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Graphs, Matroids, and Geometric Lattices

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ABSTRACT

It is shown that two triply connected graphs are isomorphic if their associated geometric lattices are isomorphic. The notion of vertex in a graph is described in terms of irreducible hyperplanes. Finally, necessary and sufficient conditions are given that a lattice be isomorphic to the geometric lattice associated with a graph.

Hassler Whitney [6] has shown that two triply connected graphs are isomorphic if their circuit structures are isomorphic. We shall show the same result by looking at the geometric lattices associated with the graphs, and we will also characterize these lattices.

In this paper a graph G shall always be a finite, non-oriented graph with no isolated vertices, loops, or multiple edges unless the contrary is stated. Associated with this graph G is a geometric lattice L, which has been called a bond-lattice by Rota [2], and a circuit structure M called a polygon-matroid by Tutte [4]. The geometric lattice L can be described perhaps most easily in terms of the incidence matrix of the edges and vertices, columns corresponding to edges. The columns can be thought of as 1-dimensional subspaces of the column space mod 2 which correspond to points of the associated projective space. The set of all joins of these points forms a geometric lattice ordered by inclusion. (See [1] for the theory of geometric lattices and Lemma 3, p. 84, for the construction of the lattice of joins.) A minimal dependent set of atoms in the lattice determines a circuit in the graph and conversely. Another way of describing the lattice L is in terms of a closure operator, a set S of edges being closed if the adjunction of any single edge will not create any new circuits. The lattice L is the lattice of closed sets. As was shown by Whitney [7], matroids and therefore geometric lattices can be characterized in terms of circuits as well as independent sets of atoms. In this paper a *hyperplane* shall be an element in the lattice covered by the maximum element I.

DEFINITION 1. *G is said to be "non-separable" if, for any two complementary non-empty subsets S and T of the edges of G ($E(G)$), there exists a circuit of G whose edge set meets both S and T. This is equivalent to the statement that the matroid associated with G is connected (see* [4]) *and that the geometric lattice is irreducible.*

DEFINITION 2. *G is said to be a "triply connected" graph if G is non-separable, and if the subgraph obtained by deleting any vertex v and all of the edges through v (the "star" through v) is also non-separable.*

DEFINITION 3. *In a geometric lattice L, a hyperplane h is said to be "irreducible" if and only if* $[0, h]$ *is an irreducible lattice.*

THEOREM 1. *Let G be a triply connected graph with L its associated geometric lattice. If v is any vertex of G, then the set of edges complementary to the star through v (the "star-complement" of v) join to make an irreducible hyperplane in L. Conversely, every irreducible hyperplane is associated with a vertex v in this manner.*

Proof. It is obvious that the star-complement of v is a closed set. If we adjoin an edge e through v, then any other edge through v lies in some circuit containing e since L is irreducible[1] (merely because G is non-separable), and this circuit must be composed solely of e and edges of the star-complement of v in addition to the other edge. Hence the star-complement of v joins to a hyperplane in L. This hyperplane is irreducible since G is assumed to be triply connected.

Let us consider the converse. Let h be an irreducible hyperplane in L. There must therefore exist an edge e with vertices v_1 and v_2 which does not lie within h. Suppose that there exists an edge f within h which contains v_2 and an edge k within h which contains v_1. Let the other vertices of f and k be v_3 and v_4, respectively. It is impossible that $v_3 = v_4$ since that would imply that e was within h. Since h is irreducible, there exists a circuit C in G which contains k and f. Now either C is of the form $(v_1, v_4),\ldots, (v_i, v_3), (v_3, v_2), (v_2, v_j),\ldots, (v_m, v_1)$ or of the form $(v_1, v_4),\ldots, (v_i, v_2), (v_2, v_3), (v_3, v_j),\ldots, (v_m, v_1)$ where all of the edges lie within h.

[1] This follows readily from the McLaughlin-Sasaki-Fujiwara theorem, which states that any two atoms in an irreducible geometric lattice have a common complement. Note that the comment in [1, p. 94] about pseudo-perspectivity and perspectivity is not correct.

In the first case we can form the circuit $(v_1, v_4),..., (v_i, v_3)$, (v_3, v_2), (v_2, v_1), and in the second case we can form the circuit $(v_1, v_4),..., (v_i, v_2)$, (v_2, v_1). In both cases this implies that $(v_2, v_1) = e$ is within h which is a contradiction. Thus one of the vertices, say v_1, must have the property that none of the edges through it lies within h. Since h is a hyperplane, the set of edges it contains cannot be a proper subset of the closed set determined by the star-complement of v_1. Thus h is determined by the star-complement of v_1.

REMARK. The proof of the converse did not require that G be triply connected. Thus an irreducible hyperplane always determines a vertex, and a vertex in a non-separable graph always determines a hyperplane.

COROLLARY 1. *In a triply connected graph G, an edge e contains a vertex v if and only if the atom associated with e in L is not within the irreducible hyperplane associated with the star-complement of v.*

COROLLARY 2. *In the geometric lattice L associated with a triply connected graph G, an atom lies in all but exactly two irreducible hyperplanes. The set of irreducible hyperplanes within which it lies is non-empty.*

COROLLARY 3. *In a lattice L associated with a triply connected graph G, there is at most one atom which is a common complement to two irreducible hyperplanes.*

THEOREM 2. *Let G_1 and G_2 be two triply connected graphs associated with geometric lattices L_1 and L_2 which are isomorphic. Then G_1 and G_2 are isomorphic. More strongly, if α is the isomorphism between L_1 and L_2, then an isomorphism β exists between G_1 and G_2 which agrees with α when applied to the edges (atoms).*

Proof. Define β so that β agrees with α when applied to the edges. If v_1 is a vertex in G_1, its star-complement determines an irreducible hyperplane h_1 in L_1 which is mapped by α onto h_2 in L_2. But then h_2 is associated with the star-complement of a vertex v_2 in G_2. Define $\beta(v_1) = v_2$. The mapping β is obviously a 1-1 mapping from the vertices of G_1 onto the vertices of G_2. If e_1 in G_1 joins v_1 and v_1', then e_1 is a common complement of the irreducible hyperplanes associated with v_1 and v_1' and therefore $\beta(e_1)$ connects $\beta(v_1)$ and $\beta(v_1')$. If v_1 and v_1' are not connected by an edge, then their associated irreducible hyperplanes do not have a common complement which is an atom, and therefore $\beta(v_1)$ and $\beta(v_1')$ are not connected. Thus β is an isomorphism between G_1 and G_2.

THEOREM 3. *If G_1 and G_2 are graphs with G_1 triply connected, and if there is an isomorphism between their associated geometric lattices L_1 and L_2, then G_2 is also triply connected and hence isomorphic to G_1.*

Proof. Since L_2 is irreducible, G_2 must certainly be non-separable. Suppose that v_2 is a vertex in G_2. Let e_2 be any edge through v_2, say $e_2 = (v_2, v_2')$. Now e_2 corresponds to e_1 in L_1, and e_1 is a complement of exactly two irreducible hyperplanes h_1 and h_1' in L_1. Therefore e_2 is a complement of exactly two irreducible hyperplanes h_2 and h_2' in L_2. By the remark after Theorem 1, h_2 and h_2' determine two vertices in G_2 each of which must be incident with e_2 because e_2 is a complement of h_2 and h_2'. Thus these vertices are v_2 and v_2'. Hence the graph obtained by deleting v_2 and the edges through it is non-separable since the hyperplane h_2 or h_2' corresponding to v_2 is irreducible. Since this is true for any v_2 in G_2, G_2 is triply connected.

We shall now be interested in lattices satisfying the following two axioms:

AXIOM 1. $(L, +, \cdot)$ is a finite geometric lattice with a set $\mathscr{H}$ of hyperplanes such that every atom in L has exactly two complements in $\mathscr{H}$, and no two atoms have the same two complements.[2]

AXIOM 2. If $\mathscr{F}$ is a proper subset of $\mathscr{H}$, then

$$D(\Pi\mathscr{F}) \leqslant |\mathscr{H} - \mathscr{F}| - 1.$$

Note that $D(x)$ is the dimension of x in L, where we assume that $D(0) = 0$.

Given now a lattice L satisfying Axioms 1 and 2, we associate with L a graph G in the following manner:

(a) The vertices of G are the hyperplanes in $\mathscr{H}$.

(b) The edges of G are the atoms in L.

(c) An edge passes through a vertex if and only if the corresponding atom and hyperplane are complementary.

[2] In [5] Tutte uses the concept of a planar mesh, whose definition resembles that of the system $\mathscr{H}$ in Axiom 1, in order to describe the faces of a planar graph and thus get to the vertices of the dual graph. His peripheral polygons play a role similar to that of irreducible hyperplanes in the Corollaries and Remark after Theorem 5 (see [5, Theorems 2.6–2.8]). However, planar meshes are defined only for graphs and exist only when the graph is planar. I would like to thank the Referee for pointing out the connection between planar meshes and Axiom 1.

Observe that, because of Axiom 1, each edge passes through exactly two vertices, and no two edges pass through the same pair of vertices. Also each vertex is contained in some edge because every hyperplane has an atom for a complement. Thus G is a finite graph with no loops, multiple edges, or isolated vertices. Let L' be the lattice of edges associated with G. We shall show that L and L' are isomorphic, but first we need to prove several lemmas.

LEMMA 1. *Any set of edges in G which forms a forest determines in a* 1-1 *manner an independent set of atoms in L.*

Proof. Obviously a one edge set determines an independent set in L. Now a forest F in G has the property that through one of its vertices there passes only one edge. If we delete this vertex and edge, we obtain a new forest after we delete any possible isolated vertices. If our forest F contains $n+1$ edges p_i and several vertices h_i, we can thus select $n+1$ edges and n vertices and name them in such a manner that p_k does not touch h_m if $k > m$ and does touch it if $k = m$. This simply means that $p_k \leqslant h_m$ if $k > m$ and that p_k and h_k are complements. By induction we can assume that $p_1, \dots, p_n$ is an independent set of atoms in L. Now if $p_{n+1} \leqslant p_1 + \cdots + p_n$, then

$$\begin{aligned}
p_{n+1} &\leqslant (p_n + \cdots + p_2 + p_1)\, h_1 h_2 \cdots h_n \\
&\leqslant [(p_n + \cdots + p_2) + p_1 h_1]\, h_2 \cdots h_n \\
&\leqslant (p_n + \cdots + p_2)\, h_2 \cdots h_n \\
&\leqslant \cdots \\
&\leqslant 0.
\end{aligned}$$

This is impossible. Hence $p_1, \dots, p_{n+1}$ is an independent set of atoms in L.

LEMMA 2. *Any set of edges in G which forms a circuit determines in L a circuit (minimal dependent set) of atoms.*

Proof. Let the circuit consist of the edges $p_1, \dots, p_n$ with vertices $h_1, \dots, h_n$. We shall show that the atom p_1 lies in the span of $p_2, \dots, p_n$. Let $\mathscr{F} = \mathscr{H} - \cup h_i$. Therefore by Axiom 2, $D(\Pi\mathscr{F}) \leqslant n - 1$. But since $\Pi\mathscr{F} \geqslant p_j$ for every j, $D(p_1 + p_2 + \cdots + p_n) \leqslant n - 1$. By Lemma 1, $p_2, \dots, p_n$ is an independent set of atoms. Therefore $D(p_2 + \cdots + p_n) = n - 1$. This implies that $p_1 \leqslant p_2 + \cdots + p_n$. Hence p_1 lies in the span of $p_2, \dots, p_n$. Thus $p_1, \dots, p_n$ is a dependent set of atoms, and by Lemma 1 it is clear that any subset of it is independent. Hence $p_1, \dots, p_n$ is a minimal dependent set of atoms (a circuit of atoms).

REMARK. By applying Axiom 2 when $\mathscr{F}$ is a one element set, we see that $n + 1 \leqslant t$ where n is the dimension of I and t is the number of elements in $\mathscr{H}$.

THEOREM 4. *Any lattice L satisfying Axioms 1 and 2 is isomorphic to a geometric lattice associated with a graph G.*

Proof. We shall show that L and L' are isomorphic by showing that circuits correspond to circuits. By Lemma 2 any circuit in G has as its correspondent a circuit of atoms in L. Let C be a circuit of atoms in L and let C' be its image in L'. If C' forms an independent set of atoms in L', then, by Lemma 1, C forms an independent set of atoms in L, which is false. Therefore C' is a dependent set which must therefore contain a circuit of atoms C''. By Lemma 2, the preimage C^* of C'' must be a circuit of atoms in L, and it must also be a subset of C. By the minimality of C, $C^* = C$; whence $C'' = C'$. Thus the matroids corresponding to L and L' are isomorphic. Hence so are L and L'.

THEOREM 5. *The lattice of a non-separable graph G of more than one edge satisfies Axioms 1 and 2.*

Proof. We define $\mathscr{H}$ to be the set of hyperplanes generated by the star-complements of the vertices in G. Thus Axiom 1 is satisfied. Axiom 2 is also satisfied because intersecting star-complements is the same as deleting all of the edges through a given vertex. The number of components (counting isolated vertices) increases by at least the number of stars deleted.

COROLLARY 1. *A lattice L is isomorphic to the lattice associated with a non-separable graph G of more than one edge if and only if L satisfies Axioms 1 and 2 and is irreducible.*

COROLLARY 2. *A lattice L is isomorphic to the lattice associated with a triply connected graph G if and only if L satisfies Axiom 1 and 2, is irreducible, and the set $\mathscr{H}$ consists of precisely the irreducible hyperplanes.*

REMARK. It is to be noted that, since the elements in $\mathscr{H}$ correspond to the vertices of the graph G, $\mathscr{H}$ must always contain all of the irreducible hyperplanes and consist of nothing else if G is triply connected.

It is readily seen that the direct union of two lattices which satisfy Axioms 1 and 2 also satisfies these axioms. Now every geometric lattice is the direct union of irreducible lattices, and in graphs irreducible lattices correspond to non-separable graphs. We thus have the following result.

THEOREM 6. *A lattice L is isomorphic to the associated geometric lattice of a finite graph G if and only if it is a finite Boolean algebra, a lattice which satisfies Axioms* 1 *and* 2, *or a direct union of a finite Boolean algebra with a lattice which satisfies Axioms* 1 *and* 2.

REMARK. The reader should contrast this theorem with the theorem of Tutte [4], which states that a lattice L is isomorphic to the associated geometric lattice of a finite graph G (*cographic* in Tutte's terminology) if and only if it is a binary geometric lattice that excludes certain minors.

The alert reader may have observed that only Axiom 1 was required to construct the graph G from the lattice L. We shall now give an example to show that Axiom 2 cannot, in fact, be deduced from Axiom 1. Let G be the complete graph on four vertices. This graph determines the lattice of partitions on a four element set. We can alter the lattice by requiring that every three edge circuit and every three edge circuit plus an extra edge be an independent set. We still obtain a geometric lattice of closed sets (a set being closed if it contains all edges dependent on it) because every subset of an independent set of edges is independent and any maximal independent set of edges now contains four edges. Our system $\mathscr{H}$ is still the set of three edge circuits. But now $n + 1 \nleqslant t$ where n is the dimension of I and t is the number of elements in $\mathscr{H}$. Thus, by the remark following Lemma 2, Axiom 2 is violated.

It is to be noted that, if we assume Axiom 3—that every pair of hyperplanes in $\mathscr{H}$ has a common complement, then the graph G will be complete, and therefore L will be isomorphic to the lattice of partitions on a finite set. It was implicitly shown in [3] that a lattice L satisfying certain other properties actually satisfied Axioms 1, 2, and 3. That L was a partition lattice was deduced, however, in another way.

Our results can be applied to infinite graphs by a slight modification of Axioms 1 and 2. L would now be a geometric lattice of possibly infinite length and $\mathscr{H} - \mathscr{F}$ would be finite. We leave the details to the interested reader.

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