Non-planar diagrams and non-commutative superspace in Dijkgraaf–Vafa theory

Takeshi Morita

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Received 13 August 2005; received in revised form 29 September 2005; accepted 10 October 2005

Available online 24 October 2005

Editor: T. Yanagida

Abstract

We consider the field theory on non-commutative superspace and non-commutative spacetime that arises on D-branes in Type II superstring theory with a constant self-dual graviphoton and NS–NS B field background. \( \mathcal{N} = 1 \) supersymmetric field theories on this non-commutative space (such theories are called \( \mathcal{N} = 1/2 \) supersymmetric theories) can be reduced to supermatrix models as in hep-th/0303210. We take an appropriate commutative limit in these theories and show that holomorphic quantities in commutative field theories are equivalent to reduced models, including non-planar diagrams to which the graviphoton contributes. This is a new derivation of Dijkgraaf–Vafa theory including non-planar diagrams.

\( \odot 2005 \) Elsevier B.V.

1. Introduction

It is generally interesting and difficult to study the \( 1/\hat{N}^2 \) correction in large-\( \hat{N} \) reduced models [2]. We consider the \( 1/\hat{N}^2 \) correction in Dijkgraaf–Vafa theory [3–6], in which low energy effective theory of \( D = 4, \mathcal{N} = 1 \) supersymmetric gauge theory is equivalent to associated matrix model, in order to illuminate this problem in this Letter.

The proof of Dijkgraaf–Vafa theory in \( \mathcal{N} = 1 U(N) \) gauge theory coupled to one adjoint matter is given in [6]. It was shown there that the Schwinger–Dyson equations (the Konishi anomaly equations) of the field theory are equivalent to those of the associated matrix model for all holomorphic quantities. As a result, the field theory is equivalent to the associated matrix model as far as holomorphic quantities are concerned. The origin of this equivalence is shown in [1,7]. This is a new large-\( \hat{N} \) reduction in non-commutative superspace. \(^{1}\) As in [8], field theories on non-commutative space \( \{x^\mu, \theta, \bar{\theta}\} \) can be mapped to matrix models. In this procedure, if the original

\( \odot 2005 \) Elsevier B.V. Open access under CC BY license.

---

\( \odot 2005 \) Elsevier B.V. Open access under CC BY license.

achieved. Therefore, the $1/N^2$ expansion is meaningful only for the leading (planar) terms, which correspond to the commutative field theory. Thus, we cannot consider non-planar quantities in the argument of (1).

In this Letter, we will construct supermatrix models corresponding to field theories on non-commutative superspace \cite{1,7,9,10}

\[
\{ \theta^\alpha, \phi^\beta \} = \gamma^\alpha{}^\beta, \quad \{ \bar{\theta}^\dot{\alpha}, \bar{\phi}^\dot{\beta} \} = 0, \\
[\gamma^\mu, \gamma^\nu] = -i C^{\mu\nu}, \tag{1.2}
\]

where $\gamma^\mu = x^\mu + i \theta^\alpha \sigma^\alpha {}^\mu \bar{\theta}$. The construction of field theories on this non-commutative superspace has been achieved \cite{11} and these theories are called $\mathcal{N} = 1/2$ supersymmetric theories. Although this non-commutativity breaks the unitarity of the theory, we consider this theory on a Euclidean space and ignore this problem. In Section 2, we will show that amplitudes of non-planar diagrams disappear in usual supersymmetric field theories. When we take the commutative limit $C^{\mu\nu} \to 0$, $\gamma^\alpha{}^\beta \to 0$, while holding the ratio $\det \gamma / \det C$ finite, the non-planar diagrams contribute to the commutative field theories. In Section 3, we will show that these higher genus quantities correspond to those of the supermatrix models. Therefore, we will understand the equivalence between the commutative field theory and the supermatrix model including non-planar diagrams. If we take the ratio $\det \gamma / \det C$ to 0, we obtain the usual commutative field theory to which the non-planar diagrams do not contribute.

On the other hand, the non-commutative superspace \cite{11–14} and non-commutative spacetime \cite{8,15,16} arises on D-branes in Type II superstring theory in constant self-dual graviphoton background \cite{17}. The non-commutative parameters are given by these background fields. Then the quantity $1/N^2$ in the reduced model is also expressed in terms of these background fields and the expansion with respect to $1/N^2$ can be regarded as a development with respect to these fields. Then, it is possible to take an appropriate commutative limit. Under this limit, the commutative field theory exhibits finite non-planar diagrams to which the graviphoton contributes. This result reproduces analyses in \cite{12,18}.

2. Appearance of the non-planar diagrams in Dijkgraaf–Vafa theory

In this section, we calculate planar and non-planar diagrams in $\mathcal{N} = 1 U(N)$ gauge theory coupled to one adjoint matter. We show that the amplitudes of the non-planar diagrams disappear in commutative space \cite{5} and do not disappear in non-commutative superspace \cite{9}. Especially, we will show that the amplitudes do not either disappear under the commutative limit $C^{\mu\nu} \to 0$, $\gamma^\alpha{}^\beta \to 0$ with a fixed finite ratio $\det \gamma / \det C$. We will interpret this result as the contributions of background graviphoton field strength and $B$ field.

2.1. Diagram calculation in commutative superspace

We will calculate planar and non-planar diagrams in $\mathcal{N} = 1$ $U(N)$ theory. The action is

\[
S = \int d^4 x d^2 \theta d^2 \bar{\theta} \mathrm{Tr} (e^{-V} \Phi \bar{\Phi}) + \int d^4 x d^2 \theta \left( \mathrm{Tr} W(\Phi) + 2 \pi i \tau \mathrm{Tr} W'' W_0 \right) + \text{c.c.}, \tag{2.1}
\]

where $V$ denotes the vector superfield including the $U(N)$ gauge field, $W_0$ denotes its field strength, $\Phi$ denotes a chiral superfield in the adjoint representation of $U(N)$ and $\tau$ denotes a gauge coupling constant. $W(\Phi)$ denotes a $(m+1)$th order polynomial superpotential,

\[
W(\Phi) = \sum_{k=0}^{m} \frac{g_k}{k+1} \Phi^{k+1}. \tag{2.2}
\]

We can consider this potential in general, however, it is enough to consider the simpler superpotential,

\[
W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3. \tag{2.3}
\]

in this section.

First, we calculate two loop diagrams for matter field in this theory. Since this theory is holomorphic, the matter kinetic term ($D$-term) and the superpotential ($F$-term) are decoupled. Therefore, we can evaluate these amplitudes considering only the superpotential. Then, the propagator of the superfield $\Phi$ is $1/m$ in terms of the holomorphic quantities. The three point vertex is $g$. Using these Feynman rules, we can calculate the two loop amplitude for matter field of Fig. 1(a) as follows:

\[
\left( \frac{1}{2} + \frac{1}{6} \right) \int \frac{d^4 k d^4 d^2 \kappa}{(2\pi)^4} e^{ik\vec{x}} \left( \frac{1}{m} \right)^3 \delta^2, \tag{2.4}
\]

Here $1/2$ and $1/6$ are symmetry factors, we omitted the traces and used usual and fermionic $\delta$ functions,

\[
\delta^4(x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\vec{x}}, \quad \delta^2(\theta) = \int 4 d^2 \kappa \ e^{-i\theta \kappa}. \tag{2.5}
\]

A $\delta^4(0)\delta^2(0)$ singularity appears in Eq. (2.4) and we need to regularize it as follows [11]:

\[
\left. \delta^4(y) \delta^2(\theta) \right|_{(y,\theta) \to (0,0)} = \frac{1}{64\pi^2} (W'' W_0)^i j. \tag{2.6}
\]
Here \(i\) and \(j\) are gauge indices. This equation is a consequence of the Konishi anomaly [19]. The gauge field contributes to the matter holomorphic terms only through this anomaly. When we consider this theory as a low-energy theory of superstrings, background graviphoton field strength and \(B\) field do not contribute to this anomaly [18].

From the chiral ring properties [6],

\[
\{W_\alpha, W_\beta\} = 0, \quad \{\Phi, W_\alpha\} = 0, \quad (2.7)
\]

amplitudes of the diagrams in which more than three \(W_\alpha\) are in a single trace is zero. Considering the combination of the three traces (there are three index loops) and four \(W_\alpha\), we obtain

\[
\frac{2g^2}{3m^3} \left( \frac{3}{64\pi^2} \text{Tr} W_\alpha^2 W_\beta W_\beta + 6 \frac{1}{64\pi^2} \text{Tr} W_\alpha^2 \frac{1}{8\pi} \text{Tr} W_\beta^2 \frac{1}{8\pi} \text{Tr} W_\beta \right). \quad (2.8)
\]

Here \(N\) is the rank of the gauge group and we simply assume that the gauge symmetry is not broken by the Higgs mechanism.

Next, we calculate a non-planar diagram (b). The process is almost the same. The difference is in the number of index loops. This non-planar diagram has only one index loop. Since we must insert four \(W_\alpha\) into one index loop, this amplitude is zero because of the chiral ring properties.

\[
\frac{1}{64\pi^2} \text{Tr} W_\alpha^2 W_\beta W_\beta \text{ is replaced to glueball superfield } S \text{ in (2.8) in the low energy theory. Then we can obtain the two loop parts of the low energy effective action. This result reproduces the value in [3].}^2 \text{ Therefore, Dijkgraaf–Vafa theory can be obtained from the calculation of the Feynman rules of the superfield } \Phi, \text{ Konishi anomaly (2.6) and the chiral ring properties (2.7).}
\]

### 2.2. Diagram calculation in non-commutative superspace

In this subsection, we calculate the two loop diagrams of Fig. 1(a) and (b) in non-commutative superspace and non-commutative spacetime described by:

\[
\{\theta^\alpha, \theta^\beta\} = \gamma^{\alpha\beta}, \quad \{\gamma^\mu, \gamma^\nu\} = -i C^{\mu\nu}, \quad \{\theta^\alpha, \theta^\beta\} = \{\gamma^\mu, \gamma^\nu\} = \{\gamma^\mu, \theta^\alpha\} = 0.
\]

Here \(\gamma^{\alpha\beta}\), \(C^{\mu\nu}\) are c-numbers.

The properties of the non-commutative superspace with \(C^{\mu\nu} = 0\) is studied in [11]. In the \(F\)-terms and the \(D\)-term of (2.1), we simply replace the standard products with star products [20] given by:

\[
f(\gamma) \star g(\gamma) = \exp \left( -\frac{i}{2} C^{\mu\nu} \frac{\delta}{\delta y^\mu} \frac{\delta}{\delta y^\nu} \right) f(\gamma) g(\gamma) \bigg|_{\gamma = \gamma'}., \quad (2.10)
\]

\[
f(\theta) \star g(\theta) = \exp \left( -\frac{i}{2} \gamma^{\alpha\beta} \frac{\delta}{\delta \theta^\alpha} \frac{\delta}{\delta \theta^\beta} \right) f(\theta) g(\theta) \bigg|_{\theta = \theta'}. \quad (2.11)
\]

Although we need to treat the anti-holomorphic terms separately, it is not too serious a problem, since our interest lies in the holomorphic terms.

\[\text{The holomorphy is broken on this non-commutative super-}\]

\[\text{space[11]. The spacetime non-commutativity } C^{\mu\nu} \text{ does not prevent the holomorphy [1], but } \gamma^{\alpha\beta}\text{ does. Therefore, when one take the commutative limit } \gamma^{\alpha\beta} \to 0, C^{\mu\nu} \to 0 \text{ with the finite ratio } \text{det } \gamma/\text{det } C, \text{ the holomorphy is recovered. Since we are interested in field theories under the commutative limit, it is meaningful to consider the matter holomorphic terms in the non-commutative superspace as in the previous subsection.}
\]

Let us consider the Feynman rules of this non-commutative theory. The propagator of \(\Phi\) is the same: \(1/m\). As in the usual non-commutative field theory, the three point vertex exhibits a non-commutative phase [9],

\[
ge^{-\frac{i}{2} C^{\mu\nu} k_\mu p_\nu - \frac{1}{2} \gamma^{\alpha\beta} k_\alpha p_\beta}, \quad (2.12)
\]

where \(k_\mu\) and \(p_\mu\) are momenta and \(k_\alpha\) and \(p_\beta\) are fermionic momenta. Since this non-commutative phase disappear in the planar diagrams, the amplitude of the diagrams (a) is the same under the commutative limit \(\gamma^{\alpha\beta} \to 0, C^{\mu\nu} \to 0\). Note that one may regard the square of the \(\delta\) function as det \(\gamma/\text{det } C\) in (2.4) as we will show latter (3.14) and one may derive another amplitude proportional to det \(\gamma/\text{det } C\). However, this calculation is non-physical, since this amplitude is 0 when one takes det \(\gamma/\text{det } C\) to 0, and this result is inconsistent with the calculation of (2.8).

Next, we consider the non-planar diagram (b). In contrast to the planar diagrams, the non-commutative phase is not cancelled and the amplitude is

\[
\frac{1}{6} \text{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{d^4\kappa}{(2\pi)^4} (\frac{1}{m})^3 \frac{g^2}{6m^3} \text{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{d^4\kappa}{(2\pi)^4} \delta^4(Ck) \delta^2(\gamma\kappa) \quad (2.13)
\]

We can take the commutative limit while holding the ratio det \(\gamma/\text{det } C\) finite and this amplitude does not vanish.

As a result, we obtain the amplitudes of the usual planar diagrams, as well as that of the non-planar diagram under the commutative limit of the non-commutative superspace. What is the meaning of the non-vanishing amplitudes of the non-planar diagrams? It is the remnant of the non-commutativity in the context of the field theory. However, when we consider this theory as a low energy theory of superstring, we can regard these amplitudes as the contributions of the background graviphoton field strength and NS–NS \(B\) field.

The non-commutative superspace which we have considered arises on D-branes in Type II superstring theory in constant self-dual graviphoton field strength \(F^{\alpha\beta}\) and constant NS–NS \(B\) field background [11,12,14,16] through, for example, calculation of hybrid formalism as in [21], where

\[
\{\theta^\alpha, \theta^\beta\} = 2\alpha' F^{\alpha\beta}, \quad (2.14)
\]

\[
\{\gamma^\mu, \gamma^\nu\} = -i (2\pi \alpha')^2 B^{\mu\nu}, \quad (2.15)
\]

\[
\{\theta^\alpha, \theta^\beta\} = \{\gamma^\mu, \theta^\alpha\} = \{\gamma^\mu, \gamma^\nu\} = \{\gamma^\mu, \theta^\beta\} = 0. \quad (2.16)
\]

\[\text{\footnote{Dijkgraaf and Vafa calculate in } SU(N) \text{ theory and we do in } U(N) \text{ theory.}}\]
We take the limit $\alpha' \to 0$ while keeping a finite non-commutativity, so that the non-commutative parameters are related to $F^{\alpha\beta}$ and $B^{\mu\nu}$ as

$$\alpha' \to 0, \quad F^{\alpha\beta}, B^{\mu\nu} \to \infty,$$

$$(2\pi\alpha')^2 B^{\mu\nu} = C^{\mu\nu}, \quad 2\alpha' F^{\alpha\beta} = \gamma^{\alpha\beta}.$$  

In this non-commutative superspace, we can derive a relation, \[
\det \frac{\gamma}{\det C} = \frac{4\det F}{(2\pi)^8\alpha'^4 \det B}. \tag{2.17}
\]

Now, we can try to take the commutative limit. If we simply take $F^{\alpha\beta}$ and $B^{\mu\nu}$ to be finite, the right side of (2.17) will diverge. We need to take an appropriate limit to hold this ratio finite. We choose to take the following limit:

$$B^{\mu\nu} \to (\alpha')^{-1}, \quad F^{\alpha\beta}: \text{finite} \Rightarrow \gamma^{\alpha\beta} \to 0, \quad C^{\mu\nu} \to 0,$$  \tag{2.18}

then (2.17) is held finite as follows:

$$\det \gamma / \det C \sim \det F \sim F^2. \tag{2.19}$$

As a result, the factor of $\det \gamma / \det C$ in the amplitudes of the non-planar diagrams can be regarded as contributions of $F^2$ under this special limit. The appearance of the amplitudes of non-planar diagrams proportional to the background self-dual graviphoton field strength $F^2$ has been argued by Ooguri and Vafa in [12]. We will make some comments about relations with our study in Section 4.

3. The equivalence of the non-planar diagrams in Dijkgraaf–Vafa theory

We have shown that the amplitudes of the non-planar diagrams do not disappear. In Dijkgraaf–Vafa theory [1,3–7], the amplitudes of the planar diagrams in the supersymmetric theory are equivalent to that of the corresponding matrix model. We will show this equivalence can be maintained including the non-planar diagrams in general using the argument of large-$\hat{N}$ reduction on the non-commutative superspace [1,7].

3.1. Field theory on non-commutative superspace and their reduced model

We consider the action (2.1) with the general superpotential (2.2) on the non-commutative superspace (2.9). We map this field theory to a supermatrix model. To do so, we introduce some matrices corresponding to the non-commutative superspace,

\[
[y^\mu, y^\nu] = -i C^{\mu\nu}, \quad [C^{\mu\lambda} B_{\lambda\nu} = \delta^\mu\nu, \quad \hat{\rho}_\mu = B_{\mu\nu} y^\nu,
\]

\[
[\hat{\rho}_\mu, \hat{\rho}_\nu] = i B_{\mu\nu}, \quad [\hat{\rho}_\mu, \hat{\rho}_\nu] = i \delta^\mu \nu, \tag{3.1}
\]

\[
\{\hat{\gamma}^{\alpha}_\beta, \hat{\gamma}^{\beta}_\alpha\} = \gamma^{\alpha\beta}, \quad \gamma^{\alpha\beta} \hat{\rho}^{\beta}_\gamma = \delta^{\alpha}_\gamma, \quad \hat{\pi}_\alpha = \hat{\gamma}^{\beta}_\alpha \hat{\rho}^{\alpha}_\beta, \tag{3.2}
\]

Then, fields on the non-commutative space correspond to matrices as follows [1,8,15]:

\[
O(y) = \int d^4k \frac{e^{ik\nu y^\nu}}{(2\pi)^4} \hat{O}(k) \leftrightarrow \hat{O} = \int d^4k \frac{e^{ik\nu \tilde{y}^\nu}}{(2\pi)^4} \hat{\tilde{O}}(k), \tag{3.3}
\]

\[
Q(\gamma) = \int 4 d^2\kappa e^{-\theta^\alpha \kappa_\alpha} \hat{Q}(\kappa) \to \hat{\tilde{Q}}(\kappa) = A + \hat{\gamma}^{\alpha}_\gamma \psi^\alpha - (\theta^1 \gamma^2 - \theta^2 \gamma^1) F \tag{3.4}
\]

The differential and integral operators are also mapped as follows:

\[
-i \frac{\partial}{\partial \gamma^\alpha} O(y) \leftrightarrow \hat{\rho}^\alpha, \quad \hat{O}, \tag{3.5}
\]

\[
\int d^4y \text{Tr}_{U(n)} O(y) = (2\pi)^2 \sqrt{\det C} \text{Tr}_{U(N)} \hat{O}, \tag{3.6}
\]

\[
\frac{\delta}{\delta \gamma^\alpha} O(y, \theta) \leftrightarrow \hat{\pi}_\alpha, \quad \hat{\tilde{Q}}(\kappa), \tag{3.7}
\]

\[
\int d^2\theta \text{Tr}_{U(N)}(y) \leftrightarrow \frac{i}{8\sqrt{\det \gamma}} \text{Str} \hat{\tilde{Q}}, \tag{3.8}
\]

where Str denotes a supertrace defined as in [1,7]. Then, we can reduce the action (2.1) to

\[
S = \int d^2\theta \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} \text{Str}_{U(N)}(\hat{\Phi} e^\hat{\Phi} e^{-\hat{\Phi}})
\]

\[
+ \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}}
\]

\[
\times \{2\pi i \text{Str}_{U(N)}(\hat{W}_\alpha \hat{W}_\alpha) + \text{Str}_{U(N)} W(\hat{\Phi}) \}
\]

\[
+ \int d^2\hat{\theta} (2\pi)^2 \sqrt{\det \hat{C}}
\]

\[
\times \{ -2\pi i \hat{\tau} \text{Tr}_{U(N)}(\hat{W}_\alpha \hat{W}_\alpha) + \text{Tr}_{U(N)} \hat{W}(\hat{\phi}) \}. \tag{3.9}
\]

Here, the hat indicates that the superfield is reduced as in (3.4) and their component fields are reduced as in (3.3).\footnote{In the anti-holomorphic terms, we expand the anti-chiral superfields with respect to $\tilde{y}$ and $\tilde{\theta}$ and their component fields are mapped to matrices as in (3.3) with respect to $\hat{y}$ instead of $y$.} The matter kinetic term and anti-holomorphic terms are functions of $\hat{\theta}$. $N$ is the infinite rank of the matrices and it is related to the bosonic non-commutativity $C^{\mu\nu}$ [1]. We introduce an appropriate dimensionful constant $g_m$ in the supermatrix model that is related to the non-commutative parameters through,

\[
\hat{N} = \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}}, \tag{3.10}
\]

We can construct in this way a reduced model of the gauge theory (2.1) in a non-commutative space (2.9), which exhibits a different non-commutativity compared to the model (1.1) which was studied in [1].
3.2. Equivalence of the non-planar diagrams

As in Section 2, the matter holomorphic terms of the action (3.9) are important to understand the holomorphic parts of the low energy effective theory. Therefore, we discuss the action:

\[ S = \frac{\hat{N}}{g_m} \text{Str} U(\hat{N}) W(\hat{\Phi}), \quad (3.11) \]

and consider the associated non-commutative field theory,

\[ \int d^4 x \, d^2 \theta \, \text{Tr} \, W(\hat{\Phi}). \quad (3.12) \]

In this theory, we can show the equivalence of correlation functions:

\[ \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle = \left\langle \left\langle \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right\rangle \right\rangle \quad ** \]

(3.13)

Here we use (3.6), (3.8), (3.10) and the equation [1]:

\[ \delta^4(y) \star \delta^4(y) \delta^2(\theta) \star \delta^2(\theta) = \frac{g_m^2}{\hat{N}^2}. \quad (3.14) \]

** on the right-hand side of (3.13) indicates that we evaluate this amplitude in the non-commutative theory (3.12) as in Section 2.2. The left-hand side is evaluated in the corresponding supermatrix model (3.11). The left side can be expanded in powers of \( g_m/\hat{N} \):

\[ \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle = \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle_0 + \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle_1 + \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle_2 + \cdots. \]

(3.15)

Here the lower right indices represent the contributions of the higher genus diagrams. Correspondingly, this can be mapped to

\[ \left\langle \left\langle \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right\rangle \right\rangle \quad **
\]

\[ = \left\langle \left\langle \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right\rangle \right\rangle_{**0} + \left\langle \left\langle \frac{8 \sqrt{\text{det} \gamma}}{(2\pi)^2 \sqrt{\text{det} C}} \left( \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right) \right\rangle \right\rangle_{**1} + \left\langle \left\langle \frac{8 \sqrt{\text{det} \gamma}}{(2\pi)^2 \sqrt{\text{det} C}} \left( \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right) \right\rangle \right\rangle_{**2} + \cdots. \]

(3.16)

This means that the supermatrix model is equivalent to the non-commutative field theory for non-planar diagrams of genus \( n \).

\[ \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle_n = \left\langle \left\langle \int d^4 y \, d^2 \theta \, \text{Tr} (\delta^4(y) \delta^2(\theta) \delta^4(y) \delta^2(\theta) \hat{\Phi}^k) \right\rangle \right\rangle_{**n} \quad (3.17) \]

This equivalence has been established on the non-commutative superspace. We are interested in studying this equivalence under the commutative limits, \( C^{\mu\nu} \rightarrow 0 \), \( \gamma^{\alpha\beta} \rightarrow 0 \). When we take these limits, a \( \delta^4(0) \delta^2(0) \) singularity appears in equation (3.17) and we need to regularize it as in (2.6). Then we obtain

\[ \left\langle \left\langle \frac{g_m}{\hat{N}} \text{Str} \hat{\Phi}^k \right\rangle \right\rangle_n = \frac{1}{64\pi^2} \left( \text{Tr} W_{\alpha} W_\alpha \hat{\Phi}^{k+1} \right)_n. \quad (3.18) \]

In this equation, the left-hand side is of order \( (g_m/\hat{N})^{2n} \) and the right-hand side is of order \( (\text{det} \gamma/\text{det} C)^n \) compared to the leading order (planar diagrams). Therefore, if \( \text{det} \gamma/\text{det} C \) is finite, the contribution of non-planar diagrams in the field theory is finite, corresponding to the supermatrix model with finite \( g_m/\hat{N} \).

Using (2.17), (2.19) and (3.10), we can obtain:

\[ \frac{g_m^2}{\hat{N}^2} = - \frac{64 \, \text{det} \gamma}{(2\pi)^4 \text{det} C} = - \frac{(2) \delta \text{det} F}{(2\pi)^4 \text{det} B} \sim \text{det} F, \quad (3.19) \]

in the context of superstring background fields. Therefore, we can regard the contributions of the non-planar diagrams in the supermatrix model as that of these fields.

This field theory is commutative but it is different from the usual commutative field theory \([3,4]\) in which the non-planar diagrams do not contribute to the amplitude. However, when we take the ratio \( \text{det} \gamma/\text{det} C \) to zero, the contribution of non-planar diagrams disappear and our field theory can reproduce the calculation of the usual field theory. In this sense, our theory can be regarded as an extension of Dijkgraaf–Vafa theory.

This result is consistent with the study of symmetries and mass dimension in \([6,22]\). In these papers, they calculate some charges and mass dimensions of operators and coupling constants, and they conclude that the symmetries forbid the non-planar diagrams to contribute to the holomorphic quantities in the supersymmetric gauge theory. However, in our argument, we add the new constant \( \text{det} \gamma/\text{det} C \) which also has these charges and mass dimension. Therefore, our calculation does not contradict their arguments.

Dijkgraaf–Vafa theory shows the equivalence of the prepotential \( \mathcal{F} \) of this gauge theory and the free energy \( F_m \) of this matrix model \([3,4]\). As in \([6]\), these quantities satisfy equations

\[ \frac{\partial \mathcal{F}}{\partial g_k} = \frac{1}{k + 1} \frac{1}{64\pi^2} \left( \text{Tr} W_{\alpha} W_\alpha \hat{\Phi}^{k+1} \right), \]

\[ \frac{\partial F_m}{\partial g_k} = \frac{1}{k + 1} \frac{g_m}{\hat{N}} \left( \text{Str} \hat{\Phi}^{k+1} \right), \quad (3.20) \]

where \( g_k \) is a coupling constant in (2.2) and \( \psi \) is a fermionic parameter in the prepotential of the \( N = 1 \) field theory. \(^4\) Since these equations hold including all diagrams, the equation (3.18) shows the equivalence of \( \mathcal{F} \) and \( F_m \) including non-planar diagrams. As a result, Dijkgraaf–Vafa theory is applicable for non-planar diagrams as well.

\(^4\) A relation between quantities calculated by supermatrix model and by bosonic matrix model is discussed in \([1]\).
4. Conclusion and discussion

We have shown the equivalence between a field theory and
a supermatrix model including non-planar diagrams and un-
derstood how the graviphoton field strength and $B$ field
background contribute to these non-planar diagrams. Our approach
can be regarded as a new way to derive Dijkgraaf–Vafa the-
ory from superstring theory. It is interesting to compare our
approach with the original Dijkgraaf–Vafa approach [3,4].

Our argument is also applicable to field theories with gauge
groups that are the products of some unitary groups coupled to
adjoint, bifundamental and/or fundamental matter [7] and we
can study how the non-planar diagrams contribute to them.

Our result that the graviphoton contributes to the non-
planar diagrams can also be derived from arguments using
diagrams [12] or Schwinger–Dyson equations [18]. These argu-
ments use the C-deformation [12]:

\[ (W_\alpha,W_\beta)_{ij} = F_{\alpha\beta} \delta_{ij} \mod \hat{D}, \tag{4.1} \]

and consider the theory on the commutative space. (This de-
formation undoes the non-commutative superspace.) The back-
ground fields in the superstring theory are different, how-
ever, our non-commutative superspace approach and this C-
deformation approach give the same result in the field theory.
These two approaches should be related in some way.

The meaning of the $1/N^2$ correction is not clearly under-
stood in general reduced models. We have shown how the
graviphoton and $B$ field, which are closed superstring back-
ground, contribute to the $1/N^2$ corrections in our reduced
model. It would be interesting to extend our approach to the
graviton multiplet [23] and propose some relation between
closed string theory and reduced models.

The relation between non-planar diagrams and the gravipho-
ton has also been advocated in the $N=2$ field theory con-
text [24]. Our approach may be applied to these theories.

Acknowledgements

We are grateful to T. Kuroki and M. Bagnoud for valuable
advice and discussions on this Letter. We would also like to
thank T. Azuma and H. Kawai for useful discussions. T.M. is
supported by a Grant-in-Aid for the 21st Century COE “Center
for Diversity and Universality in Physics”.

References

For matrix models of critical string theory, see also: