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On a Result of R. R. Hall*

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An upper bound for the mean value of a non-negative submultiplicative function by R. R. Hall [3] is sharpened and generalised. Hall's inequality implies a certain rather accurate upper sieve estimate, and this aspect of Hall's result is exploited in deriving good lower bounds for $\pi(x)$ via the sieve.

1. INTRODUCTION

Throughout this note let h denote a non-negative arithmetic function that is *sub-multiplicative* in the sense that

$$h(mn) \leqslant h(m)\, h(n) \text{ if } (m, n) = 1. \tag{1.1}$$

In [3], R.R. Hall proved the following elegant result:

If h is sub-multiplicative and satisfies also

$$h(1) = 1,\ 0 \leqslant h(n) \leqslant 1 \textit{ for all } n,$$

then

$$\sum_{n \leqslant x} h(n) \leqslant e^{\gamma} x \left(1 + O\left(\frac{\log\log x}{\log x}\right)\right) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)\left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right). \tag{1.2}$$

* In memory of P. Turán.

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As Hall pointed out, special interest attaches to the following case: let K be a positive integer none of whose prime factors exceeds x, and take

$$h_K(n) := 1, \qquad (n, K) = 1,$$
$$:= 0, \qquad (n, K) > 1.$$

Hall's result evidently applies to h_K and one derives easily from (1.2) that

$$\sum_{\substack{n \leqslant x \\ (n,K)=1}} 1 \leqslant e^{\gamma} \frac{\phi(K)}{K} x \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right). \tag{1.3}$$

Now (1.3) is an upper sieve estimate, admittedly of a special kind, which has the remarkable feature of being best possible (apart from the error term) as reference to the prime number theorem shows; to put it in another way, the Selberg sieve applied to the sum on the left of (1.3) would lead to an estimate which is (essentially) *twice* that given on the right of (1.3) (cf. van Lint and Richert [6], or Halberstam and Richert [2], Chapter 3).

Such improvements of standard sieve estimates are potentially important, as will be illustrated in a simple sieve application in section 3. Therefore it is perhaps of interest to give a simpler and more transparent proof of Hall's inequality; this proof leads at the same time to a result which is more general in several respects. Moreover, Hall used the prime number theorem in his argument; this turns out to be unnecessary and we shall use instead (1.4) below, which may be considered as a (generalized) upper Čebyčev estimate.

THEOREM 1. *Let h be a non-negative sub-multiplicative arithmetic function such that $h(1) = 1$, satisfying also*

$$\sum_{p \leqslant y} h(p) \log p \leqslant \kappa y + O\left(\frac{y}{\log^2 y}\right) \qquad (y \geqslant 2) \tag{1.4}$$

for some constant $\kappa > 0$, and

$$\sum_{\substack{p^r \geqslant y \\ r \geqslant 2}} \frac{h(p^r)}{p^r} \log p^r \ll \frac{1}{\log y} \qquad (y \geqslant 2). \tag{1.5}$$

Let

$$z \geqslant 2$$

and define

$$P(z) := \prod_{p < z} p.$$

Then

$$\sum_{\substack{n \leqslant x \\ (n, P(z))=1}} h(n) \leqslant \kappa \frac{x}{\log x} m\left(\frac{x}{z}, z\right) + O\left(\frac{xm(x, z)}{\log^2 x}\right), \tag{1.6}$$

where

$$m(x, z) := \sum_{\substack{n \leqslant x \\ (n, P(z))=1}} \frac{h(n)}{n}.$$

The O-constant in (1.6) depends at most on κ and on the O-constants implied by (1.4) and (1.5).

If we put $z = 2$ in Theorem 1, the condition $(n, P(z)) = 1$ becomes void, and we obtain as a special case

THEOREM 2. *Under the assumptions of Theorem* 1 *we have*

$$\sum_{n \leqslant x} h(n) \leqslant \kappa \frac{x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) \sum_{n \leqslant x} \frac{h(n)}{n}.$$

The main feature of (1.6) is of course the reduction to $m(x, z)$, a function which can often be evaluated asymptotically (cf. also Levin and Fainleib [5]), as we shall demonstrate in one special case in section 4. However, it is immediate from (1.1) that

$$\sum_{n \leqslant x} \frac{h(n)}{n} \leqslant \prod_{p \leqslant x}\left(1 + \sum_{r=1}^{\infty} \frac{h(p^r)}{p^r}\right),$$

and by a well-known result of Mertens

$$\frac{1}{\log x} = e^{\gamma} \prod_{p \leqslant x}\left(1 - \frac{1}{p}\right)\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Hence Theorem 2 implies (1.2) on taking $\kappa = 1$, and the factor loglog x in the error term can be avoided, as had been conjectured (cf. [3]); in particular, we now obtain (1.3) in the improved form

$$\sum_{\substack{n \leqslant x \\ (n, K)=1}} 1 \leqslant e^{\gamma} \frac{\phi(K)}{K} x\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where the O-constant is absolute.

Condition (1.4) requires that $h(p)$ is at most κ on average; and it is easy to verify that (1.5) holds (in even sharper form) if we impose the Wirsing condition

$$h(p^r) \leqslant \gamma_1 \gamma_2^r \quad (r = 2, 3, 4, \ldots)$$

where γ_1, γ_2 are constants satisfying $0 \leqslant \gamma_1$, $0 \leqslant \gamma_2 < 2$.

If one requires a result like (1.6) but one attaches no importance to the quality of the final error term one may replace the hypotheses (1.4) and (1.5) by

$$\sum_{p \leqslant y} h(p) \log p \leqslant \kappa y + O\left(\frac{y}{g(y)}\right),$$

and

$$\sum_{p, r \geqslant 2} \frac{h(p^r)}{p^r} \log p^r < \infty,$$

respectively, where $g(y)$ is a positive function tending to infinity with y in such a way that

$$\sum_{l} \frac{1}{l g(l)} < \infty.$$

Then one derives, in the same way, (1.6) with $O(xm(x, z)/\log^2 x)$ replaced by $o(xm(x, z)/\log x)$.

It would be interesting to know to what extend the conditions in both theorems can be weakened without affecting the outcome. Also, it would be both interesting and important, to derive corresponding results for arbitrary intervals of length x, and/or for arithmetic progressions, at some level of generality (cf. Hall's remarks on p. 348 of [3] concerning the twin primes problem and $\pi(x + y) - \pi(x)$). This has been underlined recently by the successful application of the sieve by Iwaniec and Jutila [4] to the location of primes in short intervals.

2. Proof of Theorem 1

Throughout the proof we indicate by Σ' that the variables of summation have to be coprime with $P(z)$. Let

$$M(x, z) := \sum_{\substack{n \leqslant x \\ (n, P(z)) = 1}} h(n) = \sum_{n \leqslant x}{}' h(n),$$

and

$$I(x, z) := \int_1^x \frac{M(t, z)}{t}\, dt = \sum_{n \leqslant x}{}' h(n) \log \frac{x}{n}.$$

We observe at once that

$$M(x, z) + I(x, z) = \sum_{n\leqslant x}' h(n)\left(\log\frac{x}{n} + 1\right) \leqslant \sum_{n\leqslant x}' h(n)\frac{x}{n},$$

so that

$$M(x, z) + I(x, z) \leqslant xm(x, z),$$

which implies both

$$M(x, z) \leqslant xm(x, z), \tag{2.1}$$

and

$$I(x, z) \leqslant xm(x, z); \tag{2.2}$$

and we remark at this stage also that $m(x, z)$ is monotonic increasing in x, a fact that will be used often in the argument below.

The proof depends on the identity

$$\begin{aligned} M(x, z)\log x &= \sum_{n\leqslant x}' h(n)\log n + \sum_{n\leqslant x}' h(n)\log\frac{x}{n} \\ &= \sum_{n\leqslant x}' h(n)\sum_{p^r\| n}\log p^r + I(x, z) \\ &= \Sigma(x, z) + I(x, z), \end{aligned} \tag{2.3}$$

say. We begin by showing (cf. (1.6)) that

$$M(x, z) \ll \frac{x}{\log x} m(x, z). \tag{2.4}$$

We have immediately that

$$\Sigma(x, z) = \sum_{\substack{p^r m\leqslant x\\ (p,m)=1}}' h(p^r m)\log p^r \leqslant \sum_{p^r m\leqslant x}' h(m)\,h(p^r)\log p^r$$

by (1.1), so that

$$\begin{aligned} \Sigma(x, z) &\leqslant \sum_{pn\leqslant x}' h(n)\,h(p)\log p + \sum_{\substack{p^r n\leqslant x\\ r\geqslant 2}}' h(n)\,h(p^r)\log p^r \\ &\leqslant \sum_{n\leqslant x/z}' h(n)\sum_{z\leqslant p\leqslant x/n} h(p)\log p + \sum_{\substack{p^r\leqslant x\\ r\geqslant 2}} h(p^r)\log p^r\, M\left(\frac{x}{p^r}, z\right). \end{aligned} \tag{2.5}$$

We now apply (1.4) in the more convenient equivalent form

$$\sum_{p \leqslant y} h(p) \log p \leqslant \kappa y + O\Big(\sum_{2 \leqslant l \leqslant y} \frac{1}{\log^2 l}\Big) \qquad (y \geqslant 2);$$

then

$$\sum_{n \leqslant x/z}{}' h(n) \sum_{z < p \leqslant x/n} h(p) \log p$$

$$\leqslant \kappa x m\left(\frac{x}{z}, z\right) + O\Big(\sum_{n \leqslant x/z}{}' h(n) \sum_{2 \leqslant l \leqslant x/n} \frac{1}{\log^2 l}\Big)$$

$$\leqslant \kappa x m\left(\frac{x}{z}, z\right) + O\Big(\sum_{2 \leqslant l \leqslant x} \frac{M(x/l, z)}{\log^2 l}\Big);$$

and therefore, from (2.5),

$$\Sigma(x, z) \leqslant \kappa x m\left(\frac{x}{z}, z\right) + O\Big(\sum_{2 \leqslant l \leqslant x} \frac{M(x/l, z)}{\log^2 l}\Big)$$

$$+ \sum_{\substack{p^r \leqslant x \\ r \geqslant 2}} h(p^r) \log p^r \, M\left(\frac{x}{p^r}, z\right). \tag{2.6}$$

If now we apply the trivial bound (2.1) we have

$$\Sigma(x, z) \leqslant \kappa x m(x, z) + O\Big(x \sum_{2 \leqslant l \leqslant x} \frac{m(x/l, z)}{l \log^2 l}\Big)$$

$$+ x \sum_{\substack{p^r \leqslant x \\ r \geqslant 2}} \frac{h(p^r) \log p^r}{p^r} m\left(\frac{x}{p^r}, z\right)$$

and, since $m(y, z)$ is monotonic increasing in y,

$$\Sigma(x, z) \leqslant x m(x, z)\Big(\kappa + O(1) + \sum_{p, r \geqslant 2} \frac{h(p^r)}{p^r} \log p^r\Big) \ll x m(x, z) \tag{2.7}$$

provided only that

$$\sum_{p, r \geqslant 2} \frac{h(p^r)}{p^r} \log p^r < \infty,$$

which certainly is implied by (1.5). Now (2.3) with (2.7) and (2.2) proves (2.4).

We are able to complete the proof of the theorem on the basis of (2.3) and (2.4). First of all, by (2.4)

$$I(x, z) \ll \int_1^x \frac{m(t, z)}{\log(2t)}\, dt \ll m(x, z)\, \frac{x}{\log x}\,.$$

Hence, by (2.3),

$$M(x, z) = \frac{\Sigma(x, z)}{\log x} + O\left(\frac{xm(x, z)}{\log^2 x}\right). \tag{2.8}$$

We deal with $\Sigma(x, z)$ on the basis of (2.6). Take first the O-term. We have, by (2.4) and (2.1) (in that order) that

$$\begin{aligned}\sum_{2\leqslant l\leqslant x} \frac{M(x/l, z)}{\log^2 l} &= \Bigl(\sum_{2\leqslant l\leqslant x^{1/2}} + \sum_{x^{1/2}<l\leqslant x}\Bigr) \frac{M(x/l, z)}{\log^2 l} \\ &\ll \frac{xm(x, z)}{\log x} \sum_{2\leqslant l\leqslant x^{1/2}} \frac{1}{l\log^2 l} + xm(x, z) \sum_{l>x^{1/2}} \frac{1}{l\log^2 l} \\ &\ll \frac{xm(x, z)}{\log x}\,. \end{aligned} \tag{2.9}$$

Finally, the last expression on the right of (2.6) is at most

$$\begin{aligned}&\Bigl(\sum_{\substack{p^r\leqslant x^{1/2}\\ r\geqslant 2}} + \sum_{\substack{x^{1/2}<p^r\leqslant x\\ r\geqslant 2}}\Bigr) h(p^r) \log p^r\, M\left(\frac{x}{p^r}, z\right) \\ &\ll \frac{xm(x, z)}{\log x} \sum_{\substack{p^r\leqslant x^{1/2}\\ r\geqslant 2}} \frac{h(p^r)}{p^r} \log p^r + xm(x, z) \sum_{\substack{p^r>x^{1/2}\\ r\geqslant 2}} \frac{h(p^r)}{p^r} \log p^r \\ &\ll \frac{xm(x, z)}{\log x}\,; \end{aligned} \tag{2.10}$$

on the way we have used once more first (2.4), then (2.1) and, at the last stage, (1.5). Hence, from (2.6), (2.9) and (2.10),

$$\Sigma(x, z) \leqslant \kappa x m\left(\frac{x}{z}, z\right) + O\left(\frac{xm(x, z)}{\log x}\right),$$

and Theorem 1 now follows from (2.8).

3. A Lower Estimate for $\pi(x)$

It has often been said in criticism of sieves that not even the most sophisticated sieve can give so much as Čebyčev's lower estimate

$$\pi(x) \gg \frac{x}{\log x}. \tag{3.1}$$

Iwaniec remarked to one of us a year ago (in conversation) that if Hall's inequality is admissible as a sieve result then this criticism is no longer justified. We shall now show how to obtain a result of type (3.1).

Define

$$\pi(x, z) := \sum_{\substack{1 \leqslant n \leqslant x \\ (n, P(z))=1}} 1 \qquad (z \geqslant 2)$$

so that (for $x \neq m^2$)

$$\pi(x, x^{1/2}) = \pi(x) - \pi(x^{1/2}) + 1.$$

By Buchstab's identity ([2], Lemma 7.1), applied to the sequence $\mathscr{A} = \{n: 1 \leqslant n \leqslant x\}$, and assuming $x \geqslant x_0$ from now on, we therefore have

$$\pi(x) \geqslant \pi(x, x^{1/2}) = \pi(x, x^{1/u}) - \sum_{x^{1/u} \leqslant p < x^{1/2}} \pi\left(\frac{x}{p}, p\right) \qquad (2 \leqslant x^{1/u} < x^{1/2}) \tag{3.2}$$

and by Theorem 8.4 of [2], which, so far as information about the distribution of primes is concerned, requires nothing deeper than Mertens' prime number theory,

$$\pi(x, x^{1/u}) \geqslant \frac{x}{\log x} (e^{-\gamma} u f(u) + O(u \log^{-1/14} x)) \qquad (u > 2) \tag{3.3}$$

(for f see below). It is clear that (3.2) and (3.3) will lead to a lower bound for $\pi(x)$ provided that upper estimates of sufficient quality are available for $\pi(x/p, p)$, $x^{1/u} \leqslant p < x^{1/2}$.

Now Theorem 1, with $h(n) = 1$, reduces this problem to estimating

$$m_\pi(x, z) := \sum_{\substack{n \leqslant x \\ (n, P(z))=1}} \frac{1}{n},$$

for (1.4) is in effect a Čebyčev upper bound estimate of a quality determined by κ and (1.5) is easily checked. Hence, by (1.6),

$$\pi(x, z) \leqslant \kappa \frac{x}{\log x} m_\pi\left(\frac{x}{z}, z\right) + O\left(\frac{x m_\pi(x, z)}{\log^2 x}\right). \tag{3.4}$$

Put

$$u = \frac{\log x}{\log z}.$$

Without appealing to the prime number theorem, we shall show in section 4 that uniformly in $u > 0$

$$m_\pi(x, z) = \psi(u) + O\left(\frac{u^3}{\log x}\right) \qquad (z \geqslant 2) \tag{3.5}$$

with a continuous function $\psi(u)$ defined by

$$\psi(u) = 1, \qquad 0 < u \leqslant 1, \tag{3.6}$$

$$\psi'(u) = \frac{\psi(u-1)}{u}, \qquad u > 1. \tag{3.7}$$

From (3.6) and (3.7) we see immediately that $\psi(u)$ is always positive and increasing, and by induction we infer that

$$\psi(u-1) \leqslant u, \quad u > 1. \tag{3.8}$$

$\psi(u)$ is linked with the above function $f(u)$ via the Buchstab functions $w(u)$ and $\rho(u)$ (cf. [2], Chapter 8):

$$e^{-\gamma} uf(u) = uw(u) - \rho(u), \; u > 0, \tag{3.9}$$

where

$$uw(u) = 1, \; \rho(u) = 1, \text{ if } 0 < u \leqslant 2, \tag{3.10}$$

$$(uw(u))' = w(u-1), \; (u-1)\,\rho'(u) = -\rho(u-1), \text{ if } u \geqslant 2, \tag{3.11}$$

(at $u = 2$ the right hand derivative has to be taken).

From these definitions it is easily checked that for $u > 1$

$$\psi'(u) = w(u)$$

and

$$\psi(u-1) = uw(u). \tag{3.12}$$

From (3.4), (3.5) and (3.8) we infer now that

$$\pi(x, z) \leqslant \kappa \frac{x}{\log x}\, \psi(u-1) + O\left(\frac{x \log x}{\log^3 z}\right) \qquad (u > 2). \tag{3.13}$$

This should, in view of (3.12), be compared with the result of Buchstab [1] who proved, on the basis of the prime number theorem, that

$$\pi(x, z) = (1 + o(1)) \frac{x}{\log x} uw(u).$$

From (3.13) it follows that for $u > 2$

$$\sum_{x^{1/u} \leqslant p < x^{1/2}} \pi\left(\frac{x}{p}, p\right) \leqslant \kappa x \sum_{x^{1/u} \leqslant p < x^{1/2}} \left\{\frac{\psi(\log x/\log p - 2)}{p \log(x/p)} + O\left(\frac{\log(x/p)}{p \log^3 p}\right)\right\}. \tag{3.14}$$

Using Mertens' well-known formula

$$\sum_{p \leqslant x} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right) \tag{3.15}$$

and Stieltjes integration, we obtain readily from (3.14) and (3.8) that

$$\begin{aligned} \sum_{x^{1/u} \leqslant p < x^{1/2}} \pi\left(\frac{x}{p}, p\right) &\leqslant \kappa x \left\{\int_{x^{1/u}}^{x^{1/2}} \frac{\psi(\log x/\log v - 2)}{v \log v \log(x/v)} \, dv \right. \\ &\quad \left. + O\left(\frac{u^3}{\log^2 x}\right) + O\left(\int_{x^{1/u}}^{x^{1/2}} \frac{\log(x/v)}{v \log^4 v} \, dv\right)\right\} \\ &\leqslant \kappa \frac{x}{\log x} \int_1^{u-1} \frac{\psi(t-1)}{t} \, dt + O\left(\frac{u^3 x}{\log^2 x}\right) \\ &\leqslant \kappa \frac{x}{\log x} (\psi(u-1) - 1) + O\left(\frac{u^3 x}{\log^2 x}\right), \end{aligned} \tag{3.16}$$

where, at the last stage, we have used (3.7) and (3.6).

We substitute (3.3) and (3.16) in (3.2) and obtain by (3.9) and (3.12)

$$\pi(x) \geqslant \frac{x}{\log x} (l(u) + O(u \log^{-1/14} x) + O(u^3 \log^{-1} x)) \qquad \text{for any} \quad u > 2, \tag{3.17}$$

where

$$l(u) = uw(u) - \rho(u) - \kappa(uw(u) - 1).$$

With (3.17) we have obtained a result of type (3.1) : for (3.11) and (3.10) yield

$$uw(u) = 1 + \log(u-1), \ \rho(u) = 1 - \log(u-1), \quad 2 \leqslant u \leqslant 3,$$

so that

$$l(u) = (2 - \kappa)\log(u - 1), \quad 2 \leqslant u \leqslant 3;$$

in particular,

$$l(3) = (2 - \kappa)\log 2,$$

which is already positive for all κ in $1 \leqslant \kappa < 2$. The interval $3 \leqslant u \leqslant 4$ leads for all κ in $1 \leqslant \kappa < 2$ to a sharper estimate; here we obtain

$$l(u) = (2 - \kappa)\log(u - 1) - \kappa \int_2^{u-1} \frac{\log(t - 1)}{t}\, dt, \qquad 3 \leqslant u \leqslant 4.$$

We finally mention that for $\kappa = 1$

$$l(u) = 1 - \rho(u)$$

so that (cf. [2], Chapter 8)

$$l(u) = 1 + O(e^{-u}),$$

and here we have in fact

$$l(4) \geqslant 0.9513, \quad l(5) \geqslant 0.9950, \quad l(6) \geqslant 0.9997.$$

4. Proof of (3.5)

We put again for brevity

$$u = \frac{\log x}{\log z}.$$

For a proof of (3.5) we may assume that for some sufficiently large constant c

$$u \leqslant \left(\frac{1}{c}\log x\right)^{1/2}. \tag{4.1}$$

To see this, observe that a crude sieve estimate (cf. [2], Theorem 2.2) gives

$$\pi(t, z) \ll \frac{t}{\log z} \qquad \text{for} \quad z \leqslant t;$$

therefore, since $\pi(t, z) = 1$ if $t < z$,

$$m_\pi(x, z) = \frac{\pi(x, z)}{x} + \int_1^z \frac{dt}{t^2} + \int_z^x \frac{\pi(t, z)}{t^2}\, dt \ll \frac{1}{\log z} + 1 + \frac{\log x}{\log z}$$

so that, with (3.8),

$$m_\pi(x, z) - \psi(u) \ll u \ll \frac{u^3}{\log x},$$

if $u > (1/c \log x)^{1/2}$.

It further sufficies to consider the function

$$m(x, z) := \sum_{\substack{n<x \\ (n, P(z))=1}} \frac{1}{n},$$

since $m(x, z)$ differs from $m_\pi(x, z)$ by at most $1/x$. Then

$$m(x, z) = 1 \text{ for } z \geqslant x. \tag{4.2}$$

It can be shown in the customary way (cf. [2], Chapter 7) that $m(x, z)$ satisfies a Buchstab relation

$$m(x, z) = m(x, x^{1/r}) + \sum_{z \leqslant p < x^{1/r}} \frac{1}{p} m\left(\frac{x}{p}, p\right), \qquad z < x^{1/r} \leqslant x. \tag{4.3}$$

As in (3.16), we obtain from (3.15) and (3.8) for

$$x^{1/(r+1)} \leqslant z < x^{1/r}$$

$$\begin{aligned} \sum_{z \leqslant p < x^{1/r}} \frac{1}{p} \psi\left(\frac{\log x}{\log p} - 1\right) &= \int_z^{x^{1/r}} \frac{\psi(\log x/\log v - 1)}{v \log v} dv + O\left(\frac{r^2}{\log x}\right) \\ &= \int_r^u \frac{\psi(t-1)}{t} dt + O\left(\frac{r^2}{\log x}\right) \\ &= \psi(u) - \psi(r) + O\left(\frac{r^2}{\log x}\right), \end{aligned} \tag{4.4}$$

by (3.7). Introducing

$$\rho(x, z) = m(x, z) - \psi(u),$$

we therefore have from (4.3) and (4.4) that

$$\rho(x, z) = \rho(x, x^{1/r}) + \sum_{z \leqslant p < x^{1/r}} \frac{1}{p} \rho\left(\frac{x}{p}, p\right) + O\left(\frac{r^2}{\log x}\right),$$

or

$$|\rho(x, z)| \leqslant |\rho(x, x^{1/r})| + \sum_{x^{1/(r+1)} \leqslant p < x^{1/r}} \frac{1}{p} \left|\rho\left(\frac{x}{p}, p\right)\right| + B\frac{r^2}{\log x},$$

$$x^{1/(r+1)} \leqslant z < x^{1/r}, \quad r \geqslant 1, \tag{4.5}$$

with some absolute constant B. We shall now prove by induction that with this absolute constant B

$$|\rho(x, z)| \leqslant B \frac{r^3}{\log x} \qquad \text{for} \quad x^{1/(r+1)} \leqslant z < x^{1/r}, \quad r = 1, 2, \ldots . \tag{4.6}$$

By (4.2) and (3.6) we see that $\rho(x, z) = 0$ for $z \geqslant x$, so that (4.6) follows from (4.5) for $r = 1$. Let $r \geqslant 2$, and suppose that (4.5) has already been proved up to $r - 1$. Then, for the interval $x^{1/(r+1)} \leqslant z < x^{1/r}$, on noting that each ρ on the right-hand side of (4.5) can be estimated by (4.6) with r replaced by $r - 1$, we obtain from (4.5)

$$|\rho(x, z)| \leqslant B \frac{(r-1)^3}{\log x} + B(r-1)^3 \sum_{x^{1/(r+1)} \leqslant p < x^{1/r}} \frac{1}{p \log(x/p)} + B \frac{r^2}{\log x} . \tag{4.7}$$

For the sum we obtain, again from (3.15),

$$\begin{aligned} \sum_{x^{1/(r+1)} \leqslant p < x^{1/r}} \frac{1}{p \log(x/p)} &= \int_{x^{1/(r+1)}}^{x^{1/r}} \frac{dv}{v \log v \log(x/v)} + O\left(\frac{r}{\log^2 x}\right) \\ &= \frac{1}{\log x} \log\left(\frac{r}{r-1}\right) + O\left(\frac{r}{\log^2 x}\right) \\ &\leqslant \frac{1}{\log x} \frac{1}{r-1} + B_1 \frac{r}{\log^2 x} . \end{aligned}$$

Hence (4.7) yields

$$|\rho(x, z)| \leqslant \frac{B}{\log x} \left\{(r-1)^3 + (r-1)^2 + B_1 \frac{r(r-1)^3}{\log x} + r^2\right\}, \tag{4.8}$$

and since, by (4.1),

$$r \leqslant \frac{\log x}{\log z} = u \leqslant \left(\frac{1}{c} \log x\right)^{1/2},$$

we have

$$B_1 \frac{r(r-1)^3}{\log x} \leqslant \frac{B_1}{c} r(r-1) \leqslant r(r-1),$$

if only $c \geqslant B_1$. It therefore follows from (4.8) that

$$|\rho(x, z)| \leqslant \frac{B}{\log x} \{(r-1)^3 + (r-1)^2 + r(r-1) + r^2\} = \frac{B}{\log x} r^3.$$

This proves (4.6), and (3.5) follows because of $r \leqslant u$.

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