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MONOTONE DECOMPOSITIONS OF INVERSE LIMIT SPACES BASED ON FINITE GRAPHS*

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A $\Theta_{n,L}$ graph is defined to be a compact, connected, locally connected metric space which is not separated into more than n components by any subcontinuum and no subcontinuum is separated into more than L components by any of its subcontinua. If X is a $\Theta_{n,L}$ graph and f is a continuous surjection of X onto X, then the inverse limit space (X, f) is a Θ_n continuum (not necessarily locally connected). Furthermore (X, f) admits a unique minimal monotone, upper semicontinuous decomposition $\mathscr{D}$ such that the quotient space $(X, f)/\mathscr{D}$ is a $\Theta_{n,L}$ graph if and only if (X, f) contains no indecomposable subcontinua with nonempty interior.

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Θ_n-continue	monotone upper semicontinuous decomposition
finite graph	inverse limit space

Introduction

In a series of papers Barge and Martin [1-3] began to investigate the connection between the dynamics of a continuous interval map f and the geometry of the inverse limit space based on the interval with f as the sole bonding map. One of the tools that they used was a well-known decomposition theorem for chainable continua, see for example Bing [5, Theorem 8]. It is easy to see that the inverse limit space based on a chainable continuum with a continuous bonding map is itself chainable. Therefore Bing's result applied to the spaces Barge and Martin were investigating.

Barge and the author continued work along these lines in their paper [4] where they investigated circle maps and their associated inverse limit spaces. With circle maps the inverse limit space need not be chainable so Bing's theorem does not apply. Work on generalizing Bing's theorem lead to this paper where a decomposition theorem for inverse limit spaces based on nonchainable continua is proven.

* The results in this paper appeared in the author's Ph.D. dissertation which was submitted to the University of Wyoming, December 1987.

In a subsequent paper [8] the author uses this result to generalize many of the results of Barge and Martin to continuous maps of finite graphs. In particular it is shown that if the map has complicated dynamics, of various sorts, then the associated inverse limit space contains indecomposable subcontinua.

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1. Definitions and notation

We define a Θ_n continuum to be a compact, connected metric space X with the property that if M is any subcontinuum of X, then the complement of M in X has at most n components. A Θ_n graph is a Θ_n continuum which is also locally connected.

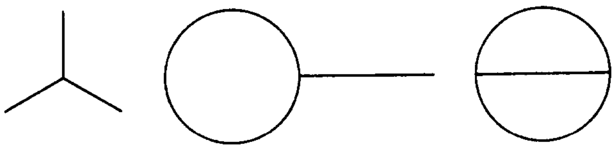
An L-od is a continuum X with the property that if M and $C_1, C_2, \ldots, C_m$ are any subcontinua of X satisfying

(i) $C_i \cap M \neq \emptyset$ for $i = 1, 2, \ldots, m$,

(ii) $C_i \not\subseteq M \cup (\bigcup_{j=1, j\neq i}^{m} C_j)$ for $i = 1, 2, \ldots, m$,

then $m \leq L$. A Θ_n continuum (or graph) which is also an L-od will be denoted by $\Theta_{n,L}$. Note that if X is a $\Theta_{n,L}$ continuum (graph), then X is a $\Theta_{n+p,L+Q}$ continuum (graph) for any positive integers p and Q. The subcontinua C_i which satisfy the conditions of the definition of an L-od for some particular subcontinuum M will be called L-od continua and M will be said to create these L-od continua.

For example, the triod (Fig. 1(a)) is a $\Theta_{3,3}$ graph, a circle is a $\Theta_{1,2}$ graph, a circle with sticker (Fig. 1(b)) is a $\Theta_{2,3}$ graph and a theta curve (Fig. 1(c)) is a $\Theta_{2,4}$ graph. Note $n \leq L$.



(a) (b) (c)

Fig. 1. (a) A triod, $\Theta_{3,3}$ graph, (b) a circle with sticker, $\Theta_{2,3}$ graph, (c) theta curve, $\Theta_{2,4}$ graph.

Associated with $f: X \to X$ is the compact, connected metric space $(X,f) = \{(x_0, x_1, \ldots) \mid x_{n+1} \in X \text{ and } x_n = f(x_{n+1})\}$ with metric

$$d((x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i},$$

where $|*|$ denotes the metric on X. We will denote elements of (X,f) by subbarred letters, as $\underline{x} = (x_0, x_1, \ldots)$. The *projection maps* π_n of (X,f) onto X given by

$\pi_n(\underset{\sim}{x}) = x_n$ are continuous. If H is a subcontinuum (compact, connected subspace) of (X, f), then $f(\pi_{n+1}(H)) = \pi_n(H)$.

If $f: X \to X$, then f induces a homeomorphism $\hat{f}: (X, f) \to (X, f)$ by $\hat{f}(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$. Notice that $f \circ \pi_n = \pi_n \circ \hat{f}$, $\pi_n = \pi_{n+1} \circ \hat{f}$, and $f \circ \pi_{n+1} = \pi_n$.

If S is a subcontinuum, then the statement that S is *decomposable* means that S is the union of two proper subcontinua. A subcontinuum which is not decomposable is said to be *indecomposable*. If p is a point of S, then the *composant of S determined by p* is the union of all proper subcontinua of S which contain p. The following two conditions are equivalent to the indecomposability of S:

(1) If H is a proper subcontinuum of S, then H contains no open sets in S.

(2) Given any two composants K_1 and K_2 of S, then either $K_1 = K_2$ or $K_1 \cap K_2 = \emptyset$.

In fact if S is indecomposable not only do the composants partition S but there are uncountably many distinct composants and each is dense in S.

We will denote the closure of a set A by either $\mathrm{cl}(A)$ or $\bar{A}$, the interior of a set by $\mathrm{int}(A)$, the complement of a set A by A^c, and the boundary of a set by $\mathrm{Bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(A^c)$.

2. Decomposition theorems

In [6] Grace and Vought proved the following:

Theorem 2.1. *Let X be a compact, metric Θ_n continuum. Then X admits a monotone, upper semicontinuous decomposition $\mathcal{D}$ such that the elements of $\mathcal{D}$ have empty interior and the quotient space $X/\mathcal{D}$ is a finite graph if and only if* $\mathrm{int}(T(H)) = \emptyset$ *for every subcontinuum H with empty interior. Furthermore the decomposition of the Θ_n continuum is given by $\mathcal{D} = \{T^{2n}(x) \mid x \in X\}$ and $\mathcal{D}$ is the unique minimal decomposition of X with respect to being monotone, upper semicontinuous and having the quotient space $X/\mathcal{D}$ be a finite graph.*

The function T in the statement of the theorem is the aposyndetic set function defined by Jones [7] as follows:

$$T(A) = A \cup \{x \in X \setminus A \mid \text{there does not exist an open set } U \text{ and a continuum } H \text{ such that } x \in U \subseteq H \subseteq X \setminus A\}.$$

Also $T^0(A) = A$ and $T^n(A) = T(T^{n-1}(A))$ for $n \geq 1$. It is known that if A is connected, then $T(A)$ is a continuum. By a minimal decomposition we mean that if $\mathcal{E}$ is any other decomposition of X which is monotone, upper semicontinuous, and for which $X/\mathcal{E}$ is a finite graph, then if $d \in \mathcal{D}$, there is an element $e = \mathcal{E}$ such that $d \subseteq e$.

Included in the proof of Theorem 2.1 of Grace and Vought [6] is the following.

Lemma 2.2. *Let X be a Θ_n continuum. If it is the case that for all subcontinua H of X with empty interior that* $\mathrm{int}(T(H)) = \emptyset$, *then X contains no indecomposable subcontinua with nonempty interior.*

In [9] Vought showed that in general the converse of this lemma is not true. We show that if X is a Θ_n graph and $f: X \to X$ is a continuous surjection, then (X, f) is a Θ_n continuum. Furthermore if (X, f) contains no indecomposable subcontinua with nonempty interior then, for all subcontinua H of (X, f) with empty interior, $\operatorname{int}(T(H)) = \emptyset$. So from Grace and Vought's theorem (X, f) would admit a monotone upper semicontinuous decomposition if and only if (X, f) contains no indecomposable subcontinua with nonempty interior.

In what follows we assume that X is a $\Theta_{n,L}$ continuum where n and L are finite and as small as possible. Also f is assumed to be a continuous surjection of X onto X.

Lemma 2.3. *Suppose $C_1, C_2, \ldots, C_K$ are subsets of (X, f) such that for any $i \in \{1, \ldots, K\}$, $C_i \not\subseteq \bigcup_{j=1, j\neq i}^{K} \bar{C}_j$; then there is an $N \in Z^+$ such that for any $i \in \{1, \ldots, K\}$ and for all $n \geq N$, $\pi_n(C_i) \not\subseteq \pi_n(\bigcup_{j=1, j\neq i}^{K} \bar{C}_j)$.*

Proof. Let i be fixed. Assume there is a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\pi_{n_k}(C_i) \subseteq \pi_{n_k}\Bigg(\bigcup_{\substack{j=1 \\ j\neq i}}^{K} \bar{C}_j\Bigg).$$

Let $\underline{x} \in C_i$. Then for each k there is a point $\underline{y}^k \in \bigcup_{j=1, j\neq i}^{K} \bar{C}_j$ such that $\pi_{n_k}(\underline{x}) = \pi_{n_k}(\underline{y}^k)$. Since $\bigcup_{j=1, j\neq i}^{K} \bar{C}_j$ is compact there is a subsequence $\{k_l\}_{l=1}^{\infty}$ such that

$$\underline{y}^{k_l} \to \underline{y} \in \bigcup_{\substack{j=1 \\ j\neq i}}^{K} \bar{C}_j$$

as $l \to \infty$. But

$$\begin{aligned} d(\underline{x}, \underline{y}^{k_l}) &= \sum_{i=0}^{\infty} |\pi_i(\underline{x}) - \pi_i(\underline{y}^{k_l})|/2^i \\ &= \sum_{i=k_l+1}^{\infty} |\pi_i(\underline{x}) - \pi_i(\underline{y}^{k_l})|/2^i \\ &\leq \operatorname{diam}(X)/2^{k_l}. \end{aligned}$$

Thus $\underline{y}^{k_l} \to \underline{x}$ as $l \to \infty$. Therefore $\underline{x} \in \bigcup_{j=1, j\neq i}^{K} \bar{C}_j$ which implies that

$$C_i \subseteq \bigcup_{\substack{j=1 \\ j\neq i}}^{K} \bar{C}_j.$$

This contradiction implies that there is a positive integer N_i such that for all $n \geq N_i$,

$$\pi_n(C_i) \not\subseteq \pi_n\Bigg(\bigcup_{\substack{j=1 \\ j\neq i}}^{K} \bar{C}_j\Bigg).$$

Since i was arbitrary let

$$N = \max_{1 \leq i \leq K} (N_i),$$

and the lemma follows. □

Lemma 2.4. *(X,f) is a Θ_m continuum with $m \leq L$, in particular m is finite.*

Proof. Assume that there is a subcontinuum M of (X,f) such that M^c has more than L components. Choose $L+1$ of these and label them $C_1, C_2, \ldots, C_{L+1}$. Then for each i, $1 \leq i \leq L+1$, $\bar{C}_i \cap M \neq \emptyset$ and

$$C_i \not\subseteq M \cup \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{L+1} \bar{C}_j \right).$$

But by Lemma 2.3 there is an N such that

$$\pi_N(C_i) \not\subseteq \pi_N\left(M \cup \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{L+1} \bar{C}_j \right) \right) = \pi_N(M) \cup \left(\bigcup_{\substack{j=1 \\ j \neq i}}^{L+1} \pi_N(\bar{C}_j) \right).$$

Also we have that $\pi_N(\bar{C}_i) \cap \pi_N(M) \neq \emptyset$. This implies that X is at least an $(L+1)$-od. This contradiction establishes the lemma. □

Lemma 2.5. *Suppose (X,f) has the property that any subcontinuum of (X,f) with nonempty interior is decomposable. Let M be a proper subcontinuum of (X,f) and let C be a component of M^c, then there exists a subcontinuum $D \subseteq C$ such that* $\operatorname{int}(D) \neq \emptyset$.

Proof. First we will show that if A is a subcontinuum of (X,f) with nonempty interior, then A is decomposable into subcontinua H and K both of which have nonempty interior.

Let A' be a subcontinuum of A that is irreducible about $\operatorname{int}(A)$. Then A' decomposes into proper subcontinua H_1 and H_2, both of which have nonempty interior since A' is irreducible about $\operatorname{int}(A)$. For $i = 1, 2$, let $K_i = \{x \in A \setminus A' \mid \text{the component of } A \setminus A' \text{ containing } x \text{ has a limit point in } H_i\}$. Then $H = H_1 \cup \bar{K}_1$ and $K = H_2 \cup \bar{K}_2$ are continua, each with nonempty interior whose union is A, and the assertion is true.

Now let M be a proper subcontinuum of (X,f) and C be a component of M^c. Then by Lemma 2.4, M^c has only finitely many components. So $\bar{C}$ has nonempty interior and by the above argument $\bar{C}$ is decomposable into subcontinua A and B each of which has nonempty interior.

Let H and K be irreducible subcontinua about $\bar{C} \setminus A$ and $\bar{C} \setminus B$ respectively. Then H and K are subcontinua of $\bar{C}$ with nonempty interior. If either H or K does not intersect M, then we have produced a subcontinuum of C with nonempty interior as desired. Otherwise we must have that $H \cap M \neq \emptyset \neq K \cap M$. Since both H and K have nonempty interiors they may be decomposed into H_1, H_2 and K_1, K_2 respectively, all having nonempty interiors. If any one of these continua does not intersect

M, then we are done. Otherwise we continue the process of decomposing each of these continua into continua with nonempty interiors. In continuing this process we will either obtain the subcontinuum that we seek or we will reach a point where we have m subcontinua each with nonempty interior and $m \geq L+1$. Each of these subcontinua will intersect M and since each has nonempty interior none will be contained in the union of M and the others. Therefore (X,f) is at least an $(L+1)$-od. It follows then from Lemma 2.3 that X is at least an $(L+1)$-od. Thus the process must have stopped before reaching this point giving us the desired subcontinuum D. $\square$

Lemma 2.6. *If A and B are compact subsets of (X,f) such that $A \cap B = \emptyset$, then there exists an $N \in Z^+$ such that for all $n \geq N$, $\pi_n(A) \cap \pi_n(B) = \emptyset$.*

Proof. Suppose there exists a subsequence $\{n_k\}$ such that $\pi_{n_k}(A) \cap \pi_{n_k}(B) \neq \emptyset$. Let $z_{n_k} \in \pi_{n_k}(A) \cap \pi_{n_k}(B)$, and $\underline{x}^k \in \pi_{n_k}^{-1}(z_{n_k}) \cap A$, and $\underline{y}^k \in \pi_{n_k}^{-1}(z_{n_k}) \cap B$. Then since A and B are compact there are subsequences $\underline{x}^{k_i} \to \underline{x} \in A$ and $\underline{y}^{k_i} \to \underline{y} \in B$ as $i \to \infty$. But as in the proof of Lemma 2.3, $d(\underline{x}^{k_i}, \underline{y}^{k_i}) \to 0$ as $i \to \infty$ so $\underline{x} = \underline{y}$. Thus $A \cap B \neq \emptyset$ contradicting the hypothesis and the lemma follows. $\square$

Theorem 2.7. *If (X,f) contains no indecomposable subcontinua with nonempty interior, then (X,f) is a Θ_n continuum.*

Proof. Suppose there exists a subcontinuum M of (X,f) such that M^c has m components where $n < m \leq L$. Label these components $C_1, C_2, \ldots, C_m$. By Lemma 2.5 there exist subcontinua $D_i \subseteq C_i$ such that $\mathrm{int}(D_i) \neq \emptyset$ for $i = 1, \ldots, m$. Let $F_i = \mathrm{cl}(\bigcup_{k=1}^{n_i} G_k)$ where the G_k are the components of $\bar{C}_i \backslash D_i$ such that $G_k \cap M \neq \emptyset$. Let $M' = M \cup (\bigcup_{i=1}^{m} F_i)$; then $(X,f) \backslash M' = \bigcup_{i=1}^{m} C_i'$ where $C_i' \subseteq C_i$, for all i and $\bar{C}_i' \cap \bar{C}_j' = \emptyset$ if $i \neq j$ since $\bar{C}_i' \cap M = \emptyset$ and $\bar{C}_i \cap \bar{C}_j \subseteq M$. But then it follows from Lemma 2.6 that there is an N such that $\pi_N(\bar{C}_i') \cap \pi_N(\bar{C}_j') = \emptyset$ if $i \neq j$. Thus $X \backslash \pi_N(M')$ has at least m components contradicting X being a Θ_n graph. $\square$

We next give examples which show that the hypothesis that (X,f) contains no indecomposable subcontinuum in Theorem 2.7 is necessary.

Example 2.8. Let $g: S^1 \to S^1$ be the function pictured in Fig. 2 taken (mod 1). If $X = S^1$, then X is a $\Theta_{1,2}$ graph but (X, g) is homeomorphic to two Knaster continua joined at their common endpoints (see Fig. 3).

The point at which the Knaster continua are joined is a subcontinuum (without interior) for which the complement consists of two components. Therefore (X, g) is a Θ_2 continuum. Note that the closure of each of these components is an indecomposable subcontinuum with nonempty interior. For inverse limits based on a circle it can be shown that if the subcontinuum being removed has interior, then its complement has at most one component.

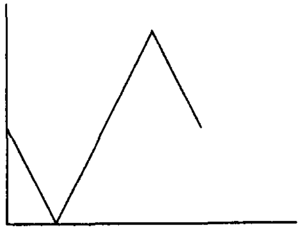


Fig. 2. $f:[0,1]\to[0,1]$ defined by $f(x)=-2x+0.5$ if $0\leq x\leq 0.25$, $f(x)=2x-0.5$ if $0.25\leq x\leq 0.75$, and $f(x)=-2x+2.5$ if $0.75\leq x\leq 1$.

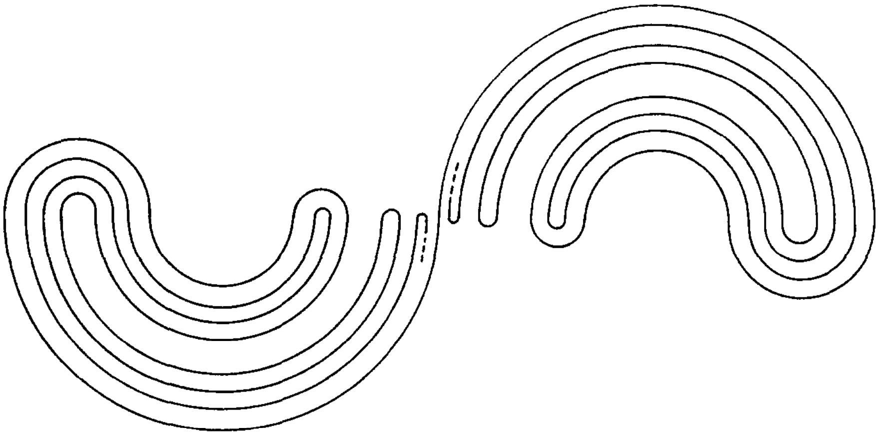


Fig. 3. Two Knaster continua joined at their common endpoint.

In the next example we show how one can start with a $\Theta_{n,L}$ graph ($n\geq 2$) and produce an inverse limit space that is a Θ_m continuum for any m between n and $2n$. Also the continuum removed to give the m components will have interior.

Example 2.9. Let X be n arcs joined at a common endpoint together with l circles (where $l+n=m$) joining the other ends of l of the arcs. Position things so that the only points of intersection are the common endpoints of the n arcs and the l points where the l circles meet the arcs. Figure 4 is an example with $n=3$, $l=2$, and $m=5$.

Let g, defined as in Example 2.8, act on each of the circles with the point where the circle meets the arc corresponding to the point $(0,0)$ in Fig. 2. The map on the

Fig. 4. A $\Theta_{3,5}$ graph.

arcs will be the identity. The inverse limit space will then be homeomorphic to n arcs joined at a common endpoint together with l copies of the continuum (X, g) from Example 2.8 joining the other ends of l of the arcs at the common endpoint of the Knaster continua (see Fig. 5). If M is the l arcs which join the Knaster continua, then M has interior and M^c has $n+l=m$ components.

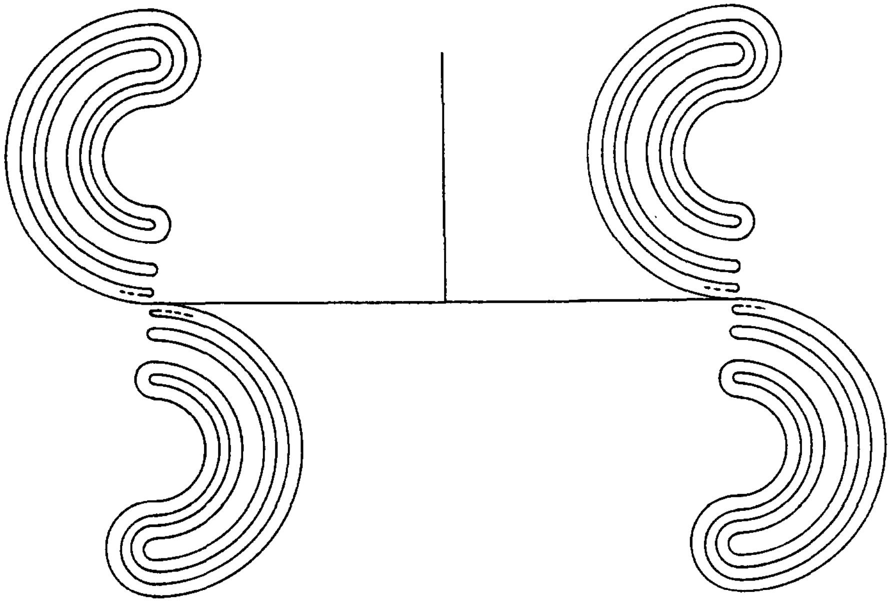


Fig. 5. Inverse limit space for Example 2.9.

The following easy lemma is needed later.

Lemma 2.10. *If $\mathscr{D}$ is a monotone, upper semicontinuous decomposition of a topological space W and P is the projection map from W onto the quotient space $W/\mathscr{D}$ and C is connected in $W/\mathscr{D}$, then $P^{-1}(C)$ is connected.*

Lemma 2.11. *If M is a subcontinuum of (X,f) with* $\mathrm{int}(M)\neq\emptyset$, *then there exists an N such that for all $n\geqslant N$,* $\mathrm{int}(\pi_n(M))\neq\emptyset$.

Proof. Assume there is a subsequence $\{n_k\}$ such that $\pi_{n_k}(M)$ has empty interior for each k. Let $\underline{x}\in\mathrm{int}(M)$ and let $\varepsilon>0$ be given. Then there is a K, dependent on ε, such that

$$\sum_{i=n_K+1}^{\infty}\frac{\mathrm{diam}(X)}{2^i}<\tfrac{1}{2}\varepsilon.$$

Also since $\mathrm{int}(\pi_{n_k}(M))=\emptyset$ there is $y_k\in X\backslash\pi_{n_k}(M)$ such that

$$\sum_{i=0}^{n_k}\frac{|\pi_i(\underline{x})-f^{n_k-i}(y_k)|}{2^i}<\tfrac{1}{2}\varepsilon.$$

Thus if $k \geq K$ and $\underline{y}^k \in \pi_{n_k}^{-1}(y_k)\setminus M$, then $d(\underline{x}, \underline{y}^k) < \varepsilon$. But since ε was arbitrary, this contradicts $\underline{x}$ being an element of the interior of M. □

Theorem 2.12. *The following are equivalent:*

(i) *If A is a subcontinuum of (X, f) with* $\operatorname{int}(A) = \emptyset$, *then* $\operatorname{int}(T(A)) = \emptyset$.

(ii) *(X, f) contains no indecomposable subcontinua with nonempty interior.*

Proof. Lemma 2.2 showed that (i)⇒(ii) is true in general. To show that (ii)⇒(i) assume that $\operatorname{int}(A) = \emptyset$ but that $\operatorname{int}(T(A)) \neq \emptyset$. We know from Lemma 2.4 that $(T(A))^c$ has only finitely many components, say $C_1, \ldots, C_m$. Then $(\bigcup_{i=1}^m \bar{C}_i)^c$ as only finitely many components since (X, f) is a Θ_n continuum. This implies that $\operatorname{int}(T(A))$ consists of at most a finite number of components in $(\bigcup_{i=1}^m \bar{C}_i)^c$. Let M be the closure of one of the components of $(\bigcup_{i=1}^m \bar{C}_i)^c$. Then $M \subseteq T(A)$ and M has nonempty interior. Therefore M is decomposable into two subcontinua H and K each having nonempty interior. Since $\operatorname{int}(H) \neq \emptyset$ and $H \subseteq T(A)$ we have that $H \cap A \neq \emptyset$. Likewise $K \cap A \neq \emptyset$.

As in the proof of Lemma 2.5 we may successively decompose H and K until we obtain at least $L+1$ subcontinua, each with nonempty interior and each intersecting A. But this would imply that (X, f) is at least an $(L+1)$-od. This contradiction implies that the interior of $T(A)$ is empty as desired. □

Collecting results we have:

Theorem 2.13. *Suppose that X is a $\Theta_{n,L}$ graph with L finite; then (X, f) contains no indecomposable subcontinua with nonempty interior if and only if (X, f) admits a minimal, monotone, upper semicontinuous decomposition $\mathscr{D} = \{T^{2n}(x) \mid x \in (X, f)\}$ such that the elements of $\mathscr{D}$ have empty interiors and the quotient space $(X, f)/\mathscr{D}$ is a finite graph.*

The next several lemmas are aimed at giving us enough information about the quotient space $(X, f)/\mathscr{D}$ so that we may use it in studying the dynamics of f on X.

Lemma 2.14. *If X is a $\Theta_{n,L}$ graph with L finite, then there exists a subcontinuum M of X such that M^c contains exactly n components and M creates exactly L, L-od continua.*

Proof. Let M be a subcontinuum of X such that M^c has exactly n components. First suppose there is a branch point p in $X\setminus M$. Let K be an irreducible continuum from M to p. Then $M \cup K$ is a subcontinuum of X and since p is a branch point $(M \cup K)^c$ has at least n components. But X is a Θ_n graph so $(M \cup K)^c$ has exactly n components. Thus we may choose M so that it contains all the branch points of X.

Let N be a subcontinuum of X which creates exactly L, L-od continua. In a manner similar to above we may show that N contains all of the branch points of

X. If M and N contain the same set of edges (an edge is an irreducible subcontinuum from one branch point to another and which contains only the two branch points that it is irreducible between), then $M \cap N$ is the subcontinuum we seek. If M and N do not contain the same set of edges let e_{ij} be an edge from branch point p_i to p_j such that $e_{ij} \subseteq M$ but $e_{ij} \not\subseteq N$. Since N contains all of the branch points and is connected there is a path from p_i to p_j in N. In fact there is only one path in N from p_i to p_j. Otherwise if there were more than one path we could remove the open middle third of one of the edges of one of the two paths connecting p_i to p_j thus leaving N a subcontinuum but creating $L+2$, L-od continua. Call this unique path from p_i to p_j in N, P. Likewise M does not contain all of the edges in P for if it did then M would contain two paths from p_i to p_j. Again we could remove an open middle third of an edge in P making X a Θ_{n+1} graph.

Let e be an edge in P such that $e \not\subseteq M$. Let $\tilde{e}$ be the middle third of e. Then $(N \cup e_{ij})\backslash\tilde{e}$ is a subcontinuum of X with L, L-od continua. Since there are only finitely many edges in X it must then be the case that we can choose M and N so that they contain the same set of edges and the lemma is proven. □

Theorem 2.15. *The decomposition space $(X,f)/\mathcal{D}$ given by Theorem* 2.13 *is a $\Theta_{n,L}$ graph.*

Proof. Let M be a subcontinuum of $(X,f)/\mathcal{D}$ such that M^c has m components and creates k, L-od continua where m and k are as large as possible. We will show that $m \leq n$ and $k \leq L$.

Suppose $m > n$. Let P be the projection map from (X,f) onto $(X,f)/\mathcal{D}$. Then by Lemma 2.10, $P^{-1}(M)$ is connected so $P^{-1}(M)$ is a subcontinuum of (X,f). But $(X,f)\backslash P^{-1}(M)$ has m components which contradicts (X,f) being a Θ_n continuum. Therefore $m \leq n$.

Now assume that $k > L$. Since $(X,f)/\mathcal{D}$ is a $\Theta_{n,k}$ graph, there is a subcontinuum M such that M creates a k-od, i.e., there exist k subcontinua $C_1, C_2, \ldots, C_k$ such that $\bigcup_{i=1}^k C_i \cup M$ is a k-od where $k > L$. The C_i, $i = 1, \ldots, k$, can be taken to be mutually disjoint since $(X,f)/\mathcal{D}$ is a graph. From Lemmas 2.3 and 2.6 there is an $N \in Z^+$ such that for all $s \geq N$, $\pi_s(P^{-1}(C_i)) \cap \pi_s(P^{-1}(C_j)) = \emptyset$ if $i \neq j$, and for all $i \in \{1, \ldots, k\}$, $\pi_s(P^{-1}(C_i)) \not\subseteq \pi_s(P^{-1}(M))$. Then $\bigcup_{i=1}^k \pi_s(P^{-1}(C_i)) \cup \pi_s(P^{-1}(M))$ is a k-od in X where $k > L$. This is a contradiction since X is a $\Theta_{n,L}$ graph and the theorem is proven. □

Lemma 2.16. *If $h:(X,f) \to (X,f)$ is a homeomorphism of (X,f) and if $h^*:(X,f)/\mathcal{D} \to (X,f)/\mathcal{D}$ is defined by $h^*(d) = \{h(\underset{\sim}{x}) \mid \underset{\sim}{x} \in d \in \mathcal{D}\}$, then h^* is a homeomorphism of (X,f).*

Proof. Recall $\mathcal{D} = \{T^{2n}(\underset{\sim}{x}) \mid \underset{\sim}{x} \in (X,f)\}$. First we show that for any nonempty set $A \subseteq (X,f)$ that $h(T(A)) = T(h(A))$. Let $y \in h(T(A))$. If $y \in h(A)$ we have $y \in T(h(A))$. Otherwise if $y \notin h(A)$ assume there is a subcontinuum K with $y \in \text{int}(K)$

and $K \cap h(A) = \emptyset$. Then $h^{-1}(y) \in \mathrm{int}(h^{-1}(K))$ and $h^{-1}(K) \cap A = \emptyset$. Therefore $h^{-1}(y) \notin T(A)$ which implies $y \notin h(T(A))$. This contradiction implies that $h(T(A)) \subseteq T(h(A))$. The other direction of the inclusion is similar.

Thus we have for all $\underset{\sim}{x} \in (X, f)$ that $h(T(\underset{\sim}{x})) = T(h(\underset{\sim}{x}))$. Assume that $h(T^{n-1}(\underset{\sim}{x})) = T^{n-1}(h(\underset{\sim}{x}))$. Then

$$h(T^n(\underset{\sim}{x})) = h(T(T^{n-1}(\underset{\sim}{x}))) = T(h(T^{n-1}(\underset{\sim}{x}))) = T(T^{n-1}(h(\underset{\sim}{x}))) = T^n(h(\underset{\sim}{x})).$$

So for any $d \in \mathscr{D}$, $h(d) \in \mathscr{D}$. Similarly for any $d \in \mathscr{D}$, $h^{-1}(d) \in \mathscr{D}$. Therefore h^* is 1-1 and onto. Finally since $\mathscr{D}$ is an upper semicontinuous decomposition of (X, f) the projection map from (X, f) onto $(X, f)/\mathscr{D}$ is closed and it follows that h^* is continuous. Therefore h^* is a homeomorphism as desired. $\square$

It is easy to see that $\hat{f}$ is a homeomorphism of (X, f). If we let $g_{\underset{\sim}{x}} \in \mathscr{D}$ be the element of the decomposition $\mathscr{D}$ which contains $\underset{\sim}{x}$ and define $\hat{\hat{f}}: (X, f)/\mathscr{D} \to (X, f)/\mathscr{D}$ by $\hat{\hat{f}}(g_{\underset{\sim}{x}}) = g_{\hat{f}(\underset{\sim}{x})}$, then it follows from Theorem 2.13 and Lemma 2.16 that $\hat{\hat{f}}$ is a homeomorphism of $(X, f)/\mathscr{D}$.

We have now shown:

Theorem 2.17. *If X is a $\Theta_{n,L}$ graph and $f: X \to X$ is a continuous surjection such that (X, f) contains no indecomposable subcontinua with nonempty interior, then $\mathscr{D} = \{T^{2n}(\underset{\sim}{x}) \mid \underset{\sim}{x} \in (X, f)\}$ is the unique minimal monotone, upper semicontinuous decomposition of (X, f) such that $(X, f)/\mathscr{D}$ is a $\Theta_{n,L}$ graph and $\hat{\hat{f}}: (X, f)/\mathscr{D} \to (X, f)/\mathscr{D}$ defined by $\hat{\hat{f}}(T^{2n}(\underset{\sim}{x})) = T^{2n}(\hat{f}(\underset{\sim}{x}))$ is a homeomorphism.*

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