Use of reduced forms in the disturbance decoupling problem

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Abstract

Specific algorithms, such as those involving the supremal of the invariant subspaces contained in a suitable subspace, are known to be able to test whether a disturbance decoupling problem (DDP) is solvable. Here, by reducing the system to its Molinari form, we obtain an alternative description of this supremal object and compute its dimension. Hence we have a general result for solving the decoupling provided that a Molinari basis is known. In particular, a necessary numerical condition for it is derived. The same technique is applied to the DDPS, that is, when stability of the decoupled closed loop system is required.

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1. Introduction

The disturbance decoupling problem (DDP) asks for the existence and the construction of a feedback in order to eliminate in the system output the effect of a disturbance acting at the input. The disturbance decoupling problem is called “with stability” (DDPS) if one imposes...
the additional constraint that the closed loop system is stable. In this paper we present some complements and improvements to the classical geometric approach.

Both problems have been largely studied in the last three decades. The geometric approach was inaugurated in 1968 by the recognition of \((A, B)-\)invariance \([1, 12, 15]\). Given a triple of matrices \((C, A, B)\) representing a system, the basic tool is the supremal of the set of \((A, B)\)-invariant subspaces contained in \(\text{Ker} C\), as well as those of the subclasses of controllability subspaces and stabilizable subspaces. Then, one obtains (see, for example \([14]\)) criteria in terms of these supremal objects in order to the DDP and DDPS be solvable, and specific algorithms to compute them and the corresponding decouplings are developed. In addition, see for example \([8, 13, 9]\) for the study on frequency domain and \([4]\) for the generalization to descriptor systems. However, in the computation of these geometric objects, numerical instabilities may be encountered with large systems (see \([11]\)). An alternative, numerically stable, approach is presented in \([3]\), based on the computation of condensed forms under orthogonal equivalence transformations. Indeed, it is valid for descriptor systems \([2]\).

Here we are mainly interested in a unified treatment of the supremal subspaces appearing in the classical geometric approach, instead of the ad-hoc construction of each of them in \([14]\). In order to do that, we consider the Molinari reduced form of the given system, and we relate the above supremal objects with the geometric structure of the four subsystems (controllable, observable, complete, autonomous) appearing in a Molinari splitting. As a significative improvement, this alternative description allows us to compute their dimensions. Moreover, the criteria for the DDP and the DDPS can be reformulated and even generalized to differentiable families of systems. Finally a necessary numerical condition for the DDP is derived.

The paper is organized as follows. In Section 2 we recall the geometric approach to the DDP in \([14]\). In Section 3 we apply it to the particular case of a reduced triple. The key point is 3.3: if a system splits into several subsystems, the study of the invariant subspaces and their supremal can be reduced to each of the subsystems. Then, the criterion for the DDP becomes quite simple (3.5).

Proposition 4.1 guarantees that these results can be transferred to a general triple \((C, A, B)\) by means of a reducing transformation. The first main result, concerning the supremal of the \((A, B)-\)invariant subspaces contained in \(\text{Ker} C\) and its application to the DDP, is obtained in 4.2. It can be generalized to families of systems differentially depending on some external parameters (4.3).

As a second application, a necessary numerical condition for the DDP to be solvable is derived in Section 5. This condition follows from the dimension of the subsystems appearing in the Molinari reduced form (5.3), which we compute using the results in \([7]\).

In an analogous way, in Section 6 we tackle the supremal of the controllability subspaces contained in \(\text{Ker} C\) 6.4 for the reduced triples, and 6.5 for the general ones.

In Section 7 we summarize (7.1) the unified geometric structure of the two supremal objects above, and we compute their dimensions (7.2).

Finally, Section 8 is devoted to the DDPS. As above, the Molinari subsystems are used to describe the supremal of the stabilizable subspaces contained in \(\text{Ker} C\), and hence to reformulate a criterion for the DDPS to be solvable.

We denote by \(\mathbb{C}\) the field of complex numbers. \(\mathbb{C}^+\) is the closed right-half complex plane and \(\mathbb{C}^-\) is the open left-half one. We write \(M_{n \times m}(\mathbb{C})\) for the vector space of matrices with \(n\) rows and \(m\) columns with entries in \(\mathbb{C}\). If \(n = m\), we write simply \(M_n(\mathbb{C})\). When we refer to a triple of matrices \((C, A, B)\) representing a system, we assume \(A \in M_n(\mathbb{C})\), \(B \in M_{n \times m}(\mathbb{C})\) and \(C \in M_{p \times n}(\mathbb{C})\). We denote by \(\sigma(\cdot)\) the spectrum of the corresponding matrix.
2. The disturbance decoupling problem (DDP)

Given the system
\[
\dot{x} = Ax(t) + Bu(t) + Sq(t), \\
z(t) = Cx(t)
\]
where the term \( q(t) \) represents a disturbance which is assumed not to be directly controllable by the controller, the system is said to be disturbance decoupled if, for each initial state \( x(0) \), the output \( z(t) \) is the same for every \( q(t) \). It is easy to see that it is equivalent to the following problem:

**Definition 2.1.** Given a system as above, the disturbance decoupling problem (DDP) consists in the existence and the construction of a feedback \( K \) such that
\[
\langle A + BK | \text{Im } S \rangle \subset \text{Ker } C.
\]

It is well known (see, for example [14]) that an approach to this problem is based on the following concepts:

**Definition 2.2.** Given a pair \( (A, B) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C}) \),

1. One says that a subspace \( W \subset \mathbb{C}^n \) is \( (A, B) \)-invariant if
   \[
   A(W) \subset W + \text{Im } B
   \]
   or, equivalently, if there is a feedback \( K \) such that
   \[
   (A + BK)(W) \subset W.
   \]
2. If \( V \subset \mathbb{C}^n \), we write \( V^* \) the largest or supremal \( (A, B) \)-invariant subspace contained in \( V \).

Then, one has the following solution to the DDP:

**Theorem 2.3** [14]. The DDP in 2.1 is solvable if and only if
\[
\text{Im } S \subset (\text{Ker } C)^*.
\]
Then a solution is provided by any feedback \( K \) such that
\[
(A + BK)(\text{Ker } C)^* \subset (\text{Ker } C)^*.
\]

Therefore, the DDP is reduced to the following matricial problem: given a triple \((C, A, B)\), we must compute the subspace \((\text{Ker } C)^*\) and feedbacks \( K \) verifying the latter relation. Solutions \( K \) are quite easy to construct provided that the DDP is solvable. For the computation of \((\text{Ker } C)^*\), one has the following algorithm:

**Theorem 2.4** [14]. Given a pair \( (A, B) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C}) \) and a subspace \( V \subset \mathbb{C}^n \), let us consider the sequence of subspaces defined by
\[
V^0 = V, \quad V^k = V \cap A^{-1}(\text{Im } B + V^{k-1}).
\]
Then, it is nonincreasing and it stabilizes in \( V^* \).
We have seen that a solution to the DDP results from applying this algorithm to $V = \text{Ker} \ C$.

Let us see some simple examples:

**Example 2.5.** In the conditions of 2.1,

1. If $C = 0$, then $\text{Ker} \ C = \mathbb{C}^n$, and hence $(\text{Ker} \ C)^* = \mathbb{C}^n$. In fact, if $C = 0$, the system is trivially disturbance decoupled because $z(t) = 0$ for all $x(t)$ and $q(t)$.
2. If $B = 0$, then $\text{Im} \ B = 0$, and hence

\[
(\text{Ker} \ C)^0 = \text{Ker} \ C \\
(\text{Ker} \ C)^1 = \text{Ker} \ C \cap A^{-1}(\text{Ker} \ C)^0 = \text{Ker} \ C \cap A^{-1}(\text{Ker} \ C) = \text{Ker} \ C \cap \text{Ker} \ CA \\
(\text{Ker} \ C)^2 = \text{Ker} \ C \cap A^{-1}(\text{Ker} \ C)^1 = \text{Ker} \ C \cap A^{-1}(\text{Ker} \ C) \cap \text{Ker} \ CA \\
= \text{Ker} \ C \cap A^{-1}(\text{Ker} \ C) \cap A^{-1}(\text{Ker} \ CA) = \text{Ker} \ C \cap \text{Ker} \ CA \cap \text{Ker} \ CA^2
\]

and so on. Hence, $(\text{Ker} \ C)^*$ is the unobservable subspace.

Actually, if $B = 0$, the $(A, B)$-invariant subspaces are just those which are $A$-invariant. Therefore, $(\text{Ker} \ C)^*$ is simply the largest $A$-invariant subspace contained in $\text{Ker} \ C$, which is the unobservable subspace (see, for example, [5]).

**3. The disturbance decoupling problem for a reduced triple**

We recall that, given a triple of matrices, it is possible to reduce it to its Molinari reduced form by means of state feedback, output injection, and bases changes in the state space, input space and output space.

**Theorem 3.1** [10]. *Given a triple $(C, A, B) \in M_{p \times n}(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$, there are matrices $(P, Q, T, F, G) \in GL_n(\mathbb{C}) \times GL_p(\mathbb{C}) \times GL_m(\mathbb{C}) \times M_{m \times n}(\mathbb{C}) \times M_{n \times p}(\mathbb{C})$ such that*

\[
A_c = PAP^{-1} + PBF + GCP^{-1}, \\
B_c = PBT, \\
C_c = QCP^{-1}
\]

*have the form*

\[
\begin{pmatrix}
A_c & B_c \\
C_c & 0
\end{pmatrix} = 
\begin{pmatrix}
A_1 & 0 & B_1 \\
A_2 & A_3 & 0 \\
0 & A_4 & B_3 \\
0 & 0 & 0
\end{pmatrix},
\]

*where*

$(A_1, B_1)$ is controllable,
$(C_2, A_2)$ is observable,
$(C_3, A_3, B_3)$ is complete.
In addition, we can assume that

\[
(A_1 \ B_1) = \begin{pmatrix}
0 & 1 & & & \\
& 0 & 1 & & \\
& & 0 & 1 & \\
& & & \ddots & \ddots \\
& & & & 0 & 1
\end{pmatrix},
\]

\[
(A_2) = \begin{pmatrix}
0 & 1 & & & \\
& 0 & 1 & & \\
& & 0 & 1 & \\
& & & \ddots & \ddots \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix},
\]

\[
(A_3 \ B_3) = \begin{pmatrix}
0 & 1 & & & \\
& 0 & 1 & & \\
& & 0 & 1 & \\
& & & \ddots & \ddots \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix},
\]

\[
(A_4) \text{ a Jordan matrix.}
\]

We assume the decomposition

\[
C^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4
\]

corresponding to the block partition of \( A_c \).

Our aim is to reduce the DDP for the triple \( (C_c, A_c, B_c) \) to each of the above subsystems. Note that

\[
\ker C_c = X_1 \oplus \ker C_2 \oplus \ker C_3 \oplus X_4.
\]

Therefore, we immediately have that

**Lemma 3.2.** Let us consider the DDP for a reduced triple \( (C_c, A_c, B_c) \) and assume \( S \) block-partitioned
\[ S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} \]

given the decomposition \( \mathbb{C}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4 \). If
\[ S_2 = S_3 = 0, \]
then the given system is trivially disturbance decoupled (with \( K_c = 0 \)).

Let us see that, in fact, it is a necessary condition. In order to prove this, we need the following lemma:

**Lemma 3.3.** Let \( (C, A, B) \) be a triple partitioned as follows:
\[
\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{pmatrix}
\]
and the corresponding decomposition \( \mathbb{C}^n = X_1 \oplus X_2 \). Then,

1. A subspace of the form 
   \[ W = W_1 \oplus W_2, \quad W_1 \subset X_1, \quad W_2 \subset X_2 \]
   is \( (A, B) \)-invariant if and only if \( W_i \) is \( (A_i, B_i) \)-invariant for \( i = 1, 2 \).
2. Given a subspace of the form 
   \[ V = V_1 \oplus V_2, \quad V_1 \subset X_1, \quad V_2 \subset X_2, \]
then 
   \[ V^* = V_1^* \oplus V_2^*. \]

**Proof.** First, we remark some quite elementary properties which will be used in the sequel. For any subspaces \( Y_1, Z_1 \subset X_1 \) and \( Y_2, Z_2 \subset X_2 \), one has

1. \( (Y_1 \oplus Y_2) + (Z_1 \oplus Z_2) = (Y_1 + Z_1) \oplus (Y_2 + Z_2) \).
2. \( Y_1 \oplus Y_2 \subset Z_1 \oplus Z_2 \) if and only if \( Y_1 \subset Z_1 \) and \( Y_2 \subset Z_2 \).
3. \( (Y_1 \oplus Y_2) \cap (Z_1 \oplus Z_2) = (Y_1 \cap Z_1) \oplus (Y_2 \cap Z_2) \).

Note that in (i) the last term is a direct sum since
\[ (Y_1 + Z_1) \cap (Y_2 + Z_2) \subset X_1 \cap X_2 = \{0\}. \]

For (ii), if \( y_1 + y_2 = z_1 + z_2 \), with \( y_i \in Y_i \) and \( z_i \in Z_i, i = 1, 2 \), then \( y_i, z_i \in X_i \). Hence, \( y_i = z_i \) since \( X_1 \) and \( X_2 \) form a direct sum.

The same argument proves the converse in (iii).

Now we tackle the proof of the assertions in the lemma.

1. \( W \) is \( (A, B) \)-invariant if and only if \( A(W) \subset W + \text{Im } B \). Clearly (see (i) above)
\[ W + \text{Im } B = (W_1 \oplus W_2) + (\text{Im } B_1 \oplus \text{Im } B_2) = (W_1 + \text{Im } B_1) \oplus (W_2 + \text{Im } B_2). \]

On the other hand,
\[ A(W) = A(W_1 \oplus W_2) = A_1(W_1) + A_2(W_2). \]

But the last term is in fact a direct sum since \( A_i(W_i) \subset X_i \) and \( X_1 \cap X_2 = \{0\} \). Hence, the initial inclusion is equivalent to (see (ii) above)
\[ A_i(W_i) \subset W_i + \text{Im } B_i, \quad i = 1, 2. \]

(2) It is enough to see that each step in algorithm 2.4 splits into the corresponding ones in \( X_1 \) and \( X_2 \).

Clearly
\[ V^0 = V_1^0 \oplus V_2^0. \]

Let us assume that
\[ V^{k-1} = V_1^{k-1} \oplus V_2^{k-1}. \]

Then, using again (i) above, we have that
\[
V^k = (V_1 \oplus V_2) \cap A^{-1}((\text{Im } B_1 \oplus \text{Im } B_2) + (V_1^{k-1} \oplus V_2^{k-1}))
\]
\[ = (V_1 \oplus V_2) \cap A^{-1}((\text{Im } B_1 + V_1^{k-1}) \oplus (\text{Im } B_2 + V_2^{k-1})). \]

We cannot apply (iii) above directly, but a similar argument works:
\[
V^k \supset (V_1 \oplus V_2) \cap (A^{-1}(\text{Im } B_1 + V_1^{k-1}) + A^{-1}(\text{Im } B_2 + V_2^{k-1}))
\]
\[ \supset (V_1 \cap A^{-1}(\text{Im } B_1 + V_1^{k-1})) \oplus (V_2 \cap A^{-1}(\text{Im } B_2 + V_2^{k-1})) \]
again recalling that \( X_1 \cap X_2 = \{0\} \).

For the converse inclusion, if \( x_1 \in V_1, x_2 \in V_2 \), then \( A(x_1) \in X_1, A(x_2) \in X_2 \). Hence, 
\[ A(x_1 + x_2) = A(x_1) + A(x_2) \in (\text{Im } B_1 + V_1^{k-1}) \oplus (\text{Im } B_2 + V_2^{k-1}) \]
implies
\[ A(x_i) \in \text{Im } B_i + V_i^{k-1}, \quad i = 1, 2. \]

Finally, for \( i = 1, 2 \),
\[ V_i \cap A^{-1}(\text{Im } B_i + V_i^{k-1}) = V_i \cap A_i^{-1}(\text{Im } B_i + V_i^{k-1}) = V_i^k. \]

In our situation, we have:

**Proposition 3.4.** Let \((C_c, A_c, B_c)\) be a reduced triple as above. Then,
\[ (\text{Ker } C_c)^* = X_1 \oplus X_4. \]

**Proof.** Clearly, as we said above,
\[ \text{Ker } C_c = X_1 \oplus \text{Ker } C_2 \oplus \text{Ker } C_3 \oplus X_4. \]

By iteratively applying the above result we have
\[ (\text{Ker } C_c)^* = X_1^* \oplus (\text{Ker } C_2)^* \oplus (\text{Ker } C_3)^* \oplus X_4^*. \]
Obviously $X_1^* = X_1$, $X_4^* = X_4$.

In example 2.5 we have seen that

$$(\ker C_2)^* = \ker C_2 \cap \ker C_2 A \cap \ker C_2 A^2 \cap \cdots$$

which, in this case, is zero since $(C_2, A_2)$ is observable.

Finally, again by the above proposition, we can reduce the computation of $(\ker C_3)^*$ to each of its blocks, which have the form

$$\begin{pmatrix} A' & B' \\ C' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

If we write $(e_1, \ldots, e_n, e_{n+1})$ the corresponding basis, we have

$$\im B' = [e_n]$$

$$(\ker C')^0 = \ker C' = [e_2, \ldots, e_n]$$

$$(\ker C')^1 = \ker C' \cap A'^{-1}(e_n) = [e_2, \ldots, e_n] \cap [e_1, e_3, \ldots, e_n]$$

$$= [e_3, \ldots, e_n]$$

$$(\ker C')^2 = \ker C' \cap A'^{-1}(e_n) = [e_2, \ldots, e_n] \cap [e_1, e_4, \ldots, e_n]$$

$$= [e_4, \ldots, e_n]$$

$$\vdots$$

$$(\ker C')^{n-1} = \ker C' \cap A'^{-1}(e_n) = [e_2, \ldots, e_n] \cap [e_1] = \{0\}.$$ 

Hence

$$(\ker C')^* = \{0\}. \quad \square$$

Therefore, Theorem 2.3 and Proposition 3.2 give:

**Corollary 3.5.** Let us consider the DDP for a reduced triple $(C_c, A_c, B_c)$, and assume $S$ block-partitioned $S = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix}$ according to the decomposition $C^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4$. Then, the DDP is solvable if and only if

$$S_2 = S_3 = 0.$$ 

In this case, we can take $K_c = 0$. 

4. The solution of the disturbance decoupling problem for a general triple by means of reducing bases

Now we tackle the case of a general triple \((C, A, B)\). Let \((C_c, A_c, B_c)\) be its Molinari reduced form, and let \(P, Q, T, F\) and \(G\) be reducing transformations as in Section 3. Our aim is to find a solution of the DDP for \((C, A, B)\) from Corollary 3.5 for \((C_c, A_c, B_c)\). To this end we state:

**Proposition 4.1.** In the above conditions,

(1) If
\[ W_c \subset \text{Ker} C_c \] is an \((A_c, B_c)\)-invariant subspace,

then
\[ W = P^{-1}W_c \subset \text{Ker} C \] is an \((A, B)\)-invariant subspace.

(2) Then, if
\[ (A_c + B_cK_c)(W_c) \subset W_c, \]

one has
\[ (A + BK)(W) \subset W, \]

where \(K = FP + T^{-1}K_cP\).

**Proof.** (1) Clearly
\[ W = P^{-1}W_c \subset P^{-1}\text{Ker} C_c = \text{Ker} C_cP = \text{Ker} QC = \text{Ker} C. \]

Thus, it is sufficient to prove (2).

(2) By hypothesis,
\[
PW \supset (PA P^{-1} + PB F + GCP^{-1} + PB T^{-1}K_c)(PW)
= P(A + B(FP + T^{-1}K_cP) + P^{-1}GC)(W)
= P(A + B(FP + T^{-1}K_cP))(W)
\]

since we have just shown that \(W \subset \text{Ker} C\).

Since \(P\) is an isomorphism, it is equivalent to
\[ W \supset (A + BK)(W). \] \(\Box\)

We conclude:

**Theorem 4.2.** Given a triple \((C, A, B)\), let \(P, Q, T, F\) and \(G\) be matrices transforming it into its Molinari reduced form, and \(\mathbb{C}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4\) the corresponding decomposition. Then,

(1) \((\text{Ker} C)^* = P^{-1}(X_1 \oplus X_4)\).

(2) A DDP is solvable if and only if
\[ \text{Im } PS \subset X_1 \oplus X_4 \]

or, equivalently, if the associated block partition has the form \[ PS = \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix} \].

**Proof.** From the above proposition and 3.4,

\[ (\text{Ker } C)^* = P^{-1}(\text{Ker } C_c)^* = P^{-1}(X_1 \oplus X_4). \]

From 2.3, the DDP for \((C, A, B)\) is solvable if and only if

\[ \text{Im } S \subset P^{-1}(X_1 \oplus X_4) \]

or, equivalently,

\[ X_1 \oplus X_4 \supset P\text{Im } S = \text{Im } PS. \quad \square \]

**Remark 4.3.** In the above conditions, bearing in mind that we can take \(K_c = 0\) (see 3.5), a theoretic solution for the DDP is \(K = FP\). However, in general it cannot be computed by a numerically stable procedure. An analogous comment is valid for Corollary 4.4.

The above theorem can be generalized to global differentiable families of triples \((C(\tau), A(\tau), B(\tau))\), \(\tau \in M\) by means of [6], where one guarantees the existence of differentiable families of reducing transformations provided that \(M\) is a contractible manifold and the family of triples has constant type, that is to say, for all \(\tau \in M\) they have the same reduced form except for the eigenvalues of \(A_4(\tau)\), which depend differentially on \(\tau \in M\), preserving the Jordan type.

**Corollary 4.4.** Let \(M\) be a contractible manifold, \((C(\tau), A(\tau), B(\tau))\), \(\tau \in M\), a differentiable family of triples having constant type, and \(S(\tau), \tau \in M\), a differentiable family of disturbance matrices. Let \(P(\tau), Q(\tau), T(\tau), F(\tau)\) and \(G(\tau), \tau \in M\), be a differentiable family of reducing transformations (see [6]). Then, the DDP is solvable for all \(\tau \in M\) if and only if

\[ P(\tau)S(\tau) = \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix}, \]

where the block partition corresponds to the constant decomposition \(\mathbb{C}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4\).

5. A necessary numerical condition for the disturbance decoupling problem

As a second application of Theorem 4.2, we can reformulate the necessary condition \(\text{rank } S \leq \dim(\text{Ker } C)^*\) for the DDP to be solvable as follows:

\[ \text{rank } S \leq \dim(\text{Ker } C)^* = \dim X_1 + \dim X_4 = n - \dim X_2 - \dim X_3. \]

Let us compute \(\dim X_i (i = 1, \ldots, 4)\) using the results in [7] in the particular case \(D = 0\).
Definition 5.1. Given a triple \((C, A, B)\), for \(0 \leq j \leq n - 1\), we consider

\[
\rho'^{co}_j = \text{rank} \begin{pmatrix}
CB & CAB & \cdots & CA^j B \\
0 & CB & \cdots & CA^{j-1} B \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & CB
\end{pmatrix},
\]

\[
\rho^{c}_j = \text{rank} \begin{pmatrix}
B & AB & A^2 B & \cdots & A^j B \\
0 & CB & CAB & \cdots & CA^{j-1} B \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & CB
\end{pmatrix},
\]

\[
\rho^{o}_j = \text{rank} \begin{pmatrix}
CA^j & CA^{j-1} B & \cdots & CAB & CB \\
CA^{j-1} & CA^{j-2} B & \cdots & CB & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
CA & CB & \cdots & 0 & 0 \\
C & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

For convenience, we take \(\rho^{co}_{-2} = \rho^{co}_{-1} = 0\).

From [7] it follows that the number of nilpotent \((j + 1)\)-blocks in \(A_3\) (see Section 3) is just

\[
\rho'^{co}_j - \rho'^{co}_{j-1} - (\rho^{co}_{j-1} - \rho^{co}_{j-2}).
\]

In particular, if it is zero, then there are no \((j + 1)\)-blocks in \(A_3\). Since the number of these indices will play an important role, we define:

Definition 5.2. We define \(\nu\) as the number of indices for which the above condition is not true, that is to say,

\[
(\rho'^{co}_j - \rho'^{co}_{j-1}) \neq (\rho^{co}_{j-1} - \rho^{co}_{j-2}) \quad \text{if} \quad j \in \{j_1, \ldots, j_\nu\}.
\]

Moreover, the sizes of the nilpotent blocks in \(A_1\) and \(A_2\) (see Section 3) are given, respectively, by the conjugate partition of

\[
\{\rho'^{c}_0 - \nu, \rho'^{c}_1 - \rho'^{c}_0 - \nu, \rho'^{c}_2 - \rho'^{c}_1 - \nu, \ldots\} \quad \text{and} \quad \{\rho^{o}_0 - \nu, \rho^{o}_1 - \rho^{o}_0 - \nu, \rho^{o}_2 - \rho^{o}_1 - \nu, \ldots\}.
\]

We conclude:

Lemma 5.3. With the above notation,

\[
\dim X_3 = \sum_{j=0}^{n-1} ((\rho'^{co}_j - \rho'^{co}_{j-1}) - (\rho^{co}_{j-1} - \rho^{co}_{j-2}))(1 + j),
\]

\[
\dim X_1 = \rho^{c}_{n-1} - n\nu,
\]

\[
\dim X_2 = \rho^{o}_{n-1} - n\nu.
\]

Then, as we have announced, from 4.2 we have:

Theorem 5.4. Given a triple \((C, A, B)\), a necessary condition in order for the DDP to be solvable is

\[
\text{rank} S \leq n - (\rho^{o}_{n-1} - n\nu) - \sum_{j=0}^{n-1} ((\rho'^{co}_j - \rho'^{co}_{j-1}) - (\rho^{co}_{j-1} - \rho^{co}_{j-2}))(1 + j).
\]
6. Supremal controllability subspace

The “controllability subspaces” are an interesting subfamily of the \((A, B)\)-invariant subspaces characterized by the fact that every state in them is reachable from the origin along a controlled trajectory contained in the subspace. Indeed:

**Definition 6.1.** Given a pair \((A, B) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C})\),

1. One says that an \((A, B)\)-invariant subspace \(W \subset \mathbb{C}^n\) is a controllability subspace of \((A, B)\) if there exists a map \(R\) such that
   \[ W = \langle A + BR | W \cap \text{Im } B \rangle. \]
2. If \(V \subset \mathbb{C}^n\), we write as \(V^*\) the largest or supremal \((A, B)\)-controllability subspace contained in \(V\).

**Remark 6.2.** The role of the supremal controllability subspace can be emphasized, for instance, if one considers a triple \((C, A, B)\) and takes \(V = \ker C\). Then, we have

\[ 0 \subset (\ker C)^*_0 \subset (\ker C)^*_1 \subset \ker C \subset \mathbb{C}^n \]

and one can see that “if state feedback control is to be chosen to make \((\ker C)^*_0\) invariant then we enjoy complete freedom of spectrum assignment on \((\ker C)^*_1\), but have no residual freedom to modify in any way the (induced) dynamic action on \((\ker C)^*_2/(\ker C)^*_0\)” [14].

The supremal controllability subspace can be computed by means of the following algorithm:

**Proposition 6.3** [14]. Given a pair \((A, B) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C})\) and a subspace \(V \subset \mathbb{C}^n\), let us consider the sequence of subspaces defined by

\[ V^*_0 = \{0\}, \quad V^*_k = V^* \cap (A(V^*_{k-1}) + \text{Im } B). \]

Then, it is increasing and it stabilizes in \(V^*_k\).

In a similar way to the supremal invariant subspace, let us obtain an expression of \((\ker C)^*_e\) by applying this algorithm to reduced triples:

**Proposition 6.4.** Let \((C_c, A_c, B_c)\) be a reduced triple as in Section 3. Then,

\[ (\ker C_c)^*_e = X_1. \]

**Proof.** We will apply the above proposition bearing in mind 3.4 and (i)–(iii) in the proof of 3.3:

\[ (\ker C_c)^*_1 = X_1 \oplus X_4 \cap (A(0) + (\text{Im } B_1 \oplus \text{Im } B_3)) = \text{Im } B_1 \]

\[ (\ker C_c)^*_2 = (X_1 \oplus X_4) \cap (A(\text{Im } B_1) + (\text{Im } B_1 \oplus \text{Im } B_3)) \]

\[ = (X_1 \oplus X_4) \cap ((\text{Im } A_1 B_1 + \text{Im } B_1) \oplus \text{Im } B_3) = \text{Im } A_1 B_1 + \text{Im } B_1 \]

and so on. Since \((A_1, B_1)\) is controllable, we have

\[ (\ker C_c)^*_e = \text{Im } B_1 + \text{Im } A_1 B_1 + \cdots = X_1. \quad \square \]
Theorem 6.5. Given a triple \((C, A, B)\), let \(P, Q, T, F\) and \(G\) be matrices transforming it to its Molinari reduced form, and let \(\mathbb{C}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4\) be the corresponding decomposition. Then,

\[
(Ker C)^*_* = P^{-1}X_1.
\]

Proof. It is sufficient to prove that if \(W_c \subset Ker C_c\) is an \((A_c, B_c)\)-controllability subspace, then \(W = P^{-1}W_c \subset Ker C\) and it is an \((A, B)\)-controllability subspace.

We have seen in 4.1 that if \(W_c \subset Ker C_c\), then \(W_c \subset Ker C\) is an \((A, B)\)-controllability subspace. Hence,

\[
W = P^{-1}W_c = (W \cap Ker C) + (A + BK)(W \cap Im B) + \cdots
\]
and so on (recall that \(W \subset Ker C\)). Hence,

\[
W = P^{-1}W_c = (W \cap Im B) + (A + BK)(W \cap Im B) + \cdots \qed
\]

7. The dimension of the maximal objects \((Ker C)^*\) and \((Ker C)^*_*\)

Summarizing 4.2 and 6.5, we have the following geometric structure, which will be extended in 8.7.

Remark 7.1. The elements of the chain

\[
0 \subset (Ker C)^* \subset Ker C \subset \mathbb{C}^n
\]
in 6.2 are characterized by

\[
Ker C = P^{-1}(X_1 \oplus Ker C_2 \oplus Ker C_3 \oplus X_4),
\]

\[
(Ker C)^* = P^{-1}(X_1 \oplus X_4),
\]

\[
(Ker C)^*_* = P^{-1}(X_1).
\]

Moreover, the computations in Section 5 give the dimension of \((Ker C)^*\) and \((Ker C)^*_*\):

Corollary 7.2. Given a triple \((C, A, B)\), with the notation in Section 5, we have

\[
\dim(Ker C)^* = n - (\rho_{n-1}^o - n) - \sum_{j=0}^{n-1}((\rho_{j+1}^o - \rho_{j+2}^o) - (\rho_{j-1}^o - \rho_{j-2}^o))(1 + j),
\]

\[
\dim(Ker C)^*_* = \rho_{n-1}^c - n.
\]
8. The disturbance decoupling problem with stability (DDPS)

In particular, the supremal controllability subspace can be applied to solve the DDP (see 2.1) when one imposes the additional constraint that the closed loop system map $A + BK$ is stable.

**Definition 8.1.** Given a system as in Section 2, the disturbance decoupling problem with stability (DDPS) consists in the existence and the construction of a feedback $K$ such that

$$\langle A + BK \mid \text{Im } S \rangle \subset \ker C,$$

$$\sigma(A + BK) \subset \mathbb{C}^-.$$

In [14], a criterion for the solvability of the DDPS is given by the subspace $V_g^*$, defined as follows:

**Definition 8.2.** Given a pair $(A, B) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$ and a subspace $V \subset \mathbb{C}^n$, we denote by $V_g^*$ the largest member of the family of subspaces $W \subset \mathbb{C}^n$, verifying that

(i) $W \subset V$.

(ii) $W$ is an $(A, B)$-invariant subspace.

(iii) There is a feedback $K$ such that

$$\sigma[(A + BK)|_W] \subset \mathbb{C}^-,$$

where $(\cdot)|_W$ means restriction to $W$.

**Remark 8.3.** Clearly, the $(A, B)$-controllability subspace verifies the above conditions. Therefore, one has:

$$0 \subset V_*^* \subset V_g^* \subset V^* \subset V \subset \mathbb{C}^n.$$

Criterion 2.3 for the solvability of the DDP is now strengthened as follows:

**Theorem 8.4** [14]. In the conditions of 8.1, assume $(A, B)$ controllable. Then, the DDPS is solvable if and only if

$$\text{Im } S \subset (\ker C)^*_g.$$

In general, $V_g^*$ can be computed as follows:

**Proposition 8.5** [14]. In the conditions of 8.2, choose any feedback $K$ such that

$$(A + BK)(V^*) \subset V^*$$

and consider the endomorphism induced in a natural way by $A + BK$ in $V^*/V_*^*$. Let

$$V^*/V_*^* = X_g \oplus X_b$$

be the decomposition by the sign of the real part of the eigenvalues ($X_g$ and $X_b$ correspond to eigenvalues in $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively). Then,

$$V_g^* = \pi^{-1}(X_g),$$
where \( \pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n / V_\ast \) is the canonical projection.

In the case of a reduced triple \((C_c, A_c, B_c)\) and \(V = \text{Ker} \ C_c\), we have
\[
V_\ast / V^{\ast}_\ast = (X_1 \oplus X_4) / X_1 \cong X_4 = (X_4)_g \oplus (X_4)_b
\]
(recall that one can take \(K_c = 0\)), where the last term is just the decomposition of \(X_4\) according to the Jordan matrix \(A_4\). Therefore,
\[
(K_c)_g^{\ast} = \pi^{-1}((X_4)_g) = X_1 \oplus (X_4)_g.
\]
For a general triple, we have:

**Theorem 8.6.** Given a triple \((C, A, B)\), let \(P, Q, T, F\) and \(G\) be matrices transforming it into its Molinari reduced form, and \(\mathbb{C}^n = X_1 \oplus X_2 \oplus X_3 \oplus X_4\) the corresponding decomposition. Let
\[
X_4 = (X_4)_g \oplus (X_4)_b
\]
be the decomposition corresponding to the eigenvalues of the Jordan matrix \(A_4\) lying in \(\mathbb{C}^+\) and \(\overline{\mathbb{C}^+}\), respectively. Then,
\[
(K_c)_g^{\ast} = P^{-1}(X_1 \oplus (X_4)_g).
\]

**Remark 8.7.** We can extend the chain in 7.1 as follows:

\[
0 \subset (K_c)_g^{\ast} \subset (K_c)_g^{\ast} \subset (K_c)^{\ast} \subset \text{Ker} \ C \subset \mathbb{C}^n
\]
\[
(K_c)_g^{\ast} = P^{-1}(X_1 \oplus (X_4)_g).
\]

In particular, criterion 8.4 yields:

**Corollary 8.8.** In the conditions of 8.6, assume \((A, B)\) controllable. Then, the DDPS in 8.1 is solvable if and only if
\[
\text{Im} \ PS \subset X_1 \oplus (X_4)_g.
\]

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**References**