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Abstract
We prove the generalized Hyers–Ulam–Rassias stability of generalized $A$-quadratic mappings of type (P) in Banach modules over a Banach $\ast$-algebra, and of generalized $A$-quadratic mappings of type (R) in Banach modules over a Banach $\ast$-algebra.

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1. Introduction

In 1940, Ulam [15] raised the following question: Under what conditions there exists an additive mapping near an approximately additive mapping?

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|$ and $\| \|$, respectively. Hyers [4] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\| f(x + y) - f(x) - f(y) \| \leq \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\| f(x) - T(x) \| \leq \epsilon$$

for all $x \in X$.

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Consider \( f : X \to Y \) to be a mapping such that \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \). Assume that there exist constants \( \epsilon \geq 0 \) and \( p \in [0, 1) \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon (\|x\|^p + \|y\|^p)
\]
for all \( x, y \in X \). Rassias [7] showed that there exists a unique \( \mathbb{R} \)-linear mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \frac{2\epsilon}{2 - 2p} \| x \|^p
\]
for all \( x \in X \).

A square norm on an inner product space satisfies the important parallelogram equality
\[
\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2.
\]
The functional equation
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [14] for mappings \( f : E_1 \to E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group. In [3], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [8–13].

Throughout this paper, assume that \( A \) is a Banach \(*\)-algebra, and that \( X \) and \( Y \) are left Banach \( A \)-modules with norms \( \| \cdot \| \) and \( \| \cdot \| \), respectively.

**Definition 1.1** [16,17]. A mapping \( f : X \to Y \) is called an \( A \)-quadratic mapping if \( f : X \to Y \) satisfies
\[
Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y), \quad (1.i)
\]
\[
Q(ax) = aQ(x)a^* \quad (1.ii)
\]
for all \( a \in A \) and all \( x, y \in X \).

In this paper, we prove the generalized Hyers–Ulam–Rassias stability of generalized \( A \)-quadratic mappings of type (P), defined in [5,6], in Banach modules over a Banach \(*\)-algebra, and of generalized \( A \)-quadratic mappings of type (R), defined in [5,6], in Banach modules over a Banach \(*\)-algebra.

### 2. Generalized quadratic mappings in Banach modules

We are going to prove the generalized Hyers–Ulam–Rassias stability of generalized \( A \)-quadratic mappings of type (P) in Banach modules over a Banach \(*\)-algebra.
Definition 2.1 [1, Definition 10.1.1]. A mapping $Q : X \rightarrow Y$ is called a generalized $A$-quadratic mapping of type (P) if

$$Q(ax) = aQ(x)a^*,$$

$$Q\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} Q\left(\sum_{i=1}^{n} x_i - nx_j\right) = n\left(\sum_{i=1}^{n} Q(x_i) + \sum_{i<j, j=3}^{n} \sum_{i=2}^{n-1} Q(x_i - x_j)\right)$$

for all $a \in A$ and all $x, x_1, \ldots, x_n \in X$.

Remark 1. When $n = 2$, it reduces to an $A$-quadratic mapping.

Theorem 2.2. Let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^n \rightarrow [0, \infty)$ such that

$$\tilde{\varphi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} \frac{1}{n^{2j}} \varphi(n^j x_1, \ldots, n^j x_n) < \infty,$$

$$(2.i)$$

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - nx_j\right) - n\left(\sum_{i=1}^{n} f(x_i) + \sum_{i<j, j=3}^{n} \sum_{i=2}^{n-1} f(x_i - x_j)\right)\right\| \leq \varphi(x_1, \ldots, x_n),$$

$$(2.ii)$$

$$\left\| f(ax) - a f(x) a^* \right\| \leq \varphi(x_1, \ldots, x_n),$$

$$(2.iii)$$

for all $a \in A$ and all $x, x_1, \ldots, x_n \in X$. Then there exists a unique generalized $A$-quadratic mapping $Q : X \rightarrow Y$ of type (P) such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{n^2} \tilde{\varphi}(x_1, \ldots, x),$$

$$(2.iv)$$

for all $x \in X$.

Proof. Put $x_1 = x_2 = \cdots = x_n = x$ in (2.ii). Then

$$\left\| f(nx) - n^2 f(x) \right\| \leq \varphi(x_1, \ldots, x),$$

for all $x \in X$. So

$$\left\| f(x) - \frac{1}{n^2} f(nx) \right\| \leq \frac{1}{n^2} \varphi(x_1, \ldots, x),$$

for all $x \in X$. Thus

$$\left\| \frac{1}{n^2j} f(n^j x) - \frac{1}{n^{2j+2}} f(n^{j+1} x) \right\| \leq \frac{1}{n^{2j+2}} \varphi(n^j x_1, \ldots, n^j x)$$

for all $x \in X$. Thus
for all \(x \in X\). For given integers \(l, m (0 \leq l < m)\),

\[
\left\| \frac{1}{n^{2l}} f(n^l x) - \frac{1}{n^{2m}} f(n^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{n^{2j+2}} \psi(n^j x, \ldots, n^l x) \tag{2.1}
\]

for all \(x \in X\). By (2.i), the sequence \(\{1/n^l f(n^l x)\}\) is a Cauchy sequence for all \(x \in X\).

By (2.i), the sequence \(\{1/n^l f(n^l x)\}\) is a Cauchy sequence for all \(x \in X\).

Since \(Y\) is complete, the sequence \(\{1/n^l f(n^l x)\}\) converges for all \(x \in X\).

Define \(Q : X \to Y\) by

\[
Q(x) = \lim_{j \to \infty} \frac{1}{n^{2j}} f(n^j x) \tag{2.2}
\]

for all \(x \in X\). Letting \(l = 0\) and taking \(m \to \infty\) in (2.1), one can obtain the inequality (2.iv).

It follows from (2.ii), (2.iii) and (2.2) that the mapping \(Q : X \to Y\) satisfies the properties of Definition 2.1. Hence the mapping \(Q : X \to Y\) is a generalized \(A\)-quadratic mapping of type (P), as desired.

**Corollary 2.3.** Let \(p (0 < p < 2)\) and \(\theta\) be positive real numbers. Let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) for which there exists a function \(\psi : X^n \to [0, \infty)\) such that

\[
\left\| f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=2}^{n} f\left(\sum_{i=1}^{n} x_i - n x_j\right) - n \left(\sum_{i=1}^{n} f(x_i) + \sum_{i<j, j=3}^{n} \sum_{i=2}^{n-1} f(x_i - x_j)\right)\right\| \leq \theta \sum_{j=1}^{n} ||x_j||^p,
\]

\[
\left\| f(ax) - af(x)a^*\right\| \leq n\theta ||x||^p
\]

for all \(a \in A\) and all \(x, x_1, \ldots, x_n \in X\). Then there exists a unique generalized \(A\)-quadratic mapping \(Q : X \to Y\) of type (P) such that

\[
\left\| f(x) - Q(x)\right\| \leq \frac{n\theta}{n^2 - n^p} ||x||^p
\]

for all \(x \in X\).

**Proof.** Define \(\varphi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \theta ||x_j||\), and apply Theorem 2.2. \(\square\)

Now we are going to prove the generalized Hyers–Ulam–Rassias stability of generalized \(A\)-quadratic mappings of type (R) in Banach modules over a Banach \(*\)-algebra.

**Definition 2.4** [1, Definition 10.2.1]. A mapping \(Q : X \to Y\) is called a generalized \(A\)-quadratic mapping of type (R) if \(Q(ax) = aQ(x)a^*\) for all \(a \in A\) and all \(x \in X\), and one of the following two identities holds:

\[
Q\left(\sum_{i=1}^{n} d_i x_i\right) + \sum_{1 \leq i < j \leq n} d_i d_j Q(x_i - x_j) = \left(\sum_{i=1}^{n} d_i\right) \left(\sum_{i=1}^{n} d_i Q(x_i)\right) \tag{2.v}
\]
for all $x_1, \ldots, x_n \in X$, some fixed $d_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and at least two of them are nonzero and such that $\sum_{i=1}^{n} d_i \neq 0$, $n \geq 2$:

$$Q \left( \sum_{i=1}^{n} d_i x_i \right) + \sum_{i=1}^{n} \left( d_i \left( \sum_{j=1}^{n} d_j \right) - 2d_j \right) Q(x_i)$$

$$= \sum_{1 \leq i < j \leq n} d_i d_j Q(x_i + x_j) \quad (2.\text{vi})$$

for all $x_1, \ldots, x_n \in X$, some fixed $d_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and at least three of them are nonzero and $n \geq 3$.

**Remark 2.** When $d_1 = d_2 = 1$ and $d_i = 0$, $i = 3, \ldots, n$, in (2.v), the mapping $Q : X \to Y$ reduces to an $A$-quadratic mapping.

When $x_3 = -x_2$, $d_1 = d_2 = d_3 = 1$ and $d_i = 0$, $i = 4, \ldots, n$, in (2.vi), the mapping $Q : X \to Y$ reduces to an $A$-quadratic mapping.

**Theorem 2.5.** Let $d := \sum_{i=1}^{n} d_i$ ($d_1, \ldots, d_n \in \mathbb{R}$, $d \neq 0$). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^n \to [0, \infty)$ such that

$$\tilde{\varphi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} \frac{1}{|d|^j} \varphi(d^j x_1, \ldots, d^j x_n) < \infty,$$

$$\left\| f \left( \sum_{i=1}^{n} d_i x_i \right) + \sum_{1 \leq i < j \leq n} d_i d_j f(x_i - x_j) - \left( \sum_{i=1}^{n} d_i \right) \left( \sum_{i=1}^{n} d_i f(x_i) \right) \right\| \leq \varphi(x_1, \ldots, x_n), \quad (2.\text{vii})$$

$$\left\| f(ax) - af(x)a^* \right\| \leq \varphi(x, \ldots, x) \quad n \text{ times} \quad (2.\text{viii})$$

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{|d|^2} \varphi(x, \ldots, x) \quad \text{n times} \quad (2.\text{x})$$

for all $a \in A$ and all $x, x_1, \ldots, x_n \in X$. Then there exists a unique generalized $A$-quadratic mapping $Q : X \to Y$ of type $(R)$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{|d|^2} \varphi(x, \ldots, x) \quad (2.\text{x})$$

for all $x \in X$.

**Proof.** Put $x_1 = x_2 = \cdots = x_n = x$ in (2.viii). Then

$$\left\| f(dx) - d^2 f(x) \right\| \leq \varphi(x, \ldots, x) \quad n \text{ times}$$

for all $x \in X$. So

$$\left\| f(x) - \frac{1}{d^2} f(dx) \right\| \leq \frac{1}{|d|^2} \varphi(x, \ldots, x) \quad \text{n times}$$
for all \( x \in X \). Thus
\[
\left\| \frac{1}{d^{2j}} f(d^j x) - \frac{1}{d^{2j+2}} f(d^{j+1} x) \right\| \leq \frac{1}{|d|^{2j+2}} \varphi(d^j x, \ldots, d^j x) \text{n times}
\]
for all \( x \in X \). For given integers \( l, m \) \((0 \leq l < m)\),
\[
\left\| \frac{1}{d^{2l}} f(d^l x) - \frac{1}{d^{2m}} f(d^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{|d|^{2j+2}} \varphi(d^j x, \ldots, d^j x) \text{n times}
\]
(2.3)
for all \( x \in X \). By (2.vii), the sequence \( \{1/|d|^{2j} f(d^j x)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{1/|d|^{2j} f(d^j x)\} \) converges for all \( x \in X \). Define \( Q : X \to Y \) by
\[
Q(x) = \lim_{j \to \infty} \frac{1}{d^{2j}} f(d^j x) \quad (2.4)
\]
for all \( x \in X \). Letting \( l = 0 \) and taking \( m \to \infty \) in (2.3), one can obtain the inequality (2.x). It follows from (2.viii), (2.ix) and (2.4) that the mapping \( Q : X \to Y \) satisfies
\[
Q(ax) = aQ(x)a^* \quad \text{for all } a \in A \text{ and all } x \in X \text{ and (2.v) in Definition 2.4. Hence the mapping } Q : X \to Y \text{ is a generalized } A \text{-quadratic mapping of type (R), as desired.}
\]

**Corollary 2.6.** Let \( p \) \((0 < p < 2)\) and \( \theta \) be positive real numbers. Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there exists a function \( \varphi : X^n \to [0, \infty) \) such that
\[
\left\| f\left( \sum_{i=1}^{n} d_i x_i \right) + \sum_{1 \leq i < j \leq n} d_i d_j f(x_i - x_j) - \left( \sum_{i=1}^{n} d_i \right) \left( \sum_{i=1}^{n} f(x_i) \right) \right\| \leq \theta \sum_{j=1}^{n} ||x_j||^p,
\]
\[
\| f(ax) - a f(x)a^* \| \leq n\theta ||x||^p
\]
for all \( a \in A \) and all \( x, x_1, \ldots, x_n \in X \). Then there exists a unique generalized \( A \)-quadratic mapping \( Q : X \to Y \) of type (R) such that
\[
\| f(x) - Q(x) \| \leq \frac{n\theta}{|d|^2 - |d|^p} ||x||^p
\]
for all \( x \in X \).

**Proof.** Define \( \varphi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \theta ||x_j|| \), and apply Theorem 2.5. \( \square \)

Next, we consider \( d_i = 1/n \) in (2.vi).

**Theorem 2.7.** Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there exists a function \( \varphi : X^n \to [0, \infty) \) such that
\[ \tilde{\psi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \psi(2^j x_1, \ldots, 2^j x_n) < \infty, \quad (2.xi) \]

\[ \left\| f \left( \sum_{i=1}^{n} \frac{1}{n} x_i \right) + \sum_{i=1}^{n} \frac{1}{n} \left( 1 - \frac{2}{n} \right) f(x_i) - \sum_{1 \leq i < j \leq n} \frac{1}{n^2} f(x_i + x_j) \right\| \leq \psi(x_1, \ldots, x_n), \quad (2.xii) \]

\[ \left\| f(ax) - af(x)a^* \right\| \leq \psi(x_1, \ldots, x_n) \quad (2.xiii) \]

for all \( a \in A \) and all \( x, x_1, \ldots, x_n \in X \). Then there exists a unique generalized \( A \)-quadratic mapping \( Q : X \to Y \) of type (R) such that

\[ \left\| f(x) - Q(x) \right\| \leq \frac{n}{2(n-1)} \psi(x_1, \ldots, x_n) \quad (2.xiv) \]

for all \( x \in X \).

**Proof.** Put \( x_1 = x_2 = \cdots = x_n = x \) in (2.xii). Then

\[ \left\| \left( 1 - \frac{2}{n} \right) f(x) - \frac{n-1}{2n} f(2x) \right\| \leq \psi(x_1, \ldots, x_n) \]

for all \( x \in X \). So

\[ \left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{n}{2(n-1)} \psi(x_1, \ldots, x_n) \]

for all \( x \in X \). Thus

\[ \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \leq \frac{n}{2(2j+1)(n-1)} \psi(2^j x_1, \ldots, 2^j x_n) \]

for all \( x \in X \). For given integers \( l, m \) \((0 \leq l < m)\),

\[ \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{m-j}} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{n}{2^{j+1}(n-1)} \psi(2^j x_1, \ldots, 2^j x_n) \quad (2.5) \]

for all \( x \in X \). By (2.xi), the sequence \( \{(1/2^j)f(2^j x)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{(1/2^j)f(2^j x)\} \) converges for all \( x \in X \). Define \( Q : X \to Y \) by

\[ Q(x) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x) \quad (2.6) \]

for all \( x \in X \). Letting \( l = 0 \) and taking \( m \to \infty \) in (2.5), one can obtain the inequality (2.xiv). It follows from (2.xii), (2.xiii) and (2.6) that the mapping \( Q : X \to Y \) satisfies \( Q(ax) = aQ(x)a^* \) for all \( a \in A \) and all \( x \in X \) and (2.vi) in Definition 2.4 when \( d_i = 1/n \) for \( i = 1, 2, \ldots, n \). Hence the mapping \( Q : X \to Y \) is a generalized \( A \)-quadratic mapping of type (R), as desired. \( \square \)
Corollary 2.8. Let $p$ ($0 < p < 2$) and $\theta$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^n \to [0, \infty)$ such that
\[
\left\| f\left( \sum_{i=1}^{n} \frac{1}{n} x_i \right) + \sum_{i=1}^{n} \frac{1}{n} \left( 1 - \frac{2}{n} \right) f(x_i) - \sum_{1 \leq i < j \leq n} \frac{1}{n^2} f(x_i + x_j) \right\| \leq \theta \sum_{j=1}^{n} ||x_j||^p,
\]
\[
\left\| f(ax) - af(x)a^* \right\| \leq n\theta ||x||^p
\]
for all $a \in A$ and all $x, x_1, \ldots, x_n \in X$. Then there exists a unique generalized $A$-quadratic mapping $Q : X \to Y$ of type $(R)$ such that
\[
\left\| f(x) - Q(x) \right\| \leq \frac{2n^2\theta}{(4 - 2^p)(n - 1)} ||x||^p
\]
for all $x \in X$.

Proof. Define $\varphi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \theta ||x_j||$, and apply Theorem 2.7. \qed

Finally, we consider $d_i = 2/n$ in (2.vi).

Theorem 2.9. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^n \to [0, \infty)$ such that
\[
\tilde{\varphi}(x_1, \ldots, x_n) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_n) < \infty,
\]
(2.xv)
\[
\left\| f\left( \sum_{i=1}^{n} \frac{2}{n} x_i \right) + \frac{2}{n} \sum_{i=1}^{n} \left( 2 - \frac{4}{n} \right) f(x_i) - \sum_{1 \leq i < j \leq n} \frac{4}{n^2} f(x_i + x_j) \right\| \leq \varphi(x_1, \ldots, x_n),
\]
(2.xvi)
\[
\left\| f(ax) - af(x)a^* \right\| \leq \varphi(x, \ldots, x)
\]
(2.xvii)
for all $a \in A$ and all $x, x_1, \ldots, x_n \in X$. Then there exists a unique generalized $A$-quadratic mapping $Q : X \to Y$ of type $(R)$ such that
\[
\left\| f(x) - Q(x) \right\| \leq \frac{n}{4(n - 2)} \tilde{\varphi}(x, \ldots, x)
\]
(2.xviii)
for all $x \in X$.

Proof. Put $x_1 = x_2 = \cdots = x_n = x$ in (2.xvi). Then
\[
\left\| 4\left( 1 - \frac{2}{n} \right) f(x) - \frac{n-2}{n} f(2x) \right\| \leq \varphi(x, \ldots, x)
\]
for all $x \in X$. So
\[
\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{n}{4(n - 2)} \varphi(x, \ldots, x)
\]
for all \( x \in X \). Thus
\[
\left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+2}} f(2^{j+1} x) \right\| \leq \frac{n}{2^{2j+2}(n-2)} \psi(2^j x, \ldots, 2^j x)
\]
for all \( x \in X \). For given integers \( l, m \) \((0 \leq l < m)\),
\[
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{n}{2^{2j+2}(n-2)} \psi(2^j x, \ldots, 2^j x)
\]
(2.7)
for all \( x \in X \). By (2.xv), the sequence \( \{1/2^j f(2^j x)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{1/2^j f(2^j x)\} \) converges for all \( x \in X \). Define
\[
Q : X \rightarrow Y
\]
by
\[
Q(x) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x)
\]
(2.8)
for all \( x \in X \). Letting \( l = 0 \) and taking \( m \to \infty \) in (2.7), one can obtain the inequality (2.xviii). It follows from (2.xvi), (2.xvii) and (2.8) that the mapping \( Q : X \to Y \) satisfies
\[
Q(ax) = aQ(x) a^* \quad \text{for all } a \in A \text{ and all } x \in X \text{ and (2.vi) in Definition 2.4 when } d_i = 2/n \text{ for } i = 1, 2, \ldots, n.
\]
Hence the mapping \( Q : X \to Y \) is a generalized \( A \)-quadratic mapping of type (R), as desired. \( \Box \)

**Corollary 2.10.** Let \( 0 < p < 2 \) and \( \theta \) be positive real numbers. Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there exists a function \( \psi : X^n \to [0, \infty) \) such that
\[
\left\| f \left( \sum_{i=1}^{n} \frac{1}{n} x_i \right) + \sum_{i=1}^{n} \frac{1}{n} \left( 1 - \frac{2}{n} \right) f(x_i) - \sum_{1 \leq i < j \leq n} \frac{1}{n^2} f(x_i + x_j) \right\| \leq \theta \sum_{j=1}^{n} \|x_j\|^p,
\]
\[
\left\| f(ax) - af(x) a^* \right\| \leq n \theta \|x\|^p
\]
for all \( a \in A \) and all \( x, x_1, \ldots, x_n \in X \). Then there exists a unique generalized \( A \)-quadratic mapping \( Q : X \to Y \) of type (R) such that
\[
\left\| f(x) - Q(x) \right\| \leq \frac{n^2 \theta}{(4 - 2^p)(n-2)} |x|^p
\]
for all \( x \in X \).

**Proof.** Define \( \psi(x_1, \ldots, x_n) = \sum_{j=1}^{n} \theta \|x_j\| \) and apply Theorem 2.9. \( \Box \)

**References**

