature K_g and the Gaussian curvature K are easily described. Indeed since $A(u, 0) = 1$ by standard formulae, we have $K_g = A_v(u, 0)$ on L and $K = -A_{vv}(u, 0)$.

Consequently on L since $K_g = \lambda K$ we find $A_v(u, 0) = -\lambda A_{vv}(u, 0)$. With this preparation we are now ready to compute the desired second variation formula.

We denote by L_ϵ the curve parametrized by $v = \varepsilon n(u)$. Thus L_ϵ is a normal displacement of L of length $\varepsilon n(u)$. The arclength functional becomes

$$J(\varepsilon) = \int_{L_\epsilon} ds = \int (A^2 + v_u^2)^{1/2}\, du.$$

We easily find

$$J(\varepsilon) = J(0) + \varepsilon \delta J + \varepsilon^2 \delta^2 J + O(\varepsilon^3),$$

where

$$\delta J = \int K_g n\, du$$

and

$$\begin{aligned} \delta^2 J &= \int \left[\tfrac{1}{2} n_u^2 + \tfrac{1}{2}(K_g^2 - K)\, n^2 - \tfrac{1}{8}(2K_g)^2\, n^2\right] du \\ &= \tfrac{1}{2} \int (n_u^2 - Kn^2)\, du. \end{aligned}$$

We now calculate the second variation of the functional

$$V(C) = \int_{\Sigma(C)} K\, dV,$$

which determines the constraint condition in (P), assuming that the component L of C is shifted to L_ϵ by the normal displacement $v(\varepsilon) = \varepsilon n(u)$ along L. Then, letting C_ϵ be the modified curve, we have

$$V(C_\epsilon) = \int_{\Sigma(C_\epsilon)} K\, dV = V(C) + \varepsilon \delta V + \varepsilon^2 \delta^2 V + O(\varepsilon^3),$$

where

$$\begin{aligned} \delta V &= \int_C Kn\, ds \\ \delta^2 V &= \frac{1}{2} \int_C n^2 \left(\frac{\partial K}{\partial v} + \lambda K^2\right) ds, \end{aligned}$$

where $\partial K/\partial v$ is the derivative of K with respect to the inner normal. Consequently the complete second variation for the variational problem (P) is found by combining our formulas for δJ, $\delta^2 J$, δV, $\delta^2 V$ and

$$\delta^2 J(n) = \frac{1}{2} \int \left\{ \left(\frac{dn}{ds}\right)^2 - \left[\lambda \frac{\partial K}{\partial v} + K + (\lambda K)^2\right] n^2 \right\} ds$$

as required.

COROLLARY. *If $|\nabla K| < 2K^{3/2}$ on V then C is connected and hence is a geodesic.*

Proof. The quadratic polynomial in λ

$$K^2\lambda^2 + \lambda(\partial K/\partial v) + K$$

has discriminant

$$(\partial K/\partial v)^2 - 4K^3.$$

Hence if $|\partial K/\partial v| < 2K^{3/2}$ we get

$$\delta^2 J(n) = \int_C [(dn/ds)^2 - A(x)\, n^2]\, ds$$

with $A(x) > 0$.

If C is not connected, we can choose n *locally constant* with

$$\int_C Kn\, ds = 0 \qquad \textit{and now} \quad dn/ds = 0$$

identically, which yields

$$\delta^2 J(n) < 0$$

which contradicts the minimality of C.

6. Stability of Simple Closed Geodesics

In this section we investigate a weak notion of stability for the simple closed geodesic C found in Section 3, provided the manifold V described there is a small C^3 perturbation of the standard unit sphere (S^2, g). Here g is a usual metric of constant Gauss curvature. Thus, let $(S^2, \tilde{g})$ denote a C^3 perturbation of the standard sphere (S^2, g) with elements denoted $d\tilde{s}^2 = \sum \tilde{g}_{ij}\, dx_i\, dx_j$ and $ds^2 = \sum g_{ij}\, dx_i\, dx_j$ with $\sup_{ij} |g_{ij} - \tilde{g}_{ij}|_{C^3} \leqslant \varepsilon$.

We consider $(S^2, \tilde{g})$ as embedded in Euclidean space $\mathbb{R}^3$ with the same support as (S^2, g). Then we shall use the methods of geometric measure theory to prove the following

STABILITY THEOREM. *There is a function $\eta(t)$ tending to zero as $t \to 0$ with the following property.*

Let (S^2, g) be the standard sphere in $\mathbb{R}^3$ and $(S^2, \tilde{g})$ be a small C^3 perturbation of the metric g in the sense that

$$\| g - \tilde{g} \|_{C^3} = \sup_{i,j} | g_{ij}(x) - \tilde{g}_{ij}(x) |_{C^3} \leqslant \varepsilon_0$$

for a suitable absolute constant $\varepsilon_0 > 0$.

Whenever $\| \tilde{g} - g \|_{C^3} = t \leqslant \varepsilon_0$, there is a closed simple geodesic $\tilde{C}$ of minimum length on $(S^2, \tilde{g})$ and a great circle C on (S^2, g) such that the Hausdorff distance of C and $\tilde{C}$ satisfies

$$\operatorname{dist}(\tilde{C}, C) \leqslant \eta(\| g - \tilde{g} \|_{C^3}).$$

Proof. We carry out the proof in two parts. We show

(i) if $\tilde{C}$, the closed simple geodesic of smallest non-zero length (shown to exist in Section 5), is written $\tilde{C} = \partial \tilde{T}$, where $\tilde{T} = \mathbf{S}^2 \llcorner \tilde{A}$ for some open set $\tilde{A} \subset S^2$, then $M(\partial \tilde{T}) = 2\pi + \varphi(t)$, where $\varphi(t) \to 0$ as $t = \| \tilde{g} - g \|_{C^3} \to 0$;

(ii) there is a standard hemisphere T in $(S^2, \tilde{g})$ such that the mass norm

$$M(\tilde{T} - T) \to 0 \qquad \text{as} \quad t \to 0$$

and from this, with $\partial T = C$, we prove the desired null convergence of the Hausdorff distance, in terms of $\| \tilde{g} - g \|_{C^3}$.

In the following arguments we have to compare currents on $(S^2, \tilde{g})$ with currents on (S^2, g). Let

$$(S^2, \tilde{g}) \xrightarrow[\sim]{\psi} (S^2, g)$$

be a diffeomorphism such that $\tilde{g}$ is the pull-back of g by ψ; then we shall compare $\psi_{\#} \tilde{T}$ with T on (S^2, g) and $(\psi^{-1})_{\#} T$ with $\tilde{T}$ on $(S^2, \tilde{g})$.

Let $\tilde{C}$ be the closed simple geodesic shown to exist in Section 3 for the perturbed sphere $(S^2, \tilde{g})$. Clearly the isoperimetric variational principle [V] guarantees that $\tilde{C}$ is a closed simple geodesic of minimum non-zero length on $(S^2, \tilde{g})$. The current carried by $\tilde{C}$ can be written as $\tilde{C} = \partial \tilde{T}$ with $T = \mathbf{S}^2 \llcorner \tilde{A}$ for some open set $\tilde{A} \subset S^2$.

Proof of (i). We begin by proving one half of it, that is,

$$M(\partial\tilde{T}) \leqslant 2\pi + \varphi(t),$$

where $\varphi(t) \to 0$ as $t \to 0$.

This is done by an explicit construction.

Let S^+ be an upper hemisphere of S^2. As $\tilde{g} \to g$ in C^3, the Gaussian curvature $K(\tilde{g}) \to 1$ uniformly on S^2. Hence $\int K(\tilde{g}) d\|\mathbf{S}^+\| \to \int d\|\mathbf{S}^+\| = 2\pi$. By changing $\mathbf{S}^+$ slightly (by adding a small geodesic disk $\tilde{\mathbf{D}}$) into the current $\tilde{\mathbf{S}}^+ = \mathbf{S}^+ + \tilde{\mathbf{D}}$, we can ensure that

$$\int K(\tilde{g}) d\|\tilde{\mathbf{S}}^+\| = 2\pi,$$

hence $\tilde{\mathbf{S}}^+$ is admissible for problem (P) on $(S^2, \tilde{g})$. Since $M(\partial\tilde{\mathbf{D}}) \to 0$ as $t \to 0$, we obtain

$$M(\partial\tilde{\mathbf{S}}^+) \leqslant M(\partial\mathbf{S}^+) + \varepsilon,$$

where $\varepsilon \to 0$ as $t \to 0$. Now the length of $\partial\mathbf{S}^+$ in the metric g goes to 2π as $t \to 0$, hence

$$M(\partial\tilde{T}) \leqslant M(\partial\tilde{\mathbf{S}}^+) \leqslant 2\pi + \varphi(t),$$

where $\varphi(t) \to 0$ as $t \to 0$.

The second half of (i) is proved by contradiction. Suppose not, then we get a sequence of metrics $g_n \to g$ in C^3 and a sequence of solutions T_n to the variational problem with

$$\overline{\lim}\, M(\partial T_n) < 2\pi$$

and (possibly replacing T_n by its complement with reversed orientation)

$$M(T_n) \leqslant \tfrac{1}{2} M((S^2, g_n)) = 2\pi$$

as $n \to \infty$.

By the compactness theorem [7, 4.2.17, p. 414] we have a subsequence, again denoted by T_n, such that $\psi_{n\#}(T_n, \partial T_n) \to (T, \partial T)$ in terms of flat convergence. Here ψ_n is the diffeomorphism of (S^2, g_n) onto (S^2, g) and T is the limiting integral current. Moreover, by construction

$$\int K(g_n) d\|T_n\| = 2\pi.$$

Now $K(g_n) \to 1$ uniformly and

$$\psi_{n\#} T_n \to T$$

in the flat topology in the maximal dimension, therefore also in the mass norm. We have

$$\begin{aligned}\int K(g)d\|T\| &= \int d\|T\| \\ &= \lim \int d\|T_n\| \\ &= \lim \left\{\int K(g_n)d\|T_n\| - \int (K(g_n)-1)d\|T_n\|\right\} \\ &= 2\pi.\end{aligned}$$

Moreover by lower semicontinuity and by our initial hypothesis

$$M(\partial T) \leqslant \lim M(\partial T_n) < 2\pi.$$

Since $\int K(g)d\|T\| = 2\pi$, the pair $(T, \partial T)$ is admissible for the variational problem (P) on S^2. On the other hand, the length of all geodesics on (S^2, g) is 2π, hence $M(\partial T) \geqslant 2\pi$, which contradicts the inequality $M(\partial T) < 2\pi$ we have obtained earlier. This completes the proof of statement (i).

Now we prove the existence of a standard hemisphere T_0 in $(S^2, \tilde{g})$ such that

$$M(\tilde{T} - T_0) \leqslant \eta(t),$$

where $\eta(t)$ is a function such that $\eta(t) \to 0$ as $t \to 0$.

Suppose not, then there is a sequence of metrics $g_n \to g$, a sequence of geodesics ∂T_n associated with solutions T_n of the variational problem (P) (defined in Section 4) and a fixed positive constant η_0 such that

$$\inf_{T_0} M(T_n - T_0) \geqslant \eta_0,$$

where T_0 runs over all currents supported by hemispheres of (S^2, g_n). By the compactness theorem again, we may suppose $\psi_{n\#}(T_n, \partial T_n) \to (T, \partial T)$ in the flat norm hence $\psi_{n\#} T_n \to T$ in the mass norm, by our earlier remark. This shows that

$$\int K(g)d\|T\| = 2\pi.$$

Hence $T \in \Sigma$ (relative to g). Also by lower semicontinuity

$$M(\partial T) \leqslant \underline{\lim}\, M(\partial T_n) \leqslant 2\pi$$

On the other hand $M(\partial T) \geqslant 2\pi$ because simple geodesics on (S^2, g) have length 2π. Thus $M(\partial T) = 2\pi$ and so T solves the Poincaré variational problem (P) for (S^2, g).

This shows T is a hemisphere and

$$M(\psi_{n\#} T_n - T) \to 0.$$

Since $\psi_n \to$ identity in the C^3 topology, we have on (S^2, g_n) that

$$M(\psi_{n\#}^{-1} T - T_0) \to 0,$$

where T_0 is the hemisphere T but now viewed as a hemisphere of (S^2, g_n). Finally

$$M(T_n - T_0) \leqslant M(T_n - \psi_{n\#}^{-1} T) + M(\psi_{n\#}^{-1} T - T_0)$$

and

$$M(T_n - \psi_{n\#}^{-1} T) \leqslant (\mathrm{Lip}\, \psi_n^{-1})^2 M(\psi_{n\#} T_n - T) \to 0,$$

hence $M(T_n - T_0) \to 0$, a contradiction. This proves the existence of a hemisphere T_0 close to $\tilde{T}$.

To prove the Hausdorff distance between ∂T_0 and $\partial \tilde{T}$, denoted

$$\mathrm{dist}(\mathrm{spt}\, \partial \tilde{T}, \mathrm{spt}\, \partial T_0),$$

is small, whenever $M(\tilde{T} - T_0) \leqslant \eta(t)$, for small t we argue by contradiction. Indeed suppose the contrary so that there is a sequence T_n and hemisphere T_0 on (S^2, g_n), such that $M(T_n - T_0) \to 0$ and such that for some point $a_n \in \mathrm{spt}\, T_n$ we have

$$\mathrm{dist}(a_n, \mathrm{spt}\, \partial T_0) \geqslant \delta_0 > 0.$$

The currents ∂T_n satisfy the uniform almost minimality property (with respect to the length in the metric g_n). Hence by a result of Almgren (see Lemma 4 of Bombieri [11])

$$M(\partial T_n \llcorner B(a_n, \delta_0/2)) \geqslant \delta_1 > 0$$

for some number δ_1 independent of n. Since $\psi_n \to$ *identity* in the C^3 topology we have that $\psi_{n\#} \partial T_0 \to \partial T$ in the flat topology, where T is a hemisphere on (S^2, g). Also $\psi_{n\#}(\partial T_n - \partial T_0) \to 0$ in the flat topology and thus

$$\psi_{n\#} \partial T_n \to \partial T$$

for the weak convergence. Let N be a neighborhood of $\operatorname{spt} \partial T$ such that $\psi_n^{-1}(N) \cap B(a_n, \delta_0/2) = \varnothing$. Then

$$\psi_{n\#} \partial T_n \llcorner N \to \partial T \llcorner N$$

for the weak convergence, hence

$$M(\partial T \llcorner N) \leqslant \underline{\lim}\, M(\psi_{n\#} \partial T_n \llcorner N).$$

Since $\operatorname{spt} \partial T \subset N$ we have

$$M(\partial T \llcorner N) = M(\partial T) = 2\pi;$$

also

$$M(\psi_{n\#} \partial T_n \llcorner N) \leqslant (\operatorname{Lip} \psi_n)^2\, M(\partial T_n \llcorner \psi_n^{-1}(N)).$$

Now by the disjointness of $\psi_n^{-1}(N)$ and $B(a_n, \delta_0/2)$ we have

$$\begin{aligned} M(\partial T_n) &\geqslant M(\partial T_n \llcorner \psi_n^{-1}(N)) \\ &\qquad + M(\partial T_n \llcorner B(a_n, \delta_0/2)) \\ &\geqslant M(\partial T_n \llcorner \psi_n^{-1}(N)) + \delta_1 \end{aligned}$$

and letting $n \to \infty$ we find, using (i), that

$$\begin{aligned} 2\pi &= \lim M(\partial T_n) \\ &\geqslant \delta_1 + \lim M(\partial T_n \llcorner \psi_n^{-1}(N)) \\ &\geqslant \delta_1 + M(\partial T) = \delta_1 + 2\pi, \end{aligned}$$

with $\delta_1 > 0$. This is the desired contradiction.

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