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Rings of matrix invariants in positive characteristic

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Abstract

Denote by $R_{n,m}$ the ring of invariants of m-tuples of $n \times n$ matrices ($m, n \geqslant 2$) over an infinite base field K under the simultaneous conjugation action of the general linear group. When $\mathrm{char}(K) = 0$, Razmyslov (Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974) 723) and Procesi (Adv. Math. 19 (1976) 306) established a connection between the Nagata–Higman theorem and the degree bound for generators of $R_{n,m}$. We extend this relationship to the case when the base field has positive characteristic. In particular, we show that if $0 < \mathrm{char}(K) \leqslant n$, then $R_{n,m}$ is not generated by its elements whose degree is smaller than m. A minimal system of generators of $R_{2,m}$ is determined for the case $\mathrm{char}(K) = 2$: it consists of $2^m + m - 1$ elements, and the maximum of their degrees is m. We deduce a consequence indicating that the theory of vector invariants of the special orthogonal group in characteristic 2 is not analogous to the case $\mathrm{char}(K) \neq 2$. We prove that the characterization of the $R_{n,m}$ that are complete intersections, known before when $\mathrm{char}(K) = 0$, is valid for any infinite K. We give a Cohen–Macaulay presentation of $R_{2,4}$, and analyze the difference between the cases $\mathrm{char}(K) = 2$ and $\mathrm{char}(K) \neq 2$.

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1. Introduction

Throughout this paper K is an infinite field, and $n \in \mathbb{N}$. Denote by $M_{n,m} = M(n,K) \oplus \cdots \oplus M(n,K)$ the space of m-tuples of $n \times n$ matrices over K. The general linear group $\mathrm{GL}(n,K)$ acts by simultaneous conjugation on $M_{n,m}$: for $g \in \mathrm{GL}(n,K)$, $A_i \in M(n,K)$ ($i = 1,\ldots,m$) we have

$$g(A_1,\ldots,A_m) = (gA_1g^{-1},\ldots,\ gA_mg^{-1}).$$

The coordinate ring of $M_{n,m}$ is the mn^2-variable commutative polynomial algebra

$$K_{n,m} := K[x_{r,ij} \mid 1 \leqslant i,j \leqslant n,\ r = 1,\ldots,m],$$

where $x_{r,ij}$ denotes the function mapping $(A_1,\ldots,A_m)$ to the (i,j) entry of A_r. The generators of $K_{n,m}$ are the entries of the so-called *generic matrices*

$$X_r := (x_{r,ij})_{i,j=1}^n \in M(n,K_{n,m}), \quad r = 1,\ldots,m.$$

The action of $\mathrm{GL}(n,K)$ on $M_{n,m}$ induces the action $(gf)(A) := f(g^{-1}A)$, ($g \in \mathrm{GL}(n,K)$, $f \in K_{n,m}, A \in M_{n,m}$) of $\mathrm{GL}(n,K)$ on $K_{n,m}$ by K-algebra automorphisms. In particular, $gx_{r,ij}$ is the (i,j) entry of $g^{-1}X_rg$. Our main object is the subalgebra

$$R_{n,m} = \{f \in K_{n,m} \mid \forall g \in \mathrm{GL}(n,K)\colon\ gf = f\}$$

of *polynomial invariants*.

The algebra $R_{n,m}$ has an extensive literature when $\mathrm{char}(K) = 0$. Its generators and relations are described in [35,30,32]. In particular, it is known that $R_{n,m}$ is generated by its elements of degree $\leqslant n^2$ (see [32]). Moreover, $R_{n,m}$ is Cohen–Macaulay by the Hochster–Roberts theorem [16]. For some small values of (n,m) an explicit Cohen–Macaulay presentation of $R_{n,m}$ is given in [11,38,39].

The problem of finding a characteristic free approach to $R_{n,m}$ (and the polynomial identities of matrices) is posed for example in the survey [13, Problem 1]. Significant progress was made during the last decade concerning the "symbolic" description of $R_{n,m}$ when $\mathrm{char}(K)$ is positive: Donkin [8] extended the description of the generators and Zubkov [42] the Procesi–Razmyslov theorem on the relations to this case.

In the present paper, we focus on explicit presentations (by means of generators and relations) of rings of matrix invariants in positive characteristic. In Section 6 we show that $R_{n,m}$ has a simple presentation (namely, $R_{n,m}$ is a complete intersection) precisely for those (n,m) like in characteristic zero, and there is no essential difference between the various characteristics in these simplest cases.

On the other hand, in Section 3 we make explicit a new phenomenon appearing in positive characteristic: if $0 < \mathrm{char}(K) \leqslant n$, then the degree bound for the generators of $R_{n,m}$ goes to infinity with m (see Corollary 3.2). In order to prove this we combine the idea of Razmyslov [32] and Procesi [30] with a more recent result of Kermer [17], to relate the degree bound with the so-called "Nagata–Higman Problem" (see Theorem 3.1). It seems interesting to compare this result with the following fact (see [8, p. 399] or [41, Theorem 2]): the theory of modules with good filtration implies that the

Hilbert series of $R_{n,m}$ is independent of char(K). So it is remarkable that we get the same quantitative behavior from very different combinatorial structures. Our result has also a consequence for the invariant theory of the special orthogonal group, which shows that the uniform form of the first fundamental theorem on invariants of vectors valid in the case $\mathrm{char}(K) \neq 2$ does not hold in characteristic 2, where the situation changes drastically (see Section 7).

For $R_{2,m}$ it is possible to give also a minimal system of generators; this was done in [35] when $\mathrm{char}(K) = 0$, and the result remains the same for any odd characteristic (cf. [31]). When $\mathrm{char}(K) = 2$ it has a new character as we indicated above.

In Section 5 we give a Cohen–Macaulay presentation (or Hironaka decomposition) of $R_{2,4}$ in any characteristic (when $\mathrm{char}(K) = 0$ this was done in [39]). This example serves as an illustration of the general picture mentioned above. $R_{2,m}$ was known to be Cohen–Macaulay when $\mathrm{char}(K) \neq 2$ (cf. [26]), and our computations show that $R_{2,4}$ is Cohen–Macaulay also in characteristic 2. This corrects an erroneous announcement in the abstract submitted by Kuzmin and Zubkov to the Fourth International Algebraic Conference, Novosibirsk, 2000 under the title "Cohen–Macaulay property of rings of matrix invariants".

2. Identities with trace

In this preparatory section we recall some notions and facts from the theory of polynomial identities of matrices (a good general reference to this topic is [13], though it is restricted to the case $\mathrm{char}(K) = 0$). In particular we reformulate a result of Kemer [17], and deduce consequences tailored to our application in Section 3 for the problem of generating $R_{n,m}$.

Let $Y = \{y_1, y_2, \ldots\}$ be a countable set of non-commuting variables. Denote by $\langle Y \rangle$ the free associative semigroup generated by $y_1, y_2, \ldots$. If C is a commutative ring with unity, then we write $C\langle Y \rangle$ for the *free associative C-algebra with unity* generated by Y, and we write $C^+\langle Y \rangle$ for the *free associative C-algebra without unity* generated by Y. The elements of $C\langle Y \rangle$ are non-commutative polynomials in the variables $y_1, y_2, \ldots$ with coefficients from C, and $C^+\langle Y \rangle$ is the ideal consisting of polynomials with zero constant term. The elements of $\langle Y \rangle$ are called *monomials*. When we speak about an ideal of $C^+\langle Y \rangle$, we assume that it is a C-submodule.

In particular, $K^+\langle Y \rangle$ is the free associative K-algebra without unity of countable rank. An ideal I of $K^+\langle Y \rangle$ is called a *T-ideal*, if I is stable with respect to all K-algebra endomorphisms of $K^+\langle Y \rangle$. Note that any map $Y \to K^+\langle Y \rangle$, $y_i \mapsto u_i$ extends to a unique K-algebra endomorphism of $K^+\langle Y \rangle$. So an ideal I is a T-ideal if $f(y_1, \ldots, y_m) \in I$, $u_1, \ldots, u_m \in K^+\langle Y \rangle$ imply that $f(u_1, \ldots, u_m) \in I$. This notion arose in the study of polynomial identities. Given an associative K-algebra A and $f(y_1, \ldots, y_m) \in K^+\langle Y \rangle$ we say that $f = 0$ is a *polynomial identity* (shortly PI) on A if for all $(a_1, \ldots, a_m) \in A \times \cdots \times A$ we have $f(a_1, \ldots, a_m) = 0$. The set $\{f \in K^+\langle Y \rangle \mid f = 0$ is a PI on $A\}$ is a T-ideal, called the T-ideal of identities of A. Conversely, any T-ideal I is the T-ideal of identities of the factor algebra $K^+\langle Y \rangle / I$. We denote by $\{f\}_T$ the T-ideal generated by $f \in K^+\langle Y \rangle$.

The relation between T-ideals and polynomial identities explained above implies that the following statements are equivalent for $f, h \in K^+\langle Y\rangle$:
(i) h is contained in $\{f\}_T$.
(ii) All associative K-algebras A satisfying the PI $f = 0$, satisfy also the PI $h = 0$.

Usually we express that (i) (hence (ii)) holds for f and h by saying that "the identity $h = 0$ is a consequence of $f = 0$".

The algebra $K\langle Y\rangle$ has a natural multigrading. The *multidegree* of a monomial u is $(\alpha_1, \alpha_2, \ldots)$, if the variable y_i appears in u α_i-times ($i = 1, 2, \ldots$), and the multidegree of the constants is $(0, 0, \ldots)$. Since K is infinite, a Vandermonde argument show that any T-ideal of $K^+\langle Y\rangle$ is spanned by multihomogeneous elements. We say that $f(y_1, \ldots, y_m) \in K^+\langle Y\rangle$ is *multilinear* in $y_1, \ldots, y_m$ if each of its monomials is of the form $y_{\pi(1)} \cdots y_{\pi(m)}$ for some permutation $\pi \in \mathrm{Sym}(1, \ldots, m)$ (in other words, the multidegree of f is $(\underbrace{1, \ldots, 1}_{m}, 0, 0, \ldots)$.

For example, consider $y_1^n \in K^+\langle Y\rangle$. Make the substitution $y_1 \mapsto y_1 + \cdots + y_n$ and take the multilinear component of $(y_1 + \cdots + y_n)^n$. We get that the *symmetric polynomial*

$$s_n(y_1, \ldots, y_n) := \sum_{\pi \in \mathrm{Sym}(1,\ldots,m)} y_{\pi(1)} \cdots y_{\pi(n)}$$

is contained in $\{y_1^n\}_T$. Conversely, $s_n(y_1, \ldots, y_1) = n! y_1^n \in \{s_n\}_T$, so if $\mathrm{char}(K) > n$, then $\{y_1^n\}_T = \{s_n\}_T$. The Nagata–Higman theorem asserts that if $\mathrm{char}(K) > n$, then the T-ideal generated by y_1^n contains $y_1 \cdots y_N$ for sufficiently large N, and the so-called Nagata–Higman problem is to determine the exact value of the minimal such N (we refer to [12] for history and results concerning this problem).

Now we enlarge the free algebra $K\langle Y\rangle$ in order to have an appropriate framework to deal with the Cayley–Hamilton identity of $n \times n$ matrices. Define an equivalence relation on the set of monomials by specifying that two monomials are equivalent if they are cyclic permutations of each other, and denote by $\bar{\mu}$ the equivalence class of $\mu \in \langle Y\rangle$. The *pure free trace ring* F is a unitary commutative polynomial K-algebra with one generator $\mathrm{tr}(\mu)$ for each equivalence class of monomials. To simplify notation we shall write $\mathrm{tr}(\mu)$ instead of $\mathrm{tr}(\bar{\mu})$. The *mixed free trace ring* is $F\langle Y\rangle$. It is endowed with the *formal trace* $\mathrm{Tr} : F\langle Y\rangle \to F$, which is the F-linear map given by $\mathrm{Tr}(\mu) = \mathrm{tr}(\mu)$ for $\mu \in \langle Y\rangle$, and $\mathrm{Tr}(1) = n$.

For $f \in F\langle Y\rangle$ we write $f = f(y_1, \ldots, y_m)$ if f is contained in the Tr-stable K-subalgebra of $F\langle Y\rangle$ generated by $y_1, \ldots, y_m$.

There is a notion of $\tilde{T}$-ideal for $F\langle Y\rangle$, analogous to the notion of T-ideal in $K^+\langle Y\rangle$. We say that a K-algebra endomorphism φ of $F\langle Y\rangle$ is Tr-*preserving* if $\mathrm{Tr}(\varphi(f)) = \varphi(\mathrm{Tr}(f))$ holds for all $f \in F\langle Y\rangle$. Any map $Y \mapsto F\langle Y\rangle$, $y_i \mapsto u_i$ extends to a unique Tr-preserving unitary K-algebra endomorphism of $F\langle Y\rangle$. We write $f(u_1, \ldots, u_m)$ for the image of $f(y_1, \ldots, y_m)$. An ideal I of $F\langle Y\rangle$ is called a $\tilde{T}$-ideal, if I is stable with respect to all Tr-preserving unitary K-algebra endomorphisms of $F\langle Y\rangle$, and I is stable with respect to the map Tr (i.e. $\mathrm{Tr}(I) \subseteq I$). We write $\{f\}_{\tilde{T}}$ for the $\tilde{T}$-ideal generated by $f \in F\langle Y\rangle$.

For $A_1,\dots,A_m \in M(n,K)$ and $f(y_1,\dots,y_m) \in F\langle Y\rangle$ the expression $f(A_1,\dots,A_m)$ has an obvious sense: we evaluate tr as the usual trace function, and 1 is substituted by the identity matrix. We say that $f(y_1,\dots,y_m)=0$ is an *identity with trace* for $M(n,K)$, if for all $A_1,\dots,A_m \in M(n,K)$ we have $f(A_1,\dots,A_m)=0$. In the special case that f is contained in F, we call the indentity $f=0$ a *pure trace identity* for $M(n,K)$. We set

$$I(M(n,K)) := \{f \in F\langle Y\rangle \mid f=0 \text{ is an identity with trace for } M(n,K)\}.$$

Clearly $I(M(n,K))$ is a $\tilde{T}$-ideal in $F\langle Y\rangle$.

The multigrading of $K\langle Y\rangle$ extends to a multigrading of $F\langle Y\rangle$: the multidegree of $\mathrm{tr}(\mu)$ is by definition the multidegree of $\mu \in \langle Y\rangle$ (in other words, the formal trace map Tr is multihomogeneous). For any subspace V of $F\langle Y\rangle$ we shall denote by $V(1^m)$ the subspace of elements of V which are multilinear in $y_1,\dots,y_m$. There is a K-basis of $F\langle Y\rangle(1^m)$ indexed by $\mathrm{Sym}(0,1,\dots,m)$: with $\pi \in \mathrm{Sym}(0,1,\dots,m)$ with cycle decomposition $\pi=(i_1,\dots,i_r)\cdots(j_1,\dots,j_s)(k_1,\dots,k_t,0)$ we associate

$$\mathrm{tr}_\pi(y_1,\dots,y_m) := \mathrm{tr}(y_{i_1}\cdots y_{i_r})\cdots\mathrm{tr}(y_{j_1}\cdots y_{j_s})y_{k_1}\cdots y_{k_t}.$$

Multilinearizing the Cayley–Hamilton identity we obtain that $d_n \in I(M(n,K))(1^n)$, where

$$d_n(y_1,\dots,y_n) := \sum_{\pi\in \mathrm{Sym}(0,1,\dots,n)} (-1)^\pi\, \mathrm{tr}_\pi(y_1,\dots,y_n).$$

The following result is due to Kemer [17]; under the assumption $\mathrm{char}(K)=0$ it was proved by Razmyslov [32] and Procesi [30]:

Theorem 2.1. *For any $m=1,2,\dots$ we have the equality*

$$I(M(n,K))(1^m) = \{d_n\}_{\tilde{T}}(1^m).$$

In other words, all multilinear identities with trace of $M(n,K)$ are consequences of the multilinearization of the Cayley–Hamilton identity.

Proof. The paper [17] works with a more general version $(\hat{F}\langle Y\rangle, \hat{\mathrm{T}}\mathrm{r})$ of $(F\langle Y\rangle,\mathrm{Tr})$, where the order n of the matrices under consideration is not a priori fixed. Namely, $\hat{F}=F[\mathrm{tr}(1)]$ is a one-variable polynomial algebra over F, and the formal trace map $\hat{\mathrm{T}}\mathrm{r}: \hat{F}\langle Y\rangle \to \hat{F}$ is the $\hat{F}$-linear map defined by $\hat{\mathrm{T}}\mathrm{r}(1)=\mathrm{tr}(1)$, and $\hat{\mathrm{T}}\mathrm{r}(\mu)=\mathrm{tr}(\mu)$ for $\mu\in\langle Y\rangle$. The meaning of the phrase "$f=0$ is an identity on $M(n,K)$" where $f\in\hat{F}\langle Y\rangle$ is defined in an obvious manner, as well as the notion of a $\tilde{T}$-ideal of $\hat{F}\langle Y\rangle$. Denote by $\hat{I}(M(n,K))$ the $\tilde{T}$-ideal of $\hat{F}\langle Y\rangle$ consisting of the elements f where $f=0$ is an identity on $M(n,K)$. The multigrading of $F\langle Y\rangle$ is extended to $\hat{F}\langle Y\rangle$ such that the multidegree of the variable $\mathrm{tr}(1)$ is zero. The statement of [17, Theorem 1] reads as follows: for all $m\in\mathbb{N}$ we have that $\hat{I}(M(n,K))(1^m)$ coincides with the degree m multilinear component of the $\tilde{T}$-ideal of $\hat{F}\langle Y\rangle$ generated by $\mathrm{tr}(1)-n$ and d_n.

There is a natural unitary K-algebra homomorphism $v:\hat{F}\langle Y\rangle\to F\langle Y\rangle$, with $v(\mathrm{tr}(1))= n, v(y_i)=y_i$ $(i=1,2,\dots)$, and $v(\mathrm{tr}(\mu))=\mathrm{tr}(\mu)$ $(\mu\in\langle Y\rangle)$. Clearly $v(\mathrm{Tr}(f))=\mathrm{Tr}(v(f))$

for all $f \in \hat{F}\langle Y \rangle$. The kernel of v is the ideal generated by $\mathrm{tr}(1) - n$ (this is actually a $\tilde{T}$-ideal, because $\mathrm{tr}(1)-n$ is fixed by all $\hat{\mathrm{Tr}}$-preserving unitary K-algebra endomorphisms of $\hat{F}\langle Y \rangle$). Moreover, v is multihomogeneous, and $\tilde{T}$-ideals of $\hat{F}\langle Y \rangle$ are mapped onto $\tilde{T}$-ideals of $F\langle Y \rangle$ (because any Tr-preserving unitary K-algebra endomorphism of $F\langle Y \rangle$ can be lifted to a corresponding endomorphism of $\hat{F}\langle Y \rangle$). In particular, $\hat{I}(M(n,K))$ is mapped onto $I(M(n,K))$ and the $\tilde{T}$-ideal of $\hat{F}\langle Y \rangle$ generated by $\mathrm{tr}(1) - n$, d_n onto the $\tilde{T}$-ideal of $F\langle Y \rangle$ generated by d_n. So our statement is just a reformulation of [17, Theorem 1]. □

Our next step is to establish a relation between $\{d_n\}_{\tilde{T}} \subset F\langle Y \rangle$ and $\{s_n\}_T \subset K^+\langle Y \rangle$. Denote by F^+ the maximal ideal of F generated by $\mathrm{Tr}(\mu)$ ($\mu \in \langle Y \rangle$). We have the decomposition $F\langle Y \rangle = F^+F\langle Y \rangle \oplus K\langle Y \rangle$ as K-vector space, the first summand is the ideal of $F\langle Y \rangle$ generated by F^+, the second summand is a K-subalgebra. Consider

$$\eta : F\langle Y \rangle \to K\langle Y \rangle,$$

the natural homomorphism whose kernel is $F^+F\langle Y \rangle$. This is multihomogeneous, and maps $\tilde{T}$-ideals onto T-ideals. The key observation is that

$$\eta(d_n(y_1,\dots,y_n)) = s_n(y_1,\dots,y_n).$$

Proposition 2.2. *$\eta(\{d_n\}_{\tilde{T}})$ coincides with $\{s_n\}_T$, the T-ideal of $K^+\langle Y \rangle$ generated by s_n.*

Proof. As an ideal, $\{d_n\}_{\tilde{T}}$ is generated by

$$\{d_n(u_1,\dots,u_n), \mathrm{Tr}(v d_n(u_1,\dots,u_n)) \mid v, u_i \in F\langle Y \rangle, i = 1,\dots,n\}. \tag{1}$$

By multilinearity of d_n it is sufficient to take $u_i \in \langle Y \rangle \cup \{1\}$ in the generating system (1). Moreover, it is sufficient to take $u_i \in \langle Y \rangle$ in (1), because $d_n(u_1,\dots,u_n) = 0$ if there is an i with $u_i = 1$. Indeed, by symmetry of d_n in its variables, it is sufficient to show that $d_n(y_1,\dots,y_{n-1},1) = 0$. Identify $\mathrm{Sym}(0,1,\dots,n-1)$ with the stabilizer of n in $\mathrm{Sym}(0,1,\dots,n)$. Observe that

$$\sum_{\tau \in \mathrm{Sym}(0,1,\dots,n)} (-1)^{\pi}\, \mathrm{tr}_{\pi}(y_1,\dots,y_n)$$

$$= \sum_{\tau \in \mathrm{Sym}(0,1,\dots,n-1)} (-1)^{\tau} \left(\mathrm{tr}_{\tau}(y_1,\dots,y_n) - \sum_{i=0}^{n-1} \mathrm{tr}_{\tau(i,n)}(y_1,\dots,y_n) \right)$$

$$= \sum_{\tau \in \mathrm{Sym}(0,1,\dots,n-1)} (-1)^{\tau} \left(\mathrm{tr}(y_n)\mathrm{tr}_{\tau}(y_1,\dots,y_{n-1}) \right.$$

$$\left. - \sum_{i=0}^{n-1} \mathrm{tr}_{\tau}\,(y_1,\dots,y_{n-1})|_{y_i \mapsto y_i y_n} \right).$$

This shows the equality $d_n(y_1,\dots,y_{n-1},1) = 0$.

Therefore $\eta(\{d_n\}_{\tilde{T}})$ is generated as an ideal by

$$\{\eta(d_n(u_1,\dots,u_n)) \mid u_i \in \langle Y\rangle,\ i=1,\dots,n\}.$$

Since $\eta(d_n)=s_n$, the ideal generated by the above set is the T-ideal $\{s_n\}_T$. □

As an immediate corollary of Theorem 2.1 and Proposition 2.2 we get the following:

Corollary 2.3. *$\eta(I(M(n,K))(1^m))=\{s_n\}_T(1^m)$ holds for all $m\in\mathbb{N}$. That is, η maps the multilinear components of $I(M(n,K))$ onto the multilinear components of the T-ideal generated by s_n in $K^+\langle Y\rangle$.*

Corollary 2.4. *The following statements are equivalent for some positive integer N:*
(i) *There exists an $h\in(F^+)^2$, multilinear in $y_1,\dots,y_{N+1}$, such that* $\mathrm{tr}(y_1\cdots y_{N+1})-h=0$ *is a pure trace identity on $M(n,K)$.*
(ii) *The T-ideal of $K^+\langle Y\rangle$ generated by $s_n(y_1,\dots,y_n)$ contains $y_1\cdots y_N$.*

Proof. The bilinear form $(A,B)\mapsto\mathrm{tr}(AB)$, $A,B\in M(n,K)$ is non-degenerate. Therefore, for $f(y_1,\dots,y_m)\in F\langle Y\rangle$ we have $f=0$ is an identity with trace on $M(n,K)$ if and only if $\mathrm{Tr}(f y_{m+1})=0$ is a pure trace identity on $M(n,K)$. So (i) is equivalent to

(i′) There exists a $b\in F^+F\langle Y\rangle$, multilinear in $y_1,\dots,y_N$, such that $y_1\cdots y_N-b$ is an identity with trace on $M(n,K)$.

The equivalence of (i′) and (ii) is an immediate consequence of Corollary 2.3. Indeed, assume first that (i′) holds. Then by Corollary 2.3, applying η to $y_1\cdots y_N-b\in I(M(n,K))(1^N)$ we get that $y_1\cdots y_N$ is contained in $\{s_n\}_T$. Conversely, assume that (ii) holds, i.e. $y_1\cdots y_N\in\{s_n\}_T(1^N)$. By Corollary 2.3 there is a $c\in I(M(n,K))(1^N)$ with $\eta(c)=y_1\cdots y_N$. Then c is of the form $c=y_1\cdots y_N-b$ with $b\in F^+F\langle Y\rangle$. □

3. Generating the ring of matrix invariants

It is proved in [8] that $R_{n,m}$ is generated by the coefficients of the characteristic polynomials of monomials of the generic matrices $X_1,\dots,X_m$. This generating system contains for example the determinants $\det(X_i)$ and traces of monomials $\mathrm{tr}(X_{i_1}\cdots X_{i_s})$.

For a subset S of $R_{n,m}$ we denote by $K[S]$ the (unitary) K-subalgebra of $R_{n,m}$ generated by S. If $K[S]=R_{n,m}$ but no proper subset of S generates $R_{n,m}$, then S is called a *minimal system of generators* (shortly m.s.o.g.) of $R_{n,m}$.

Note that $R_{n,m}$ is a graded (even $\mathbb{Z}^m$-graded) subalgebra of $K_{n,m}$ (the $\mathbb{Z}^m$-grading is obtained by giving degree $(1,0,\dots,0)$ to the coordinates of X_1, degree $(0,1,0,\dots,0)$ to the coordinates of X_2, etc.), hence it has a homogeneous (multihomogeneous) m.s.o.g. We write $R_{n,m}(1^m)$ for the multilinear component (i.e. the component of multidegree $(1,\dots,1)$) of $R_{n,m}$. By [8] it is spanned by elements of the form

$$\mathrm{tr}(X_{i_1}\cdots X_{i_r})\cdots\mathrm{tr}(X_{j_1}\cdots X_{j_s}),$$

where $(i_1,\dots,i_r)\cdots(j_1,\dots,j_s)$ is the decomposition as product of pairwise disjoint cycles of some permutation in $\mathrm{Sym}(1,\dots,m)$. Denote by R^+ the maximal ideal of

$R_{n,m}$ consisting of the elements with no degree zero component. For a subset S of homogeneous elements of $R_{n,m}$ we have $R_{n,m}=K[S]$ if and only if $S+(R^+)^2$ spans the K-vector space $R^+/(R^+)^2$. Thus, S is a homogeneous m.s.o.g. if and only if its image is a K-basis of $R^+/(R^+)^2$. This shows that the cardinality of a homogeneous m.s.o.g. depends only on $R_{n,m}$. Moreover, since $R^+/(R^+)^2$ inherits the grading (multigrading), the degrees (multidegrees) of the elements in a homogeneous (multihomogeneous) m.s.o.g. are uniquely determined by $R_{n,m}$.

We say that an element of $R_{n,m}$ is *decomposable* if it is contained in the subalgebra generated by lower degree invariants. Obviously, a homogeneous m.s.o.g. does not contain decomposable invariants. Conversely, if there is a homogeneous (multihomogeneous) non-decomposable invariant of a given degree (multidegree), then any homogeneous (multihomogeneous) m.s.o.g. must contain an element with that degree.

In the case $\mathrm{char}(K)=0$ the following result is due to Razmyslov [32] and Procesi [30] (some of the ideas have appeared already in [9], see [13] for a survey on the topic).

Theorem 3.1. *The following statements are equivalent for some positive integer N:*
(i) *The invariant* $\mathrm{tr}(X_1\cdots X_{N+1})$ *is decomposable in* $R_{n,N+1}$.
(ii) *The T-ideal in* $K^+\langle Y\rangle$ *(the free associative non-unitary K-algebra of countable rank) generated by* $\sum_{\pi\in\mathrm{Sym}(1,\dots,n)} y_{\pi(1)}\cdots y_{\pi(n)}$ *contains the element* $y_1\cdots y_N$.

Proof. Consider the surjective K-linear map $\rho: F(1^{N+1})\to R_{n,N+1}(1^{N+1})$,

$$\mathrm{tr}(y_{i_1}\cdots y_{i_r})\cdots\mathrm{tr}(y_{j_1}\cdots y_{j_s})\mapsto \mathrm{tr}(X_{i_1}\cdots X_{i_r})\cdots\mathrm{tr}(X_{j_1}\cdots X_{j_s})$$

(we substitute the variable y_i by the generic matrix X_i). It is a tautology that the kernel of ρ is $I(M(n,K))\cap F(1^{N+1})$.

Assume that the statement (i) of Corollary 2.4 holds. Applying ρ to this pure trace identity we get that $\mathrm{tr}(X_1\cdots X_{N+1})$ is decomposable in $R_{n,N+1}$.

Conversely, assume that $\mathrm{tr}(X_1\cdots X_{N+1})$ is decomposable in $R_{n,N+1}$. Then $\mathrm{tr}(X_1\cdots X_{N+1})$ is contained in the multilinear component of $(R^+_{n,N+1})^2$. This means that we have an equality in $R_{n,N+1}$ of the form $\mathrm{tr}(X_1\cdots X_{N+1})-\sum_{j\in J}\alpha_j f_j=0$, where $\alpha_j\in K$, and $f_j=\mathrm{tr}(X_{i_1}\cdots X_{i_r})\cdots\mathrm{tr}(X_{j_1}\cdots X_{j_s})$, a multilinear product of at least two traces ($j\in J$). We set $\hat f_j:=\mathrm{tr}(y_{i_1}\cdots y_{i_r})\cdots\mathrm{tr}(y_{j_1}\cdots y_{js})$, an element of $F(1^{N+1})$ mapped to f_i under ρ. Then $\mathrm{tr}(y_1\cdots y_{N+1})-\sum_{j\in J}\alpha_j\hat f_j\in I(M(n,K))(1^{N+1})$, and $\sum_{j\in J}\alpha_j\hat f_j\in(F^+)^2$.

We conclude that (i) of our Theorem 3.1 is equivalent with (i) of Corollary 2.4, which by Corollary 2.4 is equivalent with (ii). □

Denote by $N(n,K)$ the least positive integer N for which (i) and (ii) hold, if such an N exists, and $N(n,K)=\infty$, if there is no such N. If $\mathrm{char}(K)=0$, then $N(n,K)\leqslant n^2$ by [32], and $N(n,K)\geqslant n(n+1)/2$ by [20]. If $\mathrm{char}(K)>n$, then $N(n,K)\leqslant 2^n-1$ by [15].

The situation changes drastically when $0<\mathrm{char}(K)=p\leqslant n$. Then the example of [28, 5. Remarks. (I)] shows that the T-ideal generated by s_n does not contain any element of the form $y_1\cdots y_N$. Namely, take the countably generated commutative non-unitary polynomial algebra $K^+[t_1,t_2,\dots]$, and factorize by the ideal generated by

$t_1^p, t_2^p, \dots$. This algebra satisfies the PI $y^p = 0$, hence satisfies also the PI's $y^n = 0$ and its multilinearization $s_n = 0$. However, this algebra is not nilpotent. It follows that $N(n,K) = \infty$ if $0 < \mathrm{char}(K) = p \leqslant n$.

Corollary 3.2. *If* $0 < \mathrm{char}(K) \leqslant n$ *and* $n \geqslant 2$, *then* $R_{n,m}$ *is not generated by its elements of degree* $< m$.

Proof. $\mathrm{tr}(X_1 \cdots X_m)$ is not decomposable by the above remark and Theorem 3.1, hence any system of generators of $R_{n,m}$ must contain an element which has a non-zero multilinear component in $X_1, \dots, X_m$. □

Remarks. Consider the following general setup: let G be a subgroup of $\mathrm{GL}(V)$ acting on $K[V^m]$, the coordinate ring of $V^m = V \oplus \cdots \oplus V$. When $\mathrm{char}(K) = 0$ a basic theorem of Weyl [40] asserts that one gets a system of generators of $K[V^m]^G$ for $m \geqslant d = \dim(V)$ by polarizing the generators of $K[V^d]^G$. In particular, the maximal degree occurring in a h.m.s.o.g. of $K[V^m]^G$ for $m \geqslant d$ is the same as for $m = d$. This result is a consequence of the characteristic zero representation theory of the general linear group. Corollary 3.2 provides an example showing that a similar result does not hold in positive characteristic, that is, as we increase the number of copies of V, the degree bound for an m.s.o.g. may also go to infinity. Such examples with G finite are given in [34,10]. We mention that since $K_{n,m}$ is a rational $\mathrm{GL}(n,K)$-module with good filtration (see e.g. [8]), the Hilbert series $H(R_{n,m}, t)$ of the $\mathbb{N}_0$-graded algebra $R_{n,m}$ is independent from $\mathrm{char}(K)$ (see [8, p. 399] or [41, Theorem 2]). So although the combinatorial structure of $R_{n,m}$ (e.g. the number of generators) depends heavily on the characteristic of the base field, it always has the same quantitative behavior (the examples considered in [34,10] are permutation representations, so they have this feature as well).

We derived Corollary 3.2 from Kemer's result Theorem 2.1. It would be possible to use the more general result in [42], where the relations between matrix invariants are described, without restriction to the multilinear relations. However, we have chosen to apply the result in [17], whose proof is elementary.

An explicit upper degree bound for the generators of $R_{n,m}$ when $0 < \mathrm{char}(K) \leqslant n$ is given in the subsequent paper [4].

4. Minimal system of generators for 2 × 2 matrix invariants

Throughout Sections 4 and 5 we assume that $n = 2$, and $X_1, \dots, X_m$ are 2×2 generic matrices. In this special case the result of [8] cited above reads as follows:

$$R_{2,m} = K[\det(X_i), \mathrm{tr}(X_{i_1} \cdots X_{i_s}) \mid s \in \mathbb{N}; 1 \leqslant i,\ i_1, \dots, i_s \leqslant m]. \tag{2}$$

Now we shall list some equalities which hold for all 2×2 matrices X, Y, Z with entries from $K_{2,m}$. Substituting X, Y, Z by polynomials of the generic matrices $X_1, \dots, X_m$ in these equalities we obtain relations among the generators of $R_{2,m}$ given in (2).

We start with

$$2\det(X) = \mathrm{tr}^2(X) - \mathrm{tr}(X^2). \tag{3}$$

Denoting by I the 2×2 identity matrix, the Cayley–Hamilton theorem says that

$$C_2(X) = 0, \quad \text{where } C_2(X) := X^2 - \mathrm{tr}(X)X + \det(X)I. \tag{4}$$

This implies $\mathrm{tr}(C_2(X)Y) = 0$, that is

$$\mathrm{tr}(X^2Y) = \mathrm{tr}(X)\,\mathrm{tr}(XY) - \det(X)\,\mathrm{tr}(Y). \tag{5}$$

It follows from (4) that

$$H(X,Y,Z) = 0, \tag{6}$$

where

$$\begin{aligned} H(X,Y,Z) &:= \mathrm{tr}((C_2(X+Y) - C_2(X) - C_2(Y))Z) \\ &= \mathrm{tr}(XYZ) + \mathrm{tr}(XZY) - \mathrm{tr}(XY)\,\mathrm{tr}(Z) \\ &\quad - \mathrm{tr}(XZ)\,\mathrm{tr}(Y) - \mathrm{tr}(YZ)\,\mathrm{tr}(X) + \mathrm{tr}(X)\,\mathrm{tr}(Y)\,\mathrm{tr}(Z). \end{aligned}$$

Consider a monomial UX_iX_jV of the generic matrices $X_1,\dots,X_m$, where $\deg(UV) \geqslant 1$. Using the equality $H(VU,X_i,X_j) = 0$ we may replace $\mathrm{tr}(UX_iX_jV)$ by $-\mathrm{tr}(UX_jX_iV)$ modulo a decomposable invariant. Iterating this we get that for any permutation $\pi \in \mathrm{Sym}(1,\dots,s)$

$$\mathrm{tr}(X_{\pi(1)}\cdots X_{\pi(s)}) - (-1)^{\pi}\,\mathrm{tr}(X_1\cdots X_s) \text{ is decomposable in } R_{2,m}. \tag{7}$$

A corollary of (5) and (7) is that

$$\mathrm{tr}(X_{i_1}\cdots X_{i_s}) \text{ is decomposable in } R_{2,m} \text{ if } i_j = i_k \quad \text{for some } j<k. \tag{8}$$

From (2), (3), (7), (8) we obtain that

$$R_{2,m} = K[\det(X_i), \mathrm{tr}(X_{i_1}\cdots X_{i_s}) \mid 1 \leqslant i,\ s \leqslant m;\ 1 \leqslant i_1 < \cdots < i_s \leqslant m]. \tag{9}$$

Hence by the definition of $N(n,K)$ one obtains an m.s.o.g. putting the extra assumption $s \leqslant \min(m, N(n,K))$ in (9). It is well known that $N(2,K) = 3$ if $\mathrm{char}(K) \neq 2$. Indeed, the equality

$$H(X_1X_2,X_3,X_4) + H(X_2X_3,X_1,X_4) - H(X_3X_1,X_2,X_4) = 0 \tag{10}$$

shows that $2\,\mathrm{tr}(X_1X_2X_3X_4)$ is decomposable. As we noted before, $N(2,K) = \infty$ if $\mathrm{char}(K) = 2$.

Corollary 4.1. (i) *If* $\mathrm{char}(K) \neq 2$, *then*

$$\{\det(X_i), \mathrm{tr}(X_{i_1}\cdots X_{i_s}) \mid 1 \leqslant i \leqslant m;\ s \in \{1,2,3\};\ 1 \leqslant i_1 < \cdots < i_s \leqslant m\}$$

is a minimal system of generators of $R_{2,m}$.

(ii) *If* $\mathrm{char}(K) = 2$, *then*

$$\{\det(X_i), \mathrm{tr}(X_{i_1}\cdots X_{i_s}) \mid 1 \leqslant i,\ s \leqslant m;\ 1 \leqslant i_1 < \cdots < i_s \leqslant m\}$$

is a minimal system of generators of $R_{2,m}$.

Corollary 4.1(i) is due to Sibirskiĭ [35] when $\mathrm{char}(K) = 0$ and to Procesi [31] (using a different approach, see Section 7) when $\mathrm{char}(K)$ is odd. Corollary 4.1(ii) is new.

5. Cohen–Macaulay presentations

By the Noether normalization lemma any Noetherian graded algebra R contains a *homogeneous system of parameters* (shortly h.s.o.p.), i.e. a set of algebraically independent elements generating a subalgebra P (we call it the *parameter subalgebra*) over which R is a finitely generated module.

Proposition 5.1. *The following is a homogeneous system of parameters for* $R_{2,m}$:

$$\left\{ \mathrm{tr}(X_i), \det(X_i), \sum_{j<k,\, j+k=s} \mathrm{tr}(X_j X_k) \mid 1 \leqslant i \leqslant m;\ 3 \leqslant s \leqslant 2m-1 \right\}.$$

Proof. In characteristic zero this is shown in [39], and the same argument works for any characteristic: one shows that the common zero locus in $M_{2,m}$ of the above invariants is the $\mathrm{GL}(2,K)$-orbit of m-tuples of strictly upper triangular matrices, hence it coincides with the common zero locus of all homogeneous invariants of positive degree; then the result follows from the Hilbert Criterion (when $\mathrm{char}(K) > 0$ one has to replace the Ω-process in the proof of [36, Theorem 4.6.1] by [29, Lemma 3.4.2]). □

Recall that the Noetherian graded algebra R is Cohen–Macaulay if it is a free module over the parameter subalgebra P for some h.s.o.p.; in this case R is a free P-module for any h.s.o.p. By a Cohen–Macaulay presentation of a Cohen–Macaulay algebra R we mean an explicit h.s.o.p. (often called *primary invariants*) together with an explicit system of free P-module generators of R (called *secondary invariants*).

When $\mathrm{char}(K) = 0$, $R_{n,m}$ is Cohen–Macaulay by the Hochster–Roberts theorem [16]. However, when $\mathrm{char}(K) > 0$ and $n \geqslant 2$, $\mathrm{GL}(n,K)$ is not linearly reductive, and by a recent result of Kemper [18] it has a linear representation such that the corresponding ring of invariants is not Cohen–Macaulay. (If K is finite and $n \geqslant 2$, then $R_{n,m}$ is not Cohen–Macaulay for all sufficiently large m by [19, Corollary 2.4].)

The only results we found in the literature concerning the Cohen–Macaulay property of $R_{n,m}$ in positive characteristic are that $R_{2,m}$ is Cohen–Macaulay if $\mathrm{char}(K) > 2$ (cf. [26]), and $R_{3,m}$ is Cohen–Macaulay if $\mathrm{char}(K) > 3$ (cf. [27]).

As we shall see in Section 6, $R_{2,2}$ is a polynomial ring and $R_{2,3}$ is a hypersurface.

Now we shall study $R_{2,4}$ in more detail, and give a Cohen–Macaulay presentation for it. In particular, it turns out that $R_{2,4}$ is Cohen–Macaulay also in characteristic 2,[2] but we point out a difference in the structure of $R_{2,4}$ in even and odd characteristic. This illustrates the Remark after Corollary 3.2.

To simplify notation, set $R := R_{2,4}$; $R(d) := \{r \in R \mid \deg(r) \leqslant d\}$; $t_i := \mathrm{tr}(X_i)$, $u_i = \det(X_i)$, $i = 1,2,3,4$; $e_3 := \mathrm{tr}(X_1X_2)$, $e_4 := \mathrm{tr}(X_1X_3)$, $e_5 := \mathrm{tr}(X_1X_4) + \mathrm{tr}(X_2X_3)$, $e_6 := \mathrm{tr}(X_2X_4)$, $e_7 := \mathrm{tr}(X_3X_4)$; $f := \mathrm{tr}(X_1X_4)$; $g_1 := \mathrm{tr}(X_2X_3X_4)$, $g_2 := \mathrm{tr}(X_1X_3X_4)$, $g_3 := \mathrm{tr}(X_1X_2X_4)$, $g_4 := \mathrm{tr}(X_1X_2X_3)$; $h := \mathrm{tr}(X_1X_2X_3X_4)$; $P := K[t_i, u_i, e_j \mid i = 1,2,3,4;\ j = 3,4,5,6,7]$.

In the case $\mathrm{char}(K) = 0$ the following result is due to Teranishi [39, 9.1. Theorem (1)].

Theorem 5.2. *R is a free P-module generated by*

$$S = \begin{cases} \{1, f, g_1, g_2, g_3, g_4, f^2, f^3\} & \text{if } \mathrm{char}\, K \neq 2, \\ \{1, f, g_1, g_2, g_3, g_4, h, fh\} & \text{if } \mathrm{char}(K) = 2. \end{cases}$$

Proof. Denote by M the P-submodule of R generated by S. We know from Corollary 4.1 that as a P-algebra, R is generated by

$$T = \begin{cases} \{f, g_1, g_2, g_3, g_4,\} & \text{when } \mathrm{char}(K) \neq 2, \\ \{f, g_1, g_2, g_3, g_4, h\} & \text{when } \mathrm{char}(K) = 2. \end{cases}$$

First we prove that M is a subalgebra of R by verifying $TS \subset M$.

By construction M contains $R(3)$, hence $PR(3) \subseteq M$. When $\mathrm{char}(K) = 2$, the equality (10) (which was used to express $\mathrm{tr}(X_1X_2X_3X_4)$ by lower degree invariants when $\mathrm{char}(K) \neq 2$) shows that $\mathrm{tr}(X_1X_4)\,\mathrm{tr}(X_2X_3) \in PR(3)$, implying $f^2 = e_5 f - \mathrm{tr}(X_1X_4)\,\mathrm{tr}(X_2X_3) \in PR(3)$. Considering our explicit minimal system of K-algebra generators of R it follows that the factor space $R(4)/PR(3)$ is spanned by

$$\begin{cases} f^2 + PR(3) & \text{when } \mathrm{char}(K) \neq 2, \\ h + PR(3) & \text{when } \mathrm{char}(K) = 2. \end{cases}$$

Thus $R(4) \subseteq M$, consequently $PR(4) \subseteq M$.

Next we show that $R(5) \subseteq PR(4)$. Look at the degree 5 products of the algebra generators in R; the only non-obvious step is that $fg_i \in PR(4)$ for $i = 1,2,3,4$. From the identity $H(X^2Y, Z, W) + H(XYW, X, Z) - H(WX, XY, Z) = 0$ (see (6)) we get

$$\mathrm{tr}(XYZ)\,\mathrm{tr}(XW) = G(X, Y, Z, W) \quad \text{for all } X, Y, Z, W \in M(2, K_{2,4}), \tag{11}$$

[2] After first writing this paper Kuzmin and Zubkov [21] proved that $R_{2,m}$ is Cohen–Macaulay for all m in characteristic 2. $R_{n,m}$ is Cohen–Maculay for all n, m, and infinite K by results of Hashimoto [14].

where

$$G(X,Y,Z,W) := -2\,\mathrm{tr}(X^2YWZ) + \mathrm{tr}(X^2YZ)\,\mathrm{tr}(W) + \mathrm{tr}(X^2YW)\,\mathrm{tr}(Z) + \mathrm{tr}(X^2Y)\,\mathrm{tr}(ZW) - \mathrm{tr}(X^2Y)\,\mathrm{tr}(Z)\,\mathrm{tr}(W) - \mathrm{tr}(XYW)\,\mathrm{tr}(X)\,\mathrm{tr}(Z) + \mathrm{tr}(XW)\,\mathrm{tr}(XY)\,\mathrm{tr}(Z) + \mathrm{tr}(XYWZ)\,\mathrm{tr}(X) + \mathrm{tr}\,(XYW)\,\mathrm{tr}(XZ) - \mathrm{tr}(XZW)\,\mathrm{tr}(XY).$$

It follows from (5) that if U is a monomial in X_1,X_2,X_3,X_4 of degree $d-2$ and $X \in \{X_1,X_2,X_3,X_4\}$, then

$$\mathrm{tr}(X^2U) \in PR(d-1).$$

Using this observation it is easy to check that $g_4 f = G(X_1,X_2,X_3,X_4)$ is contained in $PR(4)$. Similarly, making appropriate substitutions in the identity (11) we get $g_1 f = G(X_4,X_2,X_3,X_1) \in PR(4)$, $g_3 f = e_5 g_3 - G(X_2,X_4,X_1,X_3) \in PR(4)$, $g_2 f = e_5 g_2 - G(X_3,X_4,X_1,X_2) \in PR(4)$.

The substitution $W \mapsto YW$ in (11) yields

$$\mathrm{tr}(XYZ)\,\mathrm{tr}(XYW) = G(X,Y,Z,YW). \tag{12}$$

Looking at the explicit form of G we see that

$$G(X,Y,Z,YW) \in -\,\mathrm{tr}(XZYW)\,\mathrm{tr}(XY) + PR(5).$$

For a substitution $\phi : \{X,Y,Z,W\} \to \{X_1,X_2,X_3,X_4\}$ we have

$$\mathrm{tr}(\phi(X)\phi(Z)\phi(Y)\phi(W))\,\mathrm{tr}(\phi(X)\phi(Y)) \in PR(4)$$

unless $\{\phi(X),\phi(Y)\} = \{X_1,X_4\}$ and $\{\phi(X),\phi(Y),\phi(Z),\phi(W)\} = \{X_1,X_2,X_3,X_4\}$. So varying the substitution ϕ we get equalities from (12) which show that for any $i,j \in \{1,2,3,4\}$ we have $g_i g_j \in PR(5) + fR(4)$. It follows that the factor space $R(6)/PR(5)$ is spanned by

$$\begin{cases} f^3 + PR(5) & \text{if } \mathrm{char}(K) \neq 2, \\ fh + PR(5) & \text{if } \mathrm{char}(K) = 2. \end{cases}$$

Thus by construction M contains $R(6)$, hence $PR(6) \subseteq M$.

We claim that $R(7) \subseteq PR(6)$. Since $R(1) \subset P$ and $R(5) \subseteq PR(4)$, the only products of degree 7 of the algebra generators of R for which this is not obvious are hg_i, $i = 1,2,3,4$ when $\mathrm{char}(K) = 2$. We use (11) again: $g_4 h = G(X_1,X_2,X_3,X_2X_3X_4)$, and the explicit form of G shows that $G(X_1,X_2,X_3,X_2X_3X_4) \in PR(6)$. Similarly, $g_3 h = G(X_1,X_2,X_4,X_2X_3X_4) \in PR(6)$, $g_2 h = G(X_1,X_3,X_4,X_2X_3X_4) \in PR(6)$, and $g_1 h = G(X_2,X_3,X_4,X_3X_4X_1) \in PR(6)$.

Our next step is to prove $R(8) \subseteq PR(7)$. Assume first that $\mathrm{char}(K)=2$. Since $R(1) \subset P$ and $R(5) \subseteq PR(4)$, the only products of the algebra generators we have to deal with are f^2h and h^2. We have seen already that $f^2 \in PR(3)$, thus $f^2h \in PR(3)R(4) \subseteq PR(7)$. To handle h^2 we use (11) showing

$$h^2 = G(X_1, X_2X_3, X_4, X_2X_3X_4)$$
$$\in \mathrm{tr}(X_1X_2X_3X_2X_3X_4)\,\mathrm{tr}(X_1X_4) - \mathrm{tr}(X_1X_4X_2X_3X_4)\,\mathrm{tr}(X_1X_2X_3) + PR(7). \qquad (13)$$

Here $\mathrm{tr}(X_1X_4X_2X_3X_4) \in R(5) \subseteq PR(4)$, and $\mathrm{tr}(X_1X_2X_3X_2X_3X_4)\,\mathrm{tr}(X_1X_4) \in fR(6) = f(Kfh + PR(5)) \subseteq Kf^2h + PR(7) = PR(7)$. When $\mathrm{char}(K) \neq 2$, the only product of the generators of degree 8 we have to investigate is f^4. Now take the 4×4 matrix whose (i,j) entry is

$$\mathrm{tr}\left(\left(X_i - \frac{1}{2}\mathrm{tr}(X_i)\right)\left(X_j - \frac{1}{2}\mathrm{tr}(X_j)\right)\right)$$
$$= \begin{cases} e_{i+j} - \frac{1}{2}t_it_j & \text{if } i \neq j,\ i+j \neq 5, \\ \frac{1}{2}t_i^2 - 2u_i & \text{if } i = j, \\ f - \frac{1}{2}t_1t_4 & \text{if } (i,j) = (1,4) \text{ or } (2,3), \\ e_5 - f - \frac{1}{2}t_2t_3 & \text{if } \{i,j\} = \{2,3\}. \end{cases}$$

This is the Gram matrix of four vectors lying in the three-dimensional space $sl(2,K)$ of trace zero 2×2 matrices endowed with the non-degenerate symmetric bilinear form $(A,B) \mapsto \mathrm{tr}(AB)$. Therefore the determinant of this matrix is zero, and this shows that f is integral of degree 4 over P.

Summarizing, we proved $PR(8) \subseteq PR(7) \subseteq PR(6) \subseteq M$. The only elements of TS whose degree is greater then 8 are

$$\begin{cases} g_if^3 & \text{when } \mathrm{char}(K) \neq 2, \\ g_ifh \text{ and } fh^2 & \text{when } \mathrm{char}(K) = 2. \end{cases}$$

All of them are contained in $PR(8) \subseteq M$, because $g_if^3 = (g_if)f^2 \in R(5)f^2 \subseteq f^2PR(4)$, $g_ifh = (g_if)h \in R(5)h \subseteq PR(4)h$, and $fh^2 \subseteq fPR(6)$.

So $TS \subseteq M$, hence M is a subalgebra of R containing a system of generators of R. This implies $R = M$, hence as a graded P-module, R is a homomorphic image of the free P-module with generators of degree 0,2,3,3,3,4,6. On the other hand, the Hilbert series of the graded algebra R is $(1 + t^2 + 4t^3 + t^4 + t^6)/(1-t)^4(1-t^2)^9$, the Hilbert series of the free P-module with generators of degree 0,2,3,3,3,3,4,6. Indeed, this holds when $\mathrm{char}(K)=0$ (see [39]), and since $K_{n,m}$ is a $\mathrm{GL}(n,K)$-module with good filtration, the Hilbert series of R is independent from the characteristic of K (see [8, p. 399; 41, Theorem 1]). It follows that R is a free P-module generated by S. □

6. Complete intersections

Recall that a quotient of the commutative polynomial algebra $K[x_1,\dots,x_r]$ modulo some ideal I is said to be a *complete intersection* (shortly c.i.) if I is generated by a regular sequence. The ring $K[x_1,\dots,x_r]/I$ is called a *hypersurface* if I is a principal ideal. Under the assumption $\mathrm{char}(K)=0$ Le Bruyn–Teranishi [24] characterized those (n,m) for which $R_{n,m}$ is a complete intersection. Their proof depends on some results from [22,23] concerning the étale local structure of the variety whose coordinate ring is $R_{n,m}$. The restriction on the characteristic in these latter results was removed recently in [7] (see [3] for a summary), so we are in a position to prove that the main theorem of [24] also holds over any infinite base field K.

Theorem 6.1. *The following are equivalent for* (n,m):
(i) $R_{n,m}$ *is a complete intersection.*
(ii) $R_{n,m}$ *is a hypersurface.*
(iii) (n,m) *equals* $(n,1),(1,m),(2,2),(2,3)$, or $(3,2)$.

Proof. (i) $\Rightarrow$ (iii): We shall follow the idea of [24], and replace any reference to [22,23] by the corresponding characteristic free result in [7].

Extending the base field to its algebraic closure does not influence the property whether $R_{n,m}$ is c.i. or not. So we may assume that K is algebraically closed. Suppose that $n\geqslant 2$ and $m\geqslant 2$.

First we recall some facts from commutative algebra. We refer to [25, Chapter 7] for the definition of Noetherian local rings that are complete intersections. If $R=K[V]$ is the coordinate ring of the affine variety V, then V is said to be locally c.i. in $v\in V$ if the local ring $\mathcal{O}_v(V)$ (the localization of $K[V]$ at the maximal ideal belonging to v) is c.i. Obviously if R is c.i., then R is locally c.i. in all $v\in V$. Now let $f:V\to W$ be an étale morphism of affine varieties. This means that the induced homomorphism $f_*:\hat{\mathcal{O}}_{f(v)}(W)\to\hat{\mathcal{O}}_v(V)$ between local ring completions is an isomorphism for all $v\in V$. Since a Noetherian local ring A is c.i. if and only if its completion $\hat{A}$ is c.i. (see e.g. [25, Chapter 7, Theorem 21.2 (i)]), W is locally c.i. in $f(v)$ if and only if V is locally c.i. in v.

Denote by $V_{n,m}$ the affine variety whose coordinate ring is $R_{n,m}$. It is well known that $V_{n,m}$ parameterizes the isomorphism classes of semisimple n-dimensional representations of the free algebra $L:=K\langle y_1,\dots,y_m\rangle$. Following [22] we say that a semisimple L-module S has representation type $\tau=(\mu_1,n_1;\dots;\mu_t,n_t)$, if it has t pairwise non-isomorphic simple summands $S_1,\dots,S_t$, the multiplicity of S_i in S is μ_i, and $\dim_K(S_i)=n_i$ $(i=1,\dots,t)$. The so-called local quiver Γ_τ associated with the representation type τ was introduced in [22]. Denote by V_τ the affine variety parameterizing the semi-simple representations of Γ_τ with dimension vector $\mu:=(\mu_1,\dots,\mu_t)$. We do not repeat here the construction of V_τ, because we shall need it only in some very special case worked out already in [24]. We just mention that V_τ is constructed as the affine quotient of some affine space (called the space of representations of Γ_τ with dimension vector μ) endowed with a rational action of the group $\prod_{i=1}^t \mathrm{GL}(\mu_i,K)$. We denote by $\bar{0} \in V_\tau$ the image of the origin 0 of the representation space. Let ξ be a point in $V_{n,m}$ corresponding to a semi-simple representation of type τ. Then there exists an étale morphism f from an affine open neighborhood U of $\bar{0}$ in V_τ onto an affine open neighborhood W of ξ in $V_{n,m}$ with $f(\bar{0})=\xi$ (this is shown in [22,23] when $\mathrm{char}(K)=0$; the characteristic free version is proved in [7, Corollary 4.3], see also [3, Theorem 1.2.1. (i)]). It follows that $V_{n,m}$ is locally c.i. in ξ if and only if V_τ is locally c.i. in $\bar{0}$. Therefore if $V_{n,m}$ is c.i., and there is an n-dimensional L-module of representation type τ, then V_τ is locally c.i. in $\bar{0}$.

Now choose $n_1, n_2 \in \mathbb{N}$ with $n=n_1+n_2$. The assumptions $n,m \geqslant 2$ imply that there is a representation of L of type $\sigma := (1,n_1;1,n_2)$. The local quiver Γ_σ is the directed graph consisting of two vertices v_1, v_2, $a_i := (m-1)n_i^2+1$ loops at v_i $(i=1,2)$, $b := (m-1)n_1n_2$ arrows from v_1 to v_2, and b arrows from v_2 to v_1. It is an easy matter to give the coordinate ring of V_σ explicitly (see [24]): it is an (a_1+a_2)-variable commutative polynomial algebra over C, where $C=K[x_iz_j \mid 1 \leqslant i,\ j \leqslant b]$, a subalgebra of the $(2b)$-variable polynomial algebra generated by x_i, z_j $(1 \leqslant i,j \leqslant b)$. Clearly $C \cong K[t_{ij} \mid 1 \leqslant i,\ j \leqslant b]/I_2$, where I_2 is the ideal generated by the determinants of the 2×2 minors of the generic $b \times b$ matrix (t_{ij}). So C is the coordinate ring of the variety of $b \times b$ matrices of rank $\leqslant 1$. It is well known that this variety is locally c.i. in 0 if and only if $b \leqslant 2$. It follows that V_σ is locally c.i. in $\bar{0}$ if and only if $b=(m-1)n_1n_2 \leqslant 2$, implying that (n,m) equals $(2,2)$, $(2,3)$, or $(3,2)$.

(iii) $\Rightarrow$ (ii) It is trivial that $R_{1,m}$ is an m-variable polynomial algebra, and it is well known that $R_{n,1}$ is an n-variable polynomial algebra. Our starting point is that we know the Hilbert series of $R_{2,2}$, $R_{2,3}$ and $R_{3,2}$: they were computed in the case $K=\mathbb{C}$, and as we noted in the Remark after Corollary 3.2, they remain the same for any infinite field K.

We saw in Corollary 4.1 that $R_{2,2}$ is generated by $\mathrm{tr}(X_1)$, $\det(X_1)$, $\mathrm{tr}(X_2)$, $\det(X_2)$, $\mathrm{tr}(X_1X_2)$, and these elements are clearly algebraically independent.

The Hilbert series of $R_{2,3}$ is $(1+t^3)/(1-t^3)(1-t^2)^6$ (cf. [11]). We saw in Corollary 4.1 that $\mathrm{tr}(X_i)$, $\mathrm{tr}(X_jX_k)$ $(i=1,2,3,\ 1 \leqslant j < k \leqslant 3)$, $\mathrm{tr}(X_1X_2X_3)$ generate $R_{2,3}$, and the first nine generators form an h.s.o.p. for $R_{2,3}$ by Proposition 5.1; denote by P the parameter subalgebra generated by this h.s.o.p. Therefore $\mathrm{tr}(X_1X_2X_3)$ is integral over P. We claim that the P-submodule $P + P\,\mathrm{tr}(X_1,X_2,X_3)$ of $R_{2,3}$ is free. Otherwise $\mathrm{tr}(X_1X_2X_3)$ would be contained in the quotient field of P. This is a contradiction, because being a polynomial algebra, P is integrally closed. The Hilbert series of $R_{2,3}$ shows that $R_{2,3}$ coincides with the free P-submodule generated by 1 and $\mathrm{tr}(X_1X_2X_3)$. In particular, $\mathrm{tr}(X_1X_2X_3)$ satisfies a monic quadratic equation over P. (Under the assumption $\mathrm{char}(K)=0$ this quadratic equation was determined in [11]; since the equation has integer coefficients, it is valid for any K).

The Hilbert series of $R_{3,2}$ is $(1+t^6)/(1-t)^2(1-t^2)^3(1-t^3)^4(1-t^4)$ by [38]. It is sufficient to show that $R_{3,2}$ has an h.s.o.p. whose elements have degrees 1,1,2,2,2,3,3,3,3,4. Then in the same way as for $R_{2,3}$ one can conclude that $R_{3,2}$ is a free module generated by 1 and a degree 6 invariant over the parameter subalgebra, hence this degree 6 invariant must satisfy a quadratic monic equation over the parameter subalgebra. Such an h.s.o.p. can be obtained from the results of [2]. The action of $S := \mathrm{SL}(3,K) \times \mathrm{SL}(3,K)$

on $M_{3,3}$ given by

$$(g,h)(A_1A_2A_3) := (gA_1h^{-1}, gA_2h^{-1}, gA_3h^{-1}),$$

$$(g,h) \in S, \quad A_i \in M(3,K) \ (i = 1,2,3)$$

is studied there. An h.s.o.p. for the algebra $K_{3,3}^S$ is given in [2, Theorem 3.3], take its modification given in [2, p. 186]. Moreover, the injection $M_{3,2} \to M_{3,3}$, $(A_1, A_2) \mapsto (A_1, A_2, I)$ (where I is the 3×3 identity matrix) induces a surjective homomorphism $\kappa : K_{3,3}^S \to R_{3,2}$ by [2, Proposition 4.1] (though formally it is stated there in characteristic zero, the proof is obviously characteristic free; see also [6, Theorem 3.2] for a far reaching characteristic free generalization). Our h.s.o.p. of $K_{3,3}^S$ has 11 elements, one of them is mapped to $1 \in R_{3,2}$, and the remaining 10 elements are mapped under κ to an h.s.o.p. of $R_{3,2}$, whose members have the desired degree.

(ii) $\Rightarrow$ (i) This is trivial. □

7. Special orthogonal invariants

To simplify the formulation we assume throughout Section 7 that K is algebraically closed. A basic result of classical invariant theory is the first fundamental theorem for invariants of vectors with respect to the standard action of the special orthogonal group. This result is due to Weyl [40] when $\text{char}(K) = 0$, and remains valid for any odd characteristic base field by [1]. As a consequence of the results of Section 3 we point out that the form of the first fundamental theorem for the special orthogonal group is completely different when $\text{char}(K) = 2$.

In the case $\text{char}(K) \neq 2$ Procesi [31] initiated an approach to $R_{2,m}$ via the invariant theory of $\text{SO}(3,K)$, the special orthogonal group acting on m-tuples of three-dimensional vectors. The idea is that $M(2,K)$ is the direct sum of the space of scalar matrices (on which $\text{GL}(2,K)$ acts trivially) and $sl(2,K)$, the space of trace zero matrices, on which the action of $\text{GL}(2,K)$ preserves the non-degenerate bilinear function $(A,B) \mapsto \text{tr}(AB)$. Moreover, the action of $\text{GL}(2,K)$ on $sl(2,K)$ factors through the tautological action of $\text{SO}(3,K)$. It follows that $R_{2,m}$ is an m-variable polynomial algebra over the algebra of $\text{SO}(3,K)$ invariants of m-tuples of three-dimensional vectors. So one can apply the first fundamental theorem for $\text{SO}(n,K)$-invariants given in [1] for any K with $\text{char}(K) \neq 2$.

This method does not work when $\text{char}(K) = 2$, because $sl(2,K)$ has no $\text{GL}(2,K)$-stable direct complement in $M(2,K)$. However, as it was observed in [5], there is a relation between $R_{2,m}$ and the algebra of invariants of the special orthogonal group of rank 4, which applies also in characteristic 2.

Let us recall the definition of the special orthogonal group in the case $\text{char}(K) = 2$. Let $V \cong K^n$ be an n-dimensional vector space, and $q : V \to K$ a non-degenerate quadratic form. By the *polar form* of q we mean the symmetric bilinear function $\beta(x,y) = q(x+y) - q(x) - q(y)$. (Unlike in the case $\text{char}(K) \neq 2$, here $\beta(x,x) \equiv 0$, so β does not determine q.) The assumption that q is *non-degenerate* means that if $q(x) = 0$ and $\beta(x,y) = 0$ for all y, then $x = 0$. (Note that the quadratic form $x_1^2 + \cdots + x_n^2$ on K^n

is degenerate in characteristic 2.) The group $O(V)$ of *orthogonal transformations* is defined as the subgroup of elements of $\mathrm{GL}(V)$ which preserve q. In characteristic 2 all elements of $O(V)$ have determinant 1. However, when $\dim(V)$ is even, the connected component of the identity is a subgroup of index 2 in $O(V)$; it is called the *special orthogonal group* $\mathrm{SO}(V)$, and it consists of those elements of $O(V)$ which can be written as a product of an even number of reflections (see for example [37, p. 160]). Since up to linear base change all non-degenerate quadratic forms are equivalent (K is assumed to be algebraically closed here), we speak about the group $\mathrm{SO}(V)$ without explicit reference to q.

Now let W be the four-dimensional space $M(2,K)$ endowed with the non-degenerate quadratic form $A \mapsto \det(A)$. The product of special linear groups $SL(2,K) \times SL(2,K)$ acts on W by

$$(g,h)A := gAh^{-1} \quad (A \in M(2,K), g,h \in SL(2,K)).$$

This action gives a homomorphism $\mathrm{SL}(2,K) \times \mathrm{SL}(2,K) \to \mathrm{GL}(W)$, whose image is $\mathrm{SO}(W)$ (see [37, Exercise 12.21]). Therefore we have the isomorphism

$$K_{2,m}^{SL(2,K)\times SL(2,K)} \cong K[W^m]^{\mathrm{SO}(W)},$$

where $W^m = W \oplus \cdots \oplus W$ (m-copies). Furthermore, denoting by I the 2×2 identity matrix, the injection $M_{2,m} \to M_{2,m+1}$, $(A_1,\dots,A_m) \mapsto (A_1,\dots,A_m,I)$ induces a homomorphism $K_{2,m+1} \to K_{2,m}$. By [2, Proposition 4.1] or [6, Theorem 3.2] this restricts to a surjective homomorphism $K_{2,m+1}^{\mathrm{SL}(2,K)\times\mathrm{SL}(2,K)} \to R_{2,m}$. Thus there is a surjective homomorphism

$$\kappa : K[W^{m+1}]^{\mathrm{SO}(W)} \to R_{2,m}.$$

Since κ is given by specialization of certain variables to 0 or 1, the degree of $\kappa(f)$ is not greater than the degree of f for all $f \in K[W^{m+1}]^{\mathrm{SO}(W)}$. So Corollary 4.1 can be interpreted in terms of special orthogonal invariants.

Corollary 7.1. *Let W be a four-dimensional K-vector space endowed with a non-degenerate quadratic form. If* $\mathrm{char}(K) = 2$, *then $K[W^m]^{\mathrm{SO}(W)}$, the algebra of polynomial invariants of the special orthogonal group acting on m-tuples of vectors from W, is not generated by its elements of degree $< m-1$.*

The above result is in sharp contrast with the case $\mathrm{char}(K) \neq 2$. Then by [1] the functions $(v_1,\dots,v_m) \mapsto \beta(v_i,v_j)$, $1 \leqslant i \leqslant j \leqslant n$, and $\det([v_{i_1}\dots v_{i_n}])$, $1 \leqslant i_1 < \cdots < i_n \leqslant m$ (here we think of $v_1,\dots,v_m$ as elements in K^n, and $[v_{i_1},\dots,v_{i_n}]$ is the $n\times n$ matrix whose columns are $v_{i_1},\dots,v_{i_n}$) generate $K[V^m]^{\mathrm{SO}(V)}$ ($n = \dim(V)$ is arbitrary). By Corollary 7.1 this (or some similar) uniform description of special orthogonal invariants does not work in characteristic 2. (We mention that the ring of invariants of the orthogonal group preserving the quadratic form $x_1^2 + \cdots + x_n^2$ is described in [33] for the case $\mathrm{char}(K) = 2$, and it turns out that it is generated by elements of degree 1 and 2.)

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