Spaces which are generated by discrete sets

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Abstract

Continuing the study initiated by Dow, Tkachenko, Tkachuk and Wilson, we prove that countably compact countably tight spaces, normed linear spaces in the weak topology and function spaces over \(\sigma\)-compact spaces are discretely generated. We also construct, using [CH], a compact pseudoradial space and a pseudocompact space of countable tightness which are not discretely generated.

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0. Introduction

In [5] the authors initiated a systematic study of the property of a space to be generated by its discrete subsets. Such a property turns out to be not only interesting, but also the base for many nice questions.

Fréchet, sequential and scattered spaces are generated by discrete sets, but not all spaces of countable tightness are.

A topological space \(X\) is said to be \textit{discretely generated} provided that for any set \(A \subseteq X\) and any point \(x \in A\) there exists a discrete set \(D \subseteq A\) such that \(x \in D\).

A topological space \(X\) is \textit{weakly discretely generated} provided that for any non-closed set \(A \subseteq X\) there exists a discrete set \(D \subseteq A\) such that \(D \setminus A \neq \emptyset\).

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Among other things, in [5] it is shown that every compact space is weakly discretely generated and every compact space of countable tightness is discretely generated.

In this paper, we present some further results of this type. In particular, we establish that countably compact countably tight regular spaces are discretely generated and show that the same conclusion may be false for pseudocompact spaces of countable tightness.

A couple of our results are also related to the notions of Whyburn and weakly Whyburn spaces [15].

A P-space is a space in which every $G_\delta$-set is open.

A subset $A$ of a space $X$ is $\aleph_1$-closed if $B \subseteq A$ whenever $B \subseteq A$ and $|B| \leq \aleph_1$.

The spaces considered here are always assumed to be $T_1$. Compact means compact Hausdorff.

All undefined notions can be found in [11].

A countable regular maximal space $M$ plays an important role in the sequel. The existence of such a space was first proved by El'kin [10] (see also [8] for another construction). A special feature of $M$ is to be a countable dense-in-itself regular space in which every discrete subset is closed, therefore $M$ is not weakly discretely generated.

1. Discrete generability and sequentiality like properties

In [5] it was pointed out that every sequential space is discretely generated. A natural way to weaken this assumption is to consider the spaces whose topology is defined using sequences of any possible length.

Recall that a transfinite sequence $S = \{x_\alpha: \alpha < \kappa\}$ in the space $X$ converges to a set $C \subseteq X$ provided that every neighborhood of $C$ contains a final segment of $S$. If $C = \{x\}$, then we say that $S$ converges to $x$.

A space $X$ is said to be pseudoradial (radial) if any non-closed set $A \subseteq X$ contains transfinite sequences converging to some (any) point of $A \setminus \overline{A}$.

The following assertion is a simple modification of Theorem 1 in [13].

**Lemma 1.** Let $X$ be a regular space and $C$ a closed subset of $X$. If $S \subseteq X$ is a (transfinite) sequence converging to $C$, then there exists a discrete set $D \subseteq S$ such that $C \cap \overline{D} \neq \emptyset$. Moreover, if $C = \{x\}$ then the Hausdorff separation axiom suffices.

**Proof.** Without any loss of generality, we may assume that $|S|$ is a regular cardinal and $C \subseteq X \setminus S$. Take any $x_1 \in S$ and fix disjoint open sets $U_1$ and $V_1$ in such a way that $C \subseteq U_1$ and $x_1 \in V_1$. Suppose we have already chosen points $\{x_\beta: \beta < \alpha\}$ and pairs of disjoint open sets $\{U_\beta, V_\beta\}$, $\beta < \alpha$ such that $C \subseteq U_\beta$, $x_\beta \in V_\beta$ and for any $\gamma < \alpha$ we have $x_\gamma \in (S \setminus \{x_\beta: \beta < \gamma\}) \cap \bigcap \{U_\beta: \beta < \gamma\}$. Let $D_\alpha = \{x_\beta: \beta < \alpha\}$. If $C \cap \overline{D_\alpha} \neq \emptyset$ then we stop. Otherwise, for sure $D_\alpha$ cannot be cofinal in $S$ and therefore we must have $|\alpha| < |S|$. By the fact that $S$ converges to $C$, it follows that the set $(S \setminus \overline{D_\alpha}) \cap \bigcap \{U_\beta: \beta < \alpha\}$ contains a final segment of $S$ and so we may pick a point $x_\alpha$ in it. Then we continue by choosing $U_\alpha$ and $V_\alpha$ in the obvious way. It is clear that at some stage $\gamma \leq |S|$ we must have $C \cap \overline{D_\gamma} \neq \emptyset$. Since the set $D_\gamma$ is discrete, we are done. □
Corollary 1. Every Hausdorff radial space is discretely generated.

Corollary 2. Every Hausdorff pseudoradial space is weakly discretely generated.

What makes a Hausdorff sequential space discretely generated is that a countable sequence provides indeed a strongly discrete set. Obviously, this is not the case in general of a transfinite sequence and so there are no reasons to expect that a pseudoradial space is discretely generated. Indeed, we are able to construct a compact pseudoradial space which is not discretely generated. Since this construction has a common root with another one needed later, we will postpone it to the last section.

2. The case of a P-space

Notice first that we can easily find Hausdorff P-spaces which are not weakly discretely generated. For instance, take any Hausdorff dense-in-itself P-space $X$ with the Baire property and enlarge its topology by declaring closed each meager subset of $X$.

Trying to mimic what happens in the compact case, it seems reasonable to conjecture that every regular Lindelöf P-space is weakly discretely generated. At the moment, we have only some partial answers to this.

Recall that a space has countable extent if every uncountable subset has an accumulation point. It is evident that a P-space $X$ has countable extent if and only if every set $A \subseteq X$ with $|A| = \aleph_1$ has a complete accumulation point.

Theorem 1. Let $X$ be a regular P-space of countable extent. If the tightness of $X$ does not exceed $\aleph_1$, then $X$ is discretely generated.

Proof. Let $B \subseteq X$ and $x \in \overline{B} \setminus B$. Since the space $X$ has tightness $\aleph_1$, we may take a set $A \subseteq B \setminus \{x\}$ of cardinality $\aleph_1$ such that $x \in \overline{A}$. Then, fix a family $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of closed neighborhoods of $x$ in $X$ chosen in such a way that $A \cap \bigcap \mathcal{U} = \emptyset$. Since $X$ is a P-space, we may assume that $U_\alpha \subseteq U_\beta$ whenever $\beta \leq \alpha$. Let $S$ be the subset of $X$ consisting of all points which are in the closure of some set $\{z_\alpha : \alpha < \omega_1\}$ where $z_\alpha \in U_\alpha \cap A$ for each $\alpha$. Since $X$ is a regular space of countable extent and $x \in \overline{U_\alpha} \cap A$ for each $\alpha$, it easily follows that $x \in S$. As $X$ has tightness $\aleph_1$, we may select a set $\{s_\alpha : \alpha < \omega_1\} \subseteq S$ such that $x \in \overline{\{s_\alpha : \alpha < \omega_1\}}$. Let $\{z_\beta : \beta < \omega_1\}$ be a sequence witnessing that $s_\beta \in S$ and put $D = \{z_\delta : \delta \leq \gamma < \omega_1\}$. Then $D$ is a subset of $A$ which contains each $s_\beta$ in its closure and so $x \in \overline{D}$. Furthermore, since $D \cap \bigcap \mathcal{U} = \emptyset$ and $D \setminus U_\alpha$ is countable for every $\alpha$, it follows that $D$ is discrete. ☐

Theorem 2. [CH] Every regular Lindelöf P-space of character not exceeding $\aleph_\omega$ is weakly discretely generated.

Proof. Let $X$ be a regular Lindelöf P-space with $\chi(X) \leq \aleph_\omega$ and let $A$ be a non-closed subset of $X$. Let $\kappa$ be the smallest cardinal such that there exists a nonempty $G_\delta$-set $C \subseteq \overline{A} \setminus A$. Since a regular P-space is zero-dimensional, the set $C$ can be taken as an
intersection of clopen sets. The fact that \( X \) is a P-space implies \( \omega_1 \leq \kappa \) and the fact that \( \chi(X) \leq \aleph_0 \) implies \( \kappa \leq \aleph_0 \) (just take as \( C \) any singleton in \( \overline{A} \setminus A \)). However, it will be clear at the end of the proof that \( \kappa = \aleph_0 \) cannot occur.

**Case 1.** If \( \kappa = \aleph_0 \) then select a point \( x \in \overline{A} \setminus A \), put \( C = \{ x \} \) and fix a local base \( U \) at \( x \) of cardinality \( \aleph_0 \).

**Case 2.** If \( \kappa < \aleph_0 \) then let \( U \) be a family of clopen subsets of \( X \) satisfying \( |U| = \kappa \) and \( C = \bigcap U \subseteq \overline{A} \setminus A \). Because of CH, we have \( \aleph_n = \aleph_n^\omega \) for any integer \( n \geq 1 \) and so the family \( U \) can be assumed to be closed under countable intersections. Then, the fact that \( X \) is a Lindelöf space implies that \( U \) is actually a local base of \( C \) in \( X \).

In both cases, let \( U = \{ U_\alpha: \alpha < \kappa \} \). Of course, the minimality of \( \kappa \) implies that any \( G_\lambda \) set containing \( C \) intersects \( A \) for each \( \lambda < \kappa \) and consequently, for any \( \alpha < \kappa \) we may pick a point \( x_\alpha \in \bigcap \{ U_\beta: \beta \leq \alpha \} \cap A \). It is clear that the sequence \( S = \{ x_\alpha: \alpha < \kappa \} \) converges to the set \( C \). If \( \kappa = \aleph_0 \) then \( S \) should have a countable subsequence converging to \( C \). But, in a P-space this cannot happen and that is the reason why \( \kappa \neq \aleph_0 \).

Now, Lemma 1 shows that there exists a discrete set \( D \subseteq S \) such that \( C \cap \overline{D} \neq \emptyset \). This gives \( \overline{D} \setminus A \neq \emptyset \) and we are done. \( \square \)

Some more partial results can be obtained using the principle \( P_1 \).

**P1.** asserts that if \( F \) is a family of fewer than \( 2^{\aleph_1} \) subsets of \( \omega_1 \) and if each countable intersection of members of \( F \) is uncountable, then there is an uncountable set \( D \subseteq \omega_1 \) such that \( D \setminus F \) is countable for each member \( F \) of \( F \).

It is shown in [14, Theorem 7.13, p. 286] that there is a model of ZFC satisfying \( P_1 + 2^{\aleph_1} > \aleph_2 \).

**Theorem 3 [P1].** Every Hausdorff P-space of tightness at most \( \aleph_1 \) and character less than \( 2^{\aleph_1} \) is discretely generated.

**Proof.** Let \( X \) be a Hausdorff P-space satisfying \( t(X) \leq \aleph_1 \) and \( \chi(X) < 2^{\aleph_1} \). Let \( A \subseteq X \) and \( x \in \overline{A} \setminus A \). Since \( t(X) \leq \aleph_1 \), we may assume that \(|A| = \aleph_1\). Fix a fundamental system of neighborhoods \( U \) at \( x \) such that \(|U| < 2^{\aleph_1} \). Since \( X \) is a P-space, we may apply the \( P_1 \) principle to the family \( \{ U \cap A: U \in U \} \) thus obtaining an uncountable set \( D \subseteq A \) such that \( D \setminus U \) is countable for every \( U \in U \). It is clear that \( D \) is discrete and \( \overline{D} \setminus D = \{ x \} \). \( \square \)

**Theorem 4 [P1 + \aleph_2 < 2^{\aleph_1}].** Let \( X \) be a regular P-space of countable extent. If the character of \( X \) does not exceed \( \aleph_2 \), then \( X \) is weakly discretely generated.

**Proof.** Let \( A \subseteq X \) be a non-closed set. If there exists \( B \subseteq A \) such that \(|B| = \aleph_1 \) and \( \overline{B} \setminus A \neq \emptyset \), then take \( x \in \overline{B} \setminus A \) and argue as in the proof of Theorem 3. If \( A \) is \( \aleph_1 \)-closed, then take any \( x \in A \setminus \overline{A} \) and let \( \{ U_\alpha: \alpha < \omega_2 \} \) be a fundamental system of neighborhoods at \( x \). For any \( \alpha < \omega_2 \) fix a surjection \( f: \omega_1 \to \alpha + 1 \). We may inductively choose, using the fact that we are in a regular P-space, a family \( \{ V_\beta: \beta < \omega_1 \} \) of clopen neighborhoods of \( x \) such that \( V_\beta \subseteq U_{f(\beta)} \cap \bigcap \{ V_\gamma: \gamma < \beta \} \). For any \( \beta < \omega_1 \) pick a point \( x_\beta \in V_\beta \cap A \setminus \{ x_\alpha: \gamma < \beta \} \) and let \( y_\beta \) be an accumulation point for the set
\{x_\beta^\alpha : \beta < \omega_1\}. The assumption that \(A\) is \(\aleph_1\)-closed guarantees that \(y_\alpha \in A\). Furthermore, \(y_\alpha \in U_\beta\) whenever \(\beta \leq \alpha\). The latter formula implies that the set \(D = \{y_\alpha : \alpha < \omega_2\}\) is a transfinite sequence converging to \(x\). Now, it is enough to apply Lemma 1.

The authors were not able to find a regular Lindelöf P-space which is not discretely generated.

Finally, we wish to point out that the same proofs of Theorems 3 and 4 above actually show that the involved spaces have the Whyburn and weak Whyburn property respectively (just notice that the set \(D\) appearing at the end of both proofs satisfies \(D \setminus A = \{x\}\)). Thus, they also provide a partial answer to [15, Problems 3.6 and 3.5].

3. The case of a countably compact space

Recall that a space \(X\) has countable fan tightness if for any countable family \(\{A_n : n \in \omega\}\) of subsets of \(X\) satisfying \(x \in \bigcap_{n \in \omega} A_n\) it is possible to select finite sets \(K_n \subseteq A_n\) in such a way that \(x \in \bigcup_{n \in \omega} K_n\).

Clearly, countable fan tightness implies countable tightness.

**Theorem 5.** Every Hausdorff space of countable fan tightness is discretely generated.

**Proof.** Of course, it is enough to consider the case of a countable space \(X\). Let \(A \subseteq X\) and \(x \in \overline{A}\). As \(X\) is countable and Hausdorff, there is a countable decreasing family of closed neighborhoods \(\{V_n : n \in \omega\}\) of \(x\) such that the intersection of all of them is \(\{x\}\).

As \(X\) has countable fan tightness, there is a family \(\{F_n : n \in \omega\}\) of finite subsets such that \(F_n \subseteq (V_n \cap A) \setminus \{x\}\) and \(x \in \bigcup_{n \in \omega} F_n\). It is easy to realize that the fact that the set \(\bigcup_{k \in \omega} F_k\) is finite for every \(n\) implies that the set \(\bigcup_{k \in \omega} F_k\) is discrete in \(X\). \(\square\)

Let \(C_p(X)\) be the space of continuous real-valued functions on the Tychonoff space \(X\) with the topology of pointwise convergence. It is well-known [1] that \(C_p(X)\) has countable fan tightness if and only if \(X^n\) is a Hurewicz space for each \(n \in \omega\).

It is said that \(X\) is a Hurewicz space if for any sequence \(\{\gamma_n : n \in \omega\}\) of open covers of \(X\), for each \(n \in \omega\) there exists a finite subfamily \(\mu_n \subseteq \gamma_n\) such that \(\bigcup_{n \in \omega} \mu_n\) is a cover of \(X\). In particular, every \(\sigma\)-compact space is a Hurewicz space.

The previous result and Theorem 5 allow us to derive two nice consequences.

**Corollary 3.** If \(X\) is a \(\sigma\)-compact space then \(C_p(X)\) is discretely generated.

Notice that the above corollary can fail even for separable metrizable spaces \(X\). To see this, let \(M\) be a countable regular maximal space and let \(X = C_p(M)\). Then \(X\) is a space with a countable base, but \(C_p(X)\) is not weakly discretely generated since \(M\) embeds as a closed subspace into \(C_p(X)\).

**Corollary 4.** Every normed linear space in the weak topology is discretely generated.
Proof. Let $X$ be a normed linear space and $U^*$ be the unit ball in the dual space $X^*$. The well-known theorem of Alaoglu says that $U^*$ is compact in the weak* topology on $X^*$. The result follows by observing that the space $X$ in the weak topology is just a subspace of $C_p(U^*)$. □

The major consequence of Theorem 5 comes out in combination with the next assertion, which is [2, Theorem 1].

Proposition 1. Every countably compact regular space of countable tightness has actually countable fan tightness.

Thus, we immediately get:

Theorem 6. Every countably compact regular space of countable tightness is discretely generated.

In the above theorem the regularity of the space is essential. In fact we may easily construct a countably compact Hausdorff space of countable tightness which is not discretely generated.

To do it, consider the Čech–Stone compactification $\beta M$ of a maximal countable regular space $M$. Then enlarge the topology of $\beta M$ according to [16, Theorem 1.1]. We obtain a locally countable Urysohn space $X$ having $M$ as a subspace, while the cardinality of any closed infinite subset of $X$ is still $2^\omega$. Clearly, this $X$ is a countably compact Urysohn space of countable tightness that is not discretely generated.

Much more effort is needed to show that Theorem 6 can fail for a pseudocompact Tychonoff space of countable tightness. We will deal with it in the next section.

The major open problem posed in [5] asks whether it is true that the space $2^{\omega_1}$ is not discretely generated. The authors of [5] have actually shown that this is the case by assuming either the validity of the continuum hypothesis or the existence of a L-space.

Here, we wish to add the following observation:

Proposition 2. If there exists a countable regular dense-in-itself irresolvable space $X$ of weight $\aleph_1$, then $2^{\omega_1}$ is not discretely generated.

Proof. As a consequence of [3, Theorem 1.2], if $2^{\omega_1}$ were discretely generated then any countable dense-in-itself subspace of $2^{\omega_1}$ should be resolvable. Since any zero-dimensional space of weight $\aleph_1$ embeds into $2^{\omega_1}$, the result follows. □

4. Two relevant examples

We have already observed that it is not easy to find a decent space, which is not discretely generated. This perhaps explains that our examples require longer proofs and so we decided to put them in the end of the text.
Our aim is to prove the following two theorems. Both of them need the continuum hypothesis and we do not know if they are true in ZFC.

**Theorem 7.** Assume CH. Then there is a pseudoradial compact zero-dimensional space which is not discretely generated.

**Theorem 8.** Assume CH. Then there is a pseudocompact Tychonoff space of countable tightness which is not weakly discretely generated.

Before we give proofs of both theorems, let us fix some notation for the rest of the section. Let $M$ be the product of a countable discrete space with a Cantor set, $M = \omega \times 2^\omega$. We will denote by $C_n$ the subspace $\{n\} \times 2^\omega$ of $M$.

Our plan is to adjoin a point $p$ to the space $M$ in such a way that the trace of its neighborhood system will be a special filter $F$ on $M$. The theorems will be proved then by finding two spaces $X$ and $Y$, both containing $M \cup \{p\}$. The space $X$ will be compact pseudoradial and $M \cup \{p\}$ will be a dense subspace of $X$. The space $Y$ will be pseudocompact of countable tightness and will contain $M \cup \{p\}$ as a closed nowhere dense subset.

Our first task is to construct the promised filter $F$ on $M$. It will be done in the lemma below. Some more notation, however, is needed.

A set $U \subseteq M$ is called admissible, if it is clopen and $\{n \in \omega: U \cap C_n \neq \emptyset\}$ is infinite; an admissible set $U$ is called nice, if $\{n \in \omega: U \cap C_n = \emptyset\}$ is finite. A filter $F$ on $M$ is called nice, if it has a basis consisting of nice sets. Two sets $U, V \subseteq M$ are called essentially disjoint, if $\{n \in \omega: U \cap V \cap C_n \neq \emptyset\}$ is finite, and we shall say that $U$ is essentially contained in $V$, denoting it by $U \subseteq^e V$, if $\{n \in \omega: (U \setminus V) \cap C_n \neq \emptyset\}$ is finite. Finally, two sets $U, V \subseteq M$ are called compatible, in symbols, $U \parallel V$, if the set $\{n \in \omega: (U \cap V) \cap C_n \neq \emptyset\}$ is infinite.

**Lemma 2.** Assume CH. Then there is a family $\{F_\alpha: \alpha < \omega_1\}$ consisting of clopen subsets of $M$ which satisfies the following:

(i) for every $\alpha < \omega_1$, the set $F_\alpha$ is nice;
(ii) for $\beta < \alpha < \omega_1$, $F_\alpha \subseteq^e F_\beta$;
(iii) for $\beta < \alpha < \omega_1$, the set $F_\beta \setminus F_\alpha$ is nice;
(iv) if $U$ is an open subset of $M$ such that for all $\alpha < \omega_1$, $F_\alpha$ and $U$ are compatible, then the set $\{\alpha < \omega_1: (\forall \beta < \alpha) U \parallel (F_\beta \setminus F_\alpha)\}$ is cocountable;
(v) whenever $D \subseteq M$ is nowhere dense, then there is some $\alpha < \omega_1$ such that $D \cap F_\alpha = \emptyset$. (So the filter $F$ generated by all sets $F_\alpha$, $\alpha < \omega_1$ and $M \setminus C_n$, $n \in \omega$, is a nice remote free filter on $M$.)

**Proof.** We shall proceed by transfinite recursion to $\omega_1$. Since we assume CH, we may enumerate everything we need in the following way:

Let $\{D_\alpha: 0 < \alpha < \omega_1\}$ be the list of all closed nowhere dense subsets of $M$, let $\{U_\alpha: 0 < \alpha < \omega_1\}$ be the list of all open subsets of $M$.

Let $F_0 = M$. 

The induction step. Suppose $\alpha < \omega_1$ and suppose that the sets $F_\beta$ have been found for all $\beta < \alpha$. We shall tacitly assume that each finite subset of $\{F_\beta: \beta < \alpha\}$ has a nice intersection.

Since $\alpha$ is a countable ordinal, we can write $\{H_k : k < \omega\} = \{F_\beta : \beta < \alpha\}$. Next, denote by $U_\alpha$ the set $\{U_\beta : \beta < \alpha\}$ and for each $j < \omega$, $U_\beta \cap \bigcap_{k < j} H_k$ is admissible. The set $U_\alpha$ is a countable union of clopen subsets (v).

The items (i), (ii) and (iii) have already been verified. If $F_\alpha$ is a closed nowhere dense subset $\beta < \alpha$ and (v). The set $H_0$ is nice, hence there is some $n(0) < \omega$ such that for all $n \geq n(0)$, $H_0 \cap C_n \neq \emptyset$.

Suppose that for some $k > 0$ we know an integer $n(k)$ and suppose that for every $n \geq n(k)$, $H_0 \cap H_1 \cap \cdots \cap H_k \cap C_n \neq \emptyset$.

For every $i \leq k$, the set $\{n \in \omega: C_n \cap U_i \cap \bigcap_{j \leq k} H_j\}$ is infinite. So there are integers $n(k) < m(k, 0) < m(k, 1) < \cdots < m(k, k)$ such that for each $i \leq k$, $C_{m(k, i)} \cap U_i \cap \bigcap_{j \leq k} H_j \neq \emptyset$. Also, the set $\bigcap_{j \leq k} H_j$ is nice. So there is some integer $n(k+1)$ such that $m(k, k) < n(k+1)$ and for all $n \geq n(k+1)$, the set $C_n \cap \bigcap_{j \leq k} H_j$ is nonempty.

Notice that for each $n \geq n(0)$, if $n$ is such that $n(k) \leq n < n(k+1)$, we have a nonempty clopen subset $K_n$ of $C_n$, namely $K_n = C_n \cap \bigcap_{j \leq k} H_j$, such that the clopen set $\bigcup_{n \geq n(0)} K_n$ is essentially contained in every $H_k$ and for each $k \in \omega$, it is compatible with $U_k$.

For $n \in \omega$, $n \geq n(0)$, let $L_n = K_n$ if $n \neq m(k, i)$; if $n = m(k, i)$ for some $k$ and $i$, let $L_n$ be a nonempty clopen subset of $C_n \cap U_i$. The set $D_n$ is a closed nowhere dense subset of $M$, so for each $n \geq n(0)$, $D_n \cap L_n$ is nowhere dense in $L_n$. Hence there are two disjoint nonvoid clopen subsets $M_n$, $O_n$ of $L_n$ such that $D_n \cap M_n = \emptyset$.

It remains to put $F_\alpha = \bigcup_{n \geq n(0)} M_n$.

The set $F_\alpha$ is apparently nice. If $\beta < \alpha$, then $F_\beta = H_k$ in our enumeration and $F_\alpha \setminus H_k \subseteq \bigcup_{n \geq n(k)} C_n$, which gives (ii), moreover, the set $\bigcup_{n \geq n(0)} O_n$ is nice and essentially contained in $H_k \setminus F_\alpha$, which gives (iii).

After completing the induction, the family $\{F_\alpha : \alpha < \omega_1\}$ will satisfy (i), (ii), (iii), (iv) and (v). The items (i), (ii) and (iii) have already been verified. If $D \subseteq M$ is closed nowhere dense, then $D = D_n$ for some $\alpha < \omega_1$ and from the $\alpha$th induction step we know that $F_\alpha \cap D_n = \emptyset$, hence (v) follows.

To see (iv), let $U$ be an open set such that for all $\alpha \in \omega_1$, $U \parallel F_\alpha$. There is some occurrence of the set $U$ in our list: $U = U_\alpha$ for some $\alpha < \omega_1$. We have $U \in U_\alpha$ and we have ensured that $U \parallel F_{\alpha+1}$, which immediately implies that $U \parallel F_y$ for all $y \geq \alpha + 1$.

However, the sets $O_n$ selected in the $(\alpha + 1)$st induction step and in all following steps witness to the validity of (iv).

Let $F$ be the filter on $M$ generated by $\{F_\alpha : \alpha < \omega_1\} \cup \{M \setminus C_n : n \in \omega\}$. Choose a point $p \notin M$ and define its neighborhood base in $M \cup \{p\}$ as $\{\{p\} \cup F : F \in F\}$. Since the filter $F$ has a clopen basis, the space $M \cup \{p\}$ is Tychonoff.

**Remark.** N. Fine and L. Gillman proved under the assumption of CH that $\beta \mathbb{R} \setminus \mathbb{R}$ contains remote points [12]. J.Y. Zhou and A. Dow proved under MA that a countable space with a unique non-isolated point densely embeds into a compact pseudoradial space [17,6]. Our proof presented here may be considered as a fusion and adaptation of the proofs just mentioned. The reader undoubtedly knows that remote points exist in ZFC [7–9]. However, the structure of van Douwen’s remote filters does not seem to be good for the additional demands (ii) and (iv).
Proof of Theorem 7. The underlying set of the space $X$ will be a disjoint union of $M \cup \{p\}$ and $\omega_1 \setminus \{0\}$. The topology is defined as follows:

- The space $M$ is an open subspace of $X$.
- If $0 < \alpha < \omega_1$, then the neighborhood basis at the point $\alpha$ is the family
  \[
  \left\{ (\beta, \alpha] \cup (F_\beta \setminus F_\alpha) \setminus \bigcup_{n<k} C_n : \beta < \alpha, \ k \in \omega \right\}.
  \]
- The neighborhood basis at $p$ is the family
  \[
  \left\{ (\beta, \omega_1) \cup F_\beta \setminus \bigcup_{n<k} C_n : \beta < \omega_1, \ k \in \omega \right\}.
  \]

We leave simple proofs that the space $X$ is indeed a topological space and that it is Hausdorff and zero-dimensional to the reader.

The space $X$ is compact: Choose an arbitrary infinite set $T \subseteq X$. We need to show that $T$ has a complete accumulation point. This is clear if $T \subseteq \{p\} \cup \omega_1 \setminus \{0\}$, since $\{p\} \cup \omega_1 \setminus \{0\}$ is homeomorphic to $\omega_1 + 1$. Suppose $T \subseteq M$. Then there are no problems if $T$ has a complete accumulation point in $M$. If no point from $M$ as well as the point $\{p\}$ is a complete accumulation point of $T$, then some $F_\alpha$ satisfies $|T \cap F_\alpha| < |T|$. Let $\alpha_0$ be the smallest ordinal with this property. Having $|F_{\alpha_0} \cap T| < |T|$, but $|F_\beta \cap T| \geq |T|$ for all $\beta < \alpha_0$, we see that $\alpha_0$ is the complete accumulation point of $T$. So $X$ is compact.

(An exercise for an interested reader: An alternative proof may exploit the fact that the space $X$ is the Stone space of an algebra $\mathcal{B}$, where $\mathcal{B}$ is a subalgebra of $\text{Clopen}(M)$ generated by all compact clopen subsets of $M$ and by $\{F_\alpha : \alpha < \omega_1\}$.)

The space $X$ is pseudoradial: Indeed, suppose $A \subseteq X$ is not closed. If there is $x \in X$, $x \neq p$, such that $x \in \overline{A} \setminus A$, then there is a convergent sequence ranging in $A$ with limit $x$, because $X$ is first countable in all points except $p$.

So we may suppose that $\overline{A} \setminus A = \{p\}$. Decompose $A$ into $A \cap \omega_1$, $A \cap \text{Int}_{bd_M} \overline{A}$ and $A \cap \text{bd}_M \overline{A}$. If $A \cap \omega_1$ is unbounded in $\omega_1$, then $A \cap \omega_1$ is an $\omega_1$-sequence converging to $p$. By (v), the point $p$ does not belong to the closure (taken in $M \cup \{p\}$) of the boundary $\text{bd}_M \overline{A}$, so it remains to consider the possibility $p \in A \cap \text{Int}_{bd_M} \overline{A}$. But then, by (iv) and by the definition of topology on $X$, there is some $\alpha < \omega_1$ such that $\beta \in A \cap \text{Int}_M \overline{A}$ whenever $\beta > \alpha$. Since we are assuming that $p$ is the only point in $\overline{A} \setminus A$, all those $\beta$'s belong to $A$ and therefore, as before, we see that $A \cap \omega_1$ converges to $p$.

The space $X$ is not discretely generated: The point $p$ is a cluster point of the set $M$ and there is no discrete subset of $M$ containing $p$ in its closure, because of (v).

Proof of Theorem 8. Consider the set $M \times (\omega + 1)$. Our plan is to add to this set a point $p$ in such a way that the nowhere dense set $(M \times \omega_1) \cup \{p\}$ will be homeomorphic to the space $M \cup \{p\}$ and a set of further points, which will ensure the pseudocompactness of the result.
We again need some additional notation. Let us denote by $\pi_M$ the projection of $\mathbb{M} \times \omega$ onto $\mathbb{M}$, and by $\pi_{\omega \times \omega}$ the projection from $\mathbb{M} \times \omega$ onto $\omega \times \omega$. That means, for $n, k \in \omega$, $t \in 2^\omega$, $\pi_M(t, n, k) = (n, t)$ and $\pi_{\omega \times \omega}(n, t, k) = (n, k)$.

Suppose $f, g$ are mappings, $A \subseteq [\omega]^\omega$, $f : \omega \to \omega$ and $g : A \to \omega$. We will write $f \leq^* g$ if the set $\{ n \in A : f(n) > g(n) \}$ is finite.

Fix for the rest of this proof the remote filter base $\{ F_\alpha : \alpha < \omega_1 \}$ from Lemma 2 and a scale $\{ f_\beta : \beta < \omega_1 \}$ in $\omega_1 [9]$.

Let $\alpha, \beta$ be two countable ordinals and let $G \subseteq \mathbb{M} \times \omega$ be an open set. Let us say that the set $G$ is parametrized by $\langle \alpha, \beta \rangle$, (and call the pair $\langle \alpha, \beta \rangle$ a parameter for $G$), if the following holds:

1. $\pi_M[G]$ is a clopen noncompact subset of $\mathbb{M}$;
2. $\pi_M[G] \cap F_\alpha = \emptyset$;
3. for each $\gamma < \alpha$, $\pi_M[G] \setminus F_\gamma$ is compact;
4. there is an infinite set $A \subseteq \omega$ such that $\pi_{\omega \times \omega}[G]$ is a mapping from $A$ to $\omega$;
5. $\pi_{\omega \times \omega}[G](n) < f_\beta(n)$, for each $n \in A$;
6. for all $\gamma < \beta$, $f_\gamma \leq^* \pi_{\omega \times \omega}[G]$.

Call a clopen set $G \subseteq \mathbb{M} \times \omega$ parametrized, if there is some pair of ordinals $\langle \alpha, \beta \rangle$ such that $G$ is parametrized by $\langle \alpha, \beta \rangle$.

Claim. Suppose $\{ U_i : i \in \omega \}$ is a discrete family in the topological product $\mathbb{M} \times (\omega + 1)$ such that each set $U_i$ is a clopen compact subset of $\mathbb{M} \times \omega$. Then there are ordinals $\alpha, \beta < \omega_1$ and a clopen noncompact set $G$ parametrized by $\langle \alpha, \beta \rangle$ such that $G \subseteq \bigcup \{ U_i : i \in \omega \}$.

Proof. Each set $U_i$ is compact, so it meets only finitely many clopen sets of the form $C_n \times \{ k \}$ with $n, k \in \omega$. Passing from $U_i$ to a nonempty intersection $U_i \cap (C_n \times \{ k \})$ we may and shall from now on assume that for each $i \in \omega$ there is one pair $\langle n, k \rangle$ of integers with $U_i \subseteq C_n \times \{ k \}$.

The family $\{ U_i : i \in \omega \}$ is discrete and each set $C_n \times \{ k \}$ is compact. So each set $C_n \times \{ k \}$ contains at most finitely many $U_i$’s. Passing to an infinite subset, if necessary, we may assume that for each $n, k \in \omega$ there is at most one $i \in \omega$ with $U_i \subseteq C_n \times \{ k \}$.

Given $n \in \omega$, the set of all such $k \in \omega$ that the set $C_n \times \{ k \}$ contains some $U_i$ is finite. Otherwise there would be a point in $C_n \times \{ \omega \}$ showing that the family $\{ U_i : i \in \omega \}$ is not discrete. Passing to an infinite subset once more, we shall assume that for every $n \in \omega$ there is at most one $k$ such that $C_n \times \{ k \}$ contains some $U_i$. If there already such a $k$ exists, let $f(n)$ equal to this $k$ and rename the unique $U_i \subseteq C_n \times \{ f(n) \}$ as $U_n$.

We have just defined a (possibly partial) function $f$ and it should be clear that the set $A = \text{dom } f$ is infinite.

For each $n \in A$, put $V_n = \pi_M[U_n]$. Since every set $U_n$ is clopen and compact, also every set $V_n$ is. In particular, no set $V_n$ has an isolated point. For every $n \in A$ pick a point $t_n \in V_n$. Then the set $T = \{ t_n : n \in A \} \subseteq \mathbb{M}$ is closed nowhere dense. By Lemma 2(v), there is some $\delta < \omega_1$ such that $T \cap F_\delta = \emptyset$. The set $F_\delta$ is clopen and each point $t_n$ is not isolated in a clopen $V_n$. So for every $n \in A$, the set $V_n \setminus F_\delta$ is a clopen and nonempty subset of $C_n$. 


A topology is defined by declaring the neighborhood systems of points: $\langle n, t, k \rangle$ such that $g(n) < f(\beta)(n)$: the set $\{ n \mid g(n) < f(\beta)(n) \}$ is infinite. Analogously to the previous, the nonempty set $\{ n \in B : g(n) < f(\beta)(n) \}$ must have a minimal element $\beta$.

Put $D = \{ n \in B : g(n) < f(\beta)(n) \}$. It remains to define $G = \bigcup_{n \in D} \pi^{-1}_M |W_n| \cap U_n$. It should be clear now that the set $G$ is parametrized by $(\alpha, \beta)$ and satisfies the statement of the claim. □

Let $Q$ be a maximal collection satisfying the following:

1. each set $Q \in Q$ is a parametrized subset of $\mathbb{M} \times \omega$;
2. whenever $Q, Q'$ are distinct members of $Q$, then $Q \cap Q'$ is compact.

The underlying set of the space $Y$ is the disjoint union $\mathbb{M} \times (\omega + 1) \cup Q \cup \{ p \}$. The topology is defined by declaring the neighborhood systems of points:

- The space $\mathbb{M} \times (\omega + 1)$ is an open subspace of $Y$.
- A neighborhood basis of a point $Q \in Q$ consists of all sets $\{ Q \} \cup Q \setminus D$, where $D$ is a compact clopen subset of $\mathbb{M} \times (\omega + 1)$.

Every set $Q \in Q$ is parametrized, which means that there is a pair of ordinals $(\alpha, \beta)$ such that $Q$ is parametrized by $(\alpha, \beta)$. Let us adopt the convention that for $Q \in Q$ we shall denote the corresponding ordinal pair as $(\alpha(Q), \beta(Q))$.

For $\alpha < \omega_1$, let $R_\alpha$ be the set

$$R_\alpha = \{ (n, t, k) : n \in \omega, t \in 2^\omega, k \in \omega + 1, (n, t) \notin F_\alpha, k \geq f_\alpha(n) \}$$

$$\cup \{ Q \in Q : \alpha(Q) \leq \alpha < \beta(Q) \}.$$

- A neighborhood subbasis at $p$ consists of all sets $Y \setminus R_\alpha$ for $\alpha < \omega_1$, of all sets $Y \setminus D$, where $D$ is a clopen compact subset of $\mathbb{M} \times (\omega + 1)$ and of all sets $Y \setminus (\{ Q \} \cup \{ Q \})$, where $Q \in Q$.

To see that we have already defined a topology, we must check that each set $R_\alpha$ is closed. If $Q \in Q$ does not belong to $R_\alpha$, then either $\alpha(Q) > \alpha$ or $\beta(Q) \leq \alpha$. Since the set $Q$ is parametrized, either (3) or (5) from the definition of a parametrized set implies that the set $Q \cap R_\alpha$ is compact. So the point $Q$ does not belong to the closure of $R_\alpha$. Add to this observation an obvious fact that $R_\alpha \cap (\mathbb{M} \times (\omega + 1))$ is closed in $\mathbb{M} \times (\omega + 1)$.

The set $R_\alpha \cap (\mathbb{M} \times (\omega + 1))$ is also open in $\mathbb{M} \times (\omega + 1)$. If $Q \in R_\alpha$, then $\alpha(Q) \leq \alpha$ and $\beta(Q) > \alpha$. Using (2) and (6) from the definition of a parametrized set, we see that the set $Q \setminus R_\alpha$ is compact. So the point $Q$ has a neighborhood contained in $R_\alpha$ and therefore $R_\alpha$ is open.
The space $Y$ is obviously Hausdorff. We have just checked the only nontrivial part necessary to verify that it is also zero-dimensional. So $Y$ is Tychonoff.

**The space $Y$ is pseudocompact:** Suppose not and consider some infinite discrete collection $\{U_i; i \in \omega\}$ consisting of open subsets of $Y$. Since the subset $\{p\} \cup Q \cup (\mathbb{M} \times \{\omega\})$ is nowhere dense in $Y$, we may (after some shrinking, if necessary) assume that all members of the collection are open subsets of $\mathbb{M} \times \omega$. According to Claim, there is a parametrized set $G$ contained in $\bigcup\{U_i; i \in \omega\}$. By the maximality of $Q$, there is some $Q \in \mathcal{Q}$ such that $Q \cap G$ is noncompact. Since $G \subseteq \bigcup\{U_i; i \in \omega\}$, the point $Q$ witnesses that the family $\{U_i; i \in \omega\}$ is not discrete in $Y$, a contradiction.

**The space $Y$ is countably tight:** Since the space $Y$ is first countable in all points other that $p$ and since its subspace $\mathbb{M} \times (\omega + 1)$ is hereditarily separable, we only need to check that if $p \in \overline{T}$, for some $T \subseteq Q$, then there is a countable $S \subseteq T$ such that $p \in S$.

Taking into account the shape of a neighborhood of $p$ in the subspace $\{p\} \cup Q$ and the fact that $Y$ is a $T_1$-space, we see that for any $A \in [\omega_1]^{<\omega}$ we may pick a countable infinite set $T_A \subseteq T$ such that $T_A \cap \bigcup\{R_\alpha; \alpha \in A\} = \emptyset$. Put $a_0 = 0$ and $T_0 = T[0]$. Proceeding by induction, let us assume to have already defined $a_n < \omega_1$ and a countable infinite set $T_n \subseteq T$. Then let $a_{n+1} = \sup\{\beta(Q); Q \in T_n\}$ and $T_{n+1} = \bigcup\{T_A; A \in [\omega_n]^{<\omega}\}$. We have constructed a non-decreasing chain $\{T_n; n < \omega\}$ of countable subsets of $T$ and a non-decreasing sequence of countable ordinals $\{a_n; n < \omega\}$. Put $S = \bigcup\{T_n; n < \omega\}$ and $\gamma = \sup\{a_n; n < \omega\}$. We claim that $p \in \overline{S}$. For this, it suffices to verify that for any $A \in [\omega_1]^{<\omega}$, the set $S \setminus \bigcup\{R_\alpha; \alpha \in A\}$ is infinite. Fix an integer $n$ such that $A \cap \gamma \subseteq a_n$. The definition of the sets $R_\alpha$ immediately implies that $S \cap R_\alpha = \emptyset$ for any $\gamma \leqslant \alpha < \omega_1$. Since we have $T_{A \cap \gamma} \subseteq T_{n+1} \subseteq S$, it follows that $T_{A \cap \gamma} \subseteq S \setminus \bigcup\{R_\alpha; \alpha \in A\}$. As $T_{A \cap \gamma}$ is infinite, we are done.

**The space $Y$ is not weakly discretely generated:** The definition of the topology of $Y$ easily implies that the closed subset $(\mathbb{M} \times \{\omega\}) \cup \{p\}$ of the space $Y$ is homeomorphic to $\mathbb{M} \cup \{p\}$. However, if we consider the space $\mathbb{M} \cup \{p\}$, we already know that $p \in \overline{\mathbb{M}}$, but $p$ does not belong to the closure of any discrete subset of $\mathbb{M}$, because the filter $\mathcal{F}$ is remote. □

**Remark.** The idea to use parametrization by ordinal pairs in order to get countable tightness was first suggested by O. Pavlov in the attempt to find a pseudocompact Tychonoff countably tight space without countable fan tightness. This and other results can be found in [4].

References


