ELSEVIER

SCIENCE DIRECT®

PHYSICS LETTERS B

Physics Letters B 589 (2004) 147–154

www.elsevier.com/locate/physletb

Intersecting $M2$- and $M5$-branes

K. Lechner [a,b]

[a] *Dipartimento di Fisica, Università degli Studi di Padova, Via F. Marzolo 8, 35131 Padova, Italy*
[b] *Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Via F. Marzolo 8, 35131 Padova, Italy*

Received 5 March 2004; accepted 16 March 2004

Available online 27 April 2004

Editor: L. Alvarez-Gaumé

Abstract

If an $M2$-brane intersects an $M5$-brane the canonical Wess–Zumino action is plagued by a Dirac-anomaly, i.e., a non-integer change of the action under a change of Dirac-brane. We show that this anomaly can be eliminated at the expense of a gravitational anomaly supported on the intersection manifold. Eventually we check that the last one is cancelled by the anomaly produced by the fermions present. This provides a quantum consistency check of these intersecting configurations.

PACS: 11.15.-q; 11.10.Kk; 11.30.Cp

Keywords: Dirac-branes; Anomalies; Chern-kernels

1. Introduction and summary

An $M2$- and an $M5$-brane form an electromagnetically dual pair of branes in eleven dimensions, and since $3+6-11$ is a negative number their world-volumes have generically an empty intersection. However, for exceptional brane configurations it can happen that their intersection is non-empty. Their intersection manifold $\Sigma \equiv M2 \cap M5$ can then be a manifold of dimensions $d = 0, 1, 2$ or 3. An analysis of the *quantum*-consistency of such intersections, a special case of so-called *non-transversal* intersections between two generic manifolds [1,2], is the main topic of this Letter. The are two types of quantum inconsistencies we will have to worry about: (1) gravitational ABBJ-anomalies and (2) Dirac-anomalies, i.e., (non-integer) changes of the action under a change of the Dirac-brane. As we will see these two types of anomalies are intimately related.

The relevance for M-theory of the exceptional configurations considered in this Letter, stems from the fact that eleven-dimensional supergravity admits indeed classical susy-preserving solutions, that can be interpreted as an $M2$-brane intersecting with an $M5$-brane [3–8]. These solutions are typically localized only in the *common* transverse directions of the two branes, i.e., the currents of the branes are δ-functions only in the common transverse coordinates. There exist also susy-preserving (implicit) solutions where one of the two branes is fully localized and the other is localized only in the common transverse directions [5,9]. In absence of a complete classification of all

E-mail address: kurt.lechner@pd.infn.it (K. Lechner).

0370-2693
doi:10.1016/j.physletb.2004.03.056

possible solutions, the existence of solutions where both branes are fully localized is still an open question.

Nevertheless, in this Letter we assume that from a quantum point of view both branes are fully localized. The fundamental reason for this assumption is that since $M2$ and $M5$ are dual objects, only if both currents are δ-functions on the corresponding worldvolumes there exists a consistent minimal coupling of the branes to each other, because only in this case the charge is *locally* integer and, therefore, the Dirac-brane unobservable, [10,11]. The secondary reason is that only branes with a δ-like support represent a universal type of charge distribution. The situation is similar to the $D = 4$ Julia–Zee dyons [12]. These dyons represent semi-classical solutions of the Georgi–Glashow model, whose magnetic charge is fully localized (point-like localization) while their electric charge is smeared out. Nevertheless, at the level of second quantized quantum field theory the Julia–Zee dyons appear with fully localized magnetic and electric charges [13].

While this point deserves clearly further investigation, here we take a pragmatic point of view and assume that both branes, and therefore also their possible intersection, carry well-defined worldvolumes and that the associated currents are δ-functions supported on those worldvolumes.

For generic configurations ($\Sigma = \emptyset$) the two branes are at a finite non-vanishing distance and their classical dynamics is trivially free from relative short-distance singularities, i.e., singularities due to their mutual interaction. In this case a Dirac-brane can (must) be used to describe their dynamics, and the minimal-coupling Wess–Zumino term describing the mutual interaction is independent of the Dirac-brane $\bmod 2\pi$, if Dirac's quantization condition holds.

If, on the other hand, the configuration is exceptional ($\Sigma \neq \emptyset$) the two branes stay at zero distance. In this case the mutual interaction is plagued by short-distance singularities and we will see that the canonical minimal WZ-term becomes *Dirac-brane-dependent*, i.e., it carries a (non-integer) Dirac-anomaly. In this Letter we show that for such configurations the recently developed Chern-kernel approach [11,14,15] allows to write a new (manifestly) *Dirac-brane-independent* WZ-term. However, this new WZ-term turns out to be plagued by an inflow gravitational anomaly—supported on Σ—if $d = 0$ or 2, while it is anomaly free if $d = 1$ or 3. A crucial ingredient for its construction is the new "descent-identity" (3.1).

Eventually we show that for $d = 0, 2$ the inflow gravitational anomaly on Σ is cancelled by the quantum anomaly produced by the fermions living on it. This means that the total classical + quantum effective action is (1) free from gravitational anomalies and (2) Dirac-brane-independent.

The canonical and new WZ-terms differ by a local counterterm supported on Σ, that maps the Dirac-anomaly in the gravitational anomaly, whose construction will be outlined in the concluding section.

For the anomaly cancellation mechanism of open $M2$-branes ending on $M5$-branes see [11,16].

2. Chern-kernels and Dirac-branes

In presence of a closed $M2$- and a closed $M5$-brane the Bianchi identity and equation of motion for the four-form fieldstrength of eleven-dimensional supergravity amount to[1]

$$dH = J_5, \tag{2.1}$$

$$d * H = \frac{1}{2} H H + h J_5 + J_8, \tag{2.2}$$

where J_8 (J_5) is the δ-function supported Poincaré-dual form of the electric (magnetic) brane worldvolume $M2$ ($M5$), i.e., its current; $h = db + B|_{M5}$, where b is the chiral two-form on $M5$ and B is the potential for H. The brane tensions are set to $T_{M2} = T_{M5} = 2\pi$.

In absence of the $M2$-brane ($J_8 = 0$) the basic ingredient for the construction of a consistent WZ-term for this system (see (2.6)) is the Chern-kernel. We recall now briefly the essential features of this construction [15], concentrating on the main properties of the Chern-kernel, the details of the resulting WZ-term itself, S^A_{WZ}, being unessential for what follows.

To write an action for the system above one must first solve (2.1) in terms of a potential, introducing a four-form antiderivative K for the magnetic cur-

[1] For simplicity we omit in (2.2) the gravitational curvature polynomial X_8, which corrects eleven-dimensional supergravity by the term $\int B X_8$ [17].

rent J_5,

$$dK = J_5, \qquad H = dB + K. \tag{2.3}$$

This solution is subject to the transformations (called Q-transformations in the following)

$$K' = K + dQ, \qquad B' = B - Q, \quad Q|_{M5} = 0. \tag{2.4}$$

These transformations leave the curvatures H and h invariant, and in writing an action one must ensure that this invariance, Q-invariance, remains preserved.

As shown in [11,15], the r.h.s. of (2.2) becomes a well-defined closed form and there exists a Q-invariant action if one chooses as solution for $dK = J_5$ a four-form Chern-kernel,

$$K = \frac{1}{4(4\pi)^2}\varepsilon^{a_1\cdots a_5}\hat{y}^{a_1}\mathcal{F}^{a_2a_3}\mathcal{F}^{a_4a_5},$$
$$\mathcal{F}^{ab} = F^{ab} + D\hat{y}^a D\hat{y}^b, \tag{2.5}$$

where the y^a $(a = 1,\ldots,5)$ are normal coordinates on $M5$, $\hat{y}^a = y^a/|\vec{y}|$, $D\hat{y}$ is the covariant differential w.r.t. the normal bundle $SO(5)$-connection A, and $F = dA + AA$ is its curvature. This kernel, although being invariant near $M5$, is not unique but subjected to the Q-transformations (2.4) [11]. For what follows it is important to notice that K is singular on the whole $M5$, because $\hat{y}^a$ does not admit limit when y^a goes to 0, while the three-form Q is regular and has, actually, vanishing pullback on $M5$. The four-form K can be seen as a kind of generalized Coulomb-like field (inverse-power-like singularities), or also as an angular form [18]. We recall also that in normal coordinates the $M5$-brane current reads $J_5 = dy^1\cdots dy^5\,\delta^5(y)$.

In absence of the $M2$-brane one can then write down a WZ-term giving rise to the equation of motion (2.2). It is convenient to write it as an integral over a twelve-dimensional manifold M_{12} whose boundary is the eleven-dimensional target space M_{11}, of a *closed and Q-invariant* twelve-form,[2]

$$S^A_{\mathrm{WZ}} = 2\pi\int\limits_{M_{12}} L^A_{12},$$
$$L^A_{12} = \frac{1}{6}HHH + \frac{1}{2}h\,dh\,J_5 + \frac{1}{24}P_7^{(0)}J_5, \tag{2.6}$$

[2] Eventually, to get (2.2) one has to take into account also the kinetic Born–Infeld-type action for h, see [11].

where $P_8 = dP_7^{(0)}$ is the second Pontrjagin form of the normal bundle of $M5$. We remember that the property $dL^A_{12} = 0$ ensures that, in absence of topological obstructions, S^A_{WZ} does not depend on the particular M_{12} chosen. Writing it in this way the WZ is manifestly Q-invariant, depending only on H and h, and in [15] it has been shown that L^A_{12} is a closed form and that S^A_{WZ} cancels the residual normal bundle $SO(5)$-anomaly localized on the $M5$-brane, [19].

As we observed already, for what follows the detailed form of L^A_{12} is irrelevant; what is crucial is that its consistency relies heavily on the presence of the Chern-kernel and on the corresponding solution $H = dB + K$ of the Bianchi-identity $dH = J_5$.

In presence of an $M2$-brane, $J_8 \neq 0$, one can write the WZ-term as

$$S_{\mathrm{WZ}} = S^A_{\mathrm{WZ}} + S^B_{\mathrm{WZ}} = 2\pi\int\limits_{M_{12}}\left(L^A_{12} + L^B_{12}\right),$$
$$dL^B_{12} = 0, \quad L^B_{12}\ Q\text{-invariant}, \tag{2.7}$$

where L^B_{12} must describe (1) the minimal interaction of the $M2$-brane with supergravity and (2) the mutual interaction between $M2$ and $M5$. The first interaction is canonical and corresponds to a contribution to L^B_{12} given by $dB\,J_8 = d(BJ_8)$; the presence of the second is needed because $dB\,J_8$, although being closed, is not Q-invariant. A Q-invariant completion could be achieved by adding the mutual interaction term KJ_8, leading to $dB\,J_8 + KJ_8 = HJ_8$, but this is no longer a closed form. Eventually one should have

$$L^B_{12} = HJ_8 + \cdots, \tag{2.8}$$

where the missing terms have to be (1) Q-invariant, (2) B-independent and (3) such that L^B_{12} becomes a closed form. Our main problem consists therefore in figuring out to what the missing terms correspond to.

If $M2$ and $M5$ do not intersect there is, of course, a standard procedure for writing down the missing terms above, that involves an (electric) Dirac-brane for $M2$ i.e., a four-surface D_4 whose boundary is $M2$, $\partial D_4 = M2$. Denoting the δ-function on D_4 with W_7, a seven-form, we have

$$J_8 = dW_7,$$

and one can perform the completion

$$L^B_{12} = HJ_8 - J_5W_7 = d(HW_7), \tag{2.9}$$

which satisfies all the above requirements. Of course, under a change of Dirac-brane the WZ-action

$$S^B_{\rm WZ} = 2\pi \int\limits_{M_{11}} H W_7 = 2\pi \int\limits_{D_4} H \tag{2.10}$$

changes by an integer multiple of 2π, since H has integer integrals over any closed four-manifold that does not intersect $M5$.[3]

From a twelve-dimensional point of view independence of the Dirac-brane is manifest since the Dirac-brane-dependent term $J_5 W_7$ has integer integrals over arbitrary (closed or open) manifolds.

On the other hand, if the intersection manifold Σ is non-empty the WZ-action $\int_{D_4} H$ becomes Dirac-brane-*dependent*. Indeed, under a change of Dirac-brane it changes by

$$\int\limits_{D_4'} H - \int\limits_{D_4} H = \int\limits_{S_4} H = \int\limits_{S_4} K,$$

where S_4 is a closed four-manifold. But since $M2$ intersects $M5$ also S_4 intersects $M5$ and therefore part of the flux of K stays in S_4 and part stays outside. This means that $\int_{S_4} K$ is no longer integer, and it represents a Dirac-anomaly.

From a twelve-dimensional point of view, the term $J_5 W_7$ can no longer be used to make L^B_{12} a closed form, because if Σ is non-empty then the product $J_5 W_7$ contains squares of δ-functions ($\delta(x)\delta(x)$) and becomes ill-defined; this is a consequence of the non-vanishing intersection of the normal bundles of the two branes, see below.

The canonical Dirac-brane construction must therefore be abandoned if $\Sigma \neq \emptyset$. In this case, since L^B_{12} must be closed, the first step to find out what the missing terms in (2.8) may be, consists in computing $d(H J_8) = d(K J_8)$. Now, the product $K J_8$ and its differential in the sense of distributions are well defined even if $\Sigma \neq \emptyset$ (for $d \neq 3$),[4] but the point is that one is not allowed to apply Leibnitz's rule to compute it. In fact, the result obtained using naively this rule, i.e., $d(K J_8) = J_5 J_8$, contains squares of δ-functions—for the same reasons as above—and it is ill-defined.

[3] For an alternative argument for Dirac-brane-independence, based on *integer* forms, see [20].

[4] The case $d = 3$, corresponding to $M2 \subset M5$, is in some sense trivial and will be solved separately below.

On the other hand, as we will show in the next section, the result of the evaluation of $d(K J_8)$ in the sense of distributions is well-defined, and it has a simple interpretation if expressed in terms of normal bundles.

3. Intersecting branes and normal bundles

Suppose that $\Sigma = M2 \cap M5$ is a closed manifold with dimension $d = 0, 1, 2$ or 3, and introduce its current J which is a closed $(11-d)$-form. For this case we will show that the unknown terms in (2.8) can be deduced from a new kind of "descent-identity"—as such formulated in thirteen dimensions—that involves the normal bundles of $M2$ and $M5$.

The normal bundles of $M2$, $M5$ and Σ, denoted by N_{M2}, N_{M5} and N_Σ, carry respectively fibers of dimensions 8, 5 and $11-d$. On Σ the bundles of $M2$ and $M5$ intersect to a bundle $\mathcal{N} = N_{M2} \cap N_{M5}$, whose fiber is of dimension $n = 5 + 8 - (11-d) = d+2$, with structure group $SO(n)$. For example, if the intersection is just a point, a (-1)-brane, then $n = 2$; and if $M2 \subset M5$ then $n = 5$ because in this case $\mathcal{N} = N_{M5}$.

If n is even we can define the Euler-form χ of the bundle $\mathcal{N}$, a form of degree n; if n is odd we take χ to be zero by definition. The Euler-forms of interest are then

$$\chi_2 = \frac{1}{4\pi} \varepsilon^{r_1 r_2} T^{r_1 r_2},$$

$$\chi_4 = \frac{1}{2(4\pi)^2} \varepsilon^{r_1 r_2 r_3 r_4} T^{r_1 r_2} T^{r_3 r_4},$$

where T^{rs} is the curvature of $\mathcal{N}$. Our descent notations are $\chi = d\chi^{(0)}$, $\delta\chi^{(0)} = d\chi^{(1)}$.

In going to thirteen dimensions we want to keep the degrees of the currents J_8, J_5, J unchanged. This implies that the worldvolumes of $M2$, $M5$ and Σ have to be extended respectively to five-, eight- and $(d+2)$-dimensional manifolds. This keeps the dimensions of the normal bundles, in particular the dimension of $\mathcal{N}$ and hence the degree of χ, unchanged. In absence of topological obstructions such extensions are always possible [15], and they were implicitly understood in the twelve-dimensional construction of the previous section.

The result of the computation we referred to above amounts then to a descent-identity between thirteen-forms, involving the δ-function on Σ and the Euler-form of $\mathcal{N}$,

$$d(K J_8) = J\chi, \quad \text{whenever } M2 \not\subset M5. \tag{3.1}$$

The proof is given in Appendix A. It is obvious that $d(K J_8)$ must be supported on Σ and hence proportional to J; the proportionality factor χ follows then essentially for invariance reasons. It is understood, as said above, that for $d = 1$ one takes $\chi = 0$. If $M2 \subset M5$ ($d = 3$) the product $K J_8$ is ill-defined; this case is in some sense trivial and it is solved separately below.

Our descent-identity is, actually, a *local* realization of the corresponding *cohomological* relation presented in [2].

4. Wess–Zumino action and anomaly cancellation

Given the above identity it is easy to complete the twelve-form (2.8) to make it a *Q-invariant and closed* form ($d \neq 3$):

$$\tilde{L}^B_{12} = H J_8 - J\chi^{(0)}, \tag{4.1}$$

where we introduced a standard Chern–Simons form through $\chi = d\chi^{(0)}$. The Wess–Zumino actions for the four possible intersection manifolds of $M5$ and $M2$ can then be written eventually as

$$\begin{aligned}\frac{1}{2\pi}\tilde{S}^B_{\mathrm{WZ}} &= \int_{M_{12}} \tilde{L}^B_{12} \\ &= \int_{M2} B + \begin{cases} \int_{M_{12}} (K J_8 - J_{11}\chi_1^{(0)}), & d = 0, \\ \int_{M_{12}} K J_8, & d = 1, \\ \int_{M_{12}} (K J_8 - J_9 \chi_3^{(0)}), & d = 2, \\ 0, & d = 3. \end{cases}\end{aligned} \tag{4.2}$$

We recall that the terms $K J_8$ are required for Q-invariance, and that the terms with the Euler-forms are needed to ensure independence of the particular M_{12} chosen. For $d = 0, 2$ the Euler-form is non-vanishing, while for $d = 1$ it vanishes. In this case $K J_8$ is indeed a closed form, see (3.1).

For $d = 3$ ($M2 \subset M5$) the product $K J_8 = K|_{M2} J_8$ is ill-defined, because K does not admit pullback on $M5$, and (3.1) is therefore not applicable. But in this case the term $\int_{M2} B$ is, actually, Q-invariant. Indeed, under a Q-transformation one would have $\int_{M2}(B' - B) = -\int_{M2} Q$, and this vanishes because Q vanishes on the whole $M5$, see (2.4). In other words, for $M2 \subset M5$ the form $\tilde{L}^B_{12} = dB J_8$ is already Q-invariant and closed; this explains the fourth line in (4.2).

From the list (4.2) one sees that for $d = 0, 2$ the WZ-action is plagued by a gravitational anomaly supported on the intersection manifold Σ, $\delta\tilde{S}^B_{\mathrm{WZ}} = -2\pi \int_\Sigma \chi^{(1)}$, corresponding to the inflow anomaly-polynomial

$$-2\pi\chi, \tag{4.3}$$

while for $d = 1, 3$ it is anomaly free.

On the other hand on Σ there are also fermions living, coming from the common reduction of the 32-component spinors ϑ^α, living on $M5$ and on $M2$. If d is even, these fermions are a section of the chiral spinor bundle lifted from $T(\Sigma) \oplus \mathcal{N}$, where $T(\Sigma)$ is the tangent bundle to Σ, and $\mathcal{N}$ is the intersection of the normal bundels, as above. For such fermions the anomaly can be computed as in [2], and the resulting polynomial reads

$$2\pi\left(\mathrm{ch}\left[S^+_{\mathcal{N}}\right] - \mathrm{ch}\left[S^-_{\mathcal{N}}\right]\right)\hat{A}\left[T(\Sigma)\right] = 2\pi\frac{\hat{A}[T(\Sigma)]}{\hat{A}[\mathcal{N}]}\chi, \tag{4.4}$$

where ch indicates the Chern character, $S^\pm_{\mathcal{N}}$ is the spin bundle lifted from $\mathcal{N}$ with $\pm$ chirality, and $\hat{A}$ is the roof genus. From this polynomial one has to extract the two-form part for $d = 0$, and the four-form part for $d = 2$. Since the Euler form is already a form of degree two and four respectively, the roof genera above contribute both with unity and the anomaly polynomial reduces precisely to $2\pi\chi$, cancelling the inflow.

This represents a quantum consistency check of the intersecting $M2/M5$ configurations considered in this Letter.

5. Concluding remarks

The anomaly cancellation mechanism presented in this Letter has a transparent meaning for $d = 2$, where Σ is the worldvolume of a closed string.

For $d = 0$ the intersection manifold Σ is just a point P and represents an instanton. In this case the bundle $\mathcal{N}$ is a two-plane centered on P, with structure

group $SO(2)$ and Abelian connection W^{rs}, and the inflow anomaly $-2\pi\int_\Sigma \chi^{(1)}$ reduces to $-\Lambda(P)$, where $\delta W^{rs} = d(\varepsilon^{rs}\Lambda)$. This anomaly has a clear meaning since the normal bundle transformations of $\mathcal{N}$ correspond just to rotations around P in the two-plane centered at P, and $\Lambda(P)$ is the variation of the polar angle φ of that plane, $\delta\varphi = \Lambda(P)$. What is more obscure is the meaning of (chiral) fermionic degrees of freedom on an instanton and the appearance of the corresponding quantum anomaly. In lack of this insight, above we took simply advantage from the fact that the index formula (4.4) makes sense also for $d = 0$.

Our strategy for constructing a consistent interaction between intersecting $M2$- and $M5$-branes assumes that the branes intersect "strictly", i.e., that they stay strictly at zero distance. An alternative strategy for describing intersecting branes would be to introduce a framing regularization, where the branes are moved at a finite distance ε from each other. For each finite ε the branes are non-intersecting and one could introduce consistently a Dirac-brane to describe their interaction, as explained in section two, see (2.10). However, for $\varepsilon \to 0$ this WZ-term, although remaining finite, would become Dirac-brane-dependent, as explained in the text.

Keeping then the branes strictly intersecting, there are two ways for writing a classical action. The first is $\tilde{S}^B_{\mathrm{WZ}}$ given in (4.2): it is (manifestly) Dirac-brane-independent but carries a gravitational anomaly (for $d = 0, 2$). The second is S^B_{WZ} given in (2.10): it is (manifestly) free from gravitational anomalies but it is plagued by a Dirac-anomaly. This means that there exists a local (in the sense of "Wess–Zumino") counterterm that maps the Dirac-anomaly in a gravitational anomaly and vice versa. Its implicit construction goes along the following lines. Starting point is an identity similar to (3.1),

$$d(K W_7) = K J_8 - J\Phi, \tag{5.1}$$

for some $(d+1)$-form Φ, which can be proven using, for example, the regularizations given in Appendix A of [11]; again, one is not allowed to use naively Leibnitz's rule. For $d = 3$ the term $K J_8$ in this identity has to be replaced by 0. The form Φ, supported on Σ, is diffeomorphism invariant, but depends on W_7, i.e., on the Dirac-brane D_4. Using (3.1) together with (5.1) one gets

$$d\Phi = \chi \Rightarrow \Phi - \chi^{(0)} = d\omega,$$

for some d-form ω on Σ; for d odd Φ is thus a closed form. It is then immediately seen that

$$\tilde{S}^B_{\mathrm{WZ}} = S^B_{\mathrm{WZ}} + 2\pi \int_\Sigma \omega.$$

The counterterm we searched for is $\int_\Sigma \omega$ and it is supported on Σ, as one may have expected. This proves in particular that the Dirac-anomaly itself is supported on Σ, in agreement with the fact that if $\Sigma = \emptyset$ then there is no Dirac-anomaly at all.

The configurations we have considered in this Letter are exceptional in that, a priori, a small perturbation makes the two branes again non-intersecting. The stability of these configurations can, however, be inferred from the existence of their classical-solution (semi-localized) counterparts of $D = 11$ Sugra, mentioned in the introduction, whose stability is guaranteed by supersymmetry. There exist indeed solutions for $d = 3$, preserving $1/2$ susy [3], and solutions for $d = 2$, preserving $1/4$ susy [4,6]. To our knowledge no solutions for $d = 1$ or $d = 0$ are yet known. The results of the present paper, indicating that intersecting $M2/M5$ configurations are quantum consistent for any value of d, suggest that also for $d = 0, 1$ supersymmetric classical solutions may exist. A dimensional reduction of the complete interacting system Sugra $+ M2 + M5$ to ten dimensions, analogous to the one of [21], may help to answer this question.

Acknowledgements

The author thanks M. Cariglia and P.A. Marchetti for useful discussions. This work is supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime.

Appendix A. Proof of the descent-identity

Since J_8 is the δ-function on $M2$, the first step in proving (3.1) consists in evaluating K restricted to

$M2$. After that one can compute

$$d(K J_8) = d(K|_{M2} J_8) = d(K|_{M2}) J_8. \tag{A.1}$$

Since away from $M5$ K is a closed form, the pullback $K|_{M2}$ is closed a part from (possible) δ-function contributions supported on Σ. This means that it is sufficient to evaluate $K|_{M2}$ in the vicinity of Σ. On Σ the normal bundle of $M5$ has n coordinates in common with the normal bundle of $M2$, precisely the ones of $\mathcal{N}$, so we can split the normal coordinates of $M5$ as $y^a = (y^r, y^i)$ $(r = 1, \ldots, n)$ $(i = n+1, \ldots, 5)$. Near Σ we have $y^r = 0$, while the y^i become $3 - d = 5 - n$ normal coordinates for Σ with respect to $M2$. Correspondingly the $SO(n)$-connection W of $\mathcal{N}$ is embedded in the $SO(5)$-connection A according to $A^{rs} = W^{rs}$, $T = dW + WW$.

We perform now the evaluation of $K|_{M2}$ near Σ, i.e., for $y^r = 0$ and keeping only terms that can give δ-function contributions when applying the differential, for each case separately.

$d = 0$. In this case Σ is a point in $D = 11$, corresponding to a two-dimensional surface in $D = 13$, and its current is an eleven-form $J = J_{11}$. The fiber of $\mathcal{N}$ is two-dimensional ($n = 2$), with structure-group $SO(2)$ and Euler-form $\chi_2(T)$. The identity to be proved is therefore

$$d(K J_8) = J_{11} \chi_2(T). \tag{A.2}$$

δ-function contributions localized at Σ are supported in $y^i = 0$ $(i = 3, 4, 5)$ and they can arise from the angular form

$$K_0 = \frac{1}{8\pi} \varepsilon^{ijk} \hat{y}^i \, d\hat{y}^j \, d\hat{y}^k,$$

since $dK_0 = d^3 y \, \delta^3(y)$. It is then easy to evaluate (2.5) for $y^r = 0$ $(r = 1, 2)$ and to extract the contribution proportional to K_0,

$$K|_{M2} = \frac{1}{4\pi} \varepsilon^{rs} \left(T^{rs} + \left(\delta^{ij} - 3 \hat{y}^i \hat{y}^j \right) A^{ir} A^{js} \right) K_0.$$

Since one has $d[(\delta^{ij} - 3\hat{y}^i \hat{y}^j) K_0] = 0$, when taking the differential only the first term contributes with a δ-function and one gets

$$d(K|_{M2}) = d^3 y \, \delta^3(y) \chi_2(T).$$

(A.2) follows then from (A.1) and from $d^3 y \, \delta^3(y) \times J_8 = J_{11}$.

$d = 1$. In this case Σ is a worldline in $D = 11$ and its current is a ten-form $J = J_{10}$. The fiber of $\mathcal{N}$ is three-dimensional ($n = 3$) and its Euler-form vanishes. The descent-identity reduces therefore to

$$d(K J_8) = 0.$$

As above one should extract from $K|_{M2}$, taken at $y^r = 0$ $(r = 1, 2, 3)$, contributions proportional to the angular form, that is now $K_0 = \frac{1}{2\pi} \varepsilon^{ij} \hat{y}^i \, d\hat{y}^j$ $(i, j = 4, 5)$, $dK_0 = d^2 y \, \delta^2(y)$. But since K contains only odd powers of $\hat{y}$, and K_0 is even in $\hat{y}$, in $K|_{M2}$ the angular form K_0 appears always multiplied by odd powers of $\hat{y}$, and taking the differential the current $d^2 y \, \delta^2(y)$ can never show up. This implies that $d(K|_{M2}) = 0$.

$d = 2$. In this case Σ is a two-surface in $D = 11$, and its current is a nine-form $J = J_9$. The fiber of $\mathcal{N}$ is four-dimensional ($n = 4$) with structure-group $SO(4)$ and Euler-form $\chi_4(T)$. The identity becomes then

$$d(K J_8) = J_9 \chi_4(T).$$

In this case one has $r = 1, 2, 3, 4$ and $i = 5$, and it is straightforward to evaluate (2.5) at $y^r = 0$,

$$\begin{aligned} K|_{M2} &= \frac{1}{4(4\pi)^2} \frac{y^5}{|y^5|} \varepsilon^{r_1 r_2 r_3 r_4} T^{r_1 r_2} T^{r_3 r_4} \\ &= \frac{y^5}{2|y^5|} \chi_4(T). \end{aligned}$$

The differential of the "angular-form" is here simply $d(y^5/2|y^5|) = dy^5 \, \delta(y^5)$, and one gets

$$d(K|_{M2}) = dy^5 \, \delta\left(y^5\right) \chi_4(T).$$

One concludes then as in the case $d = 0$.

$d = 3$. In this case we have $n = 5$, $J = J_8$ and $\chi = 0$, and the r.h.s. of (3.1) vanishes. On the other hand, since $\Sigma = M2 \subset M5$, the pullback $K|_{M2}$ would require to evaluate K for $y^a \to 0$. But this limit depends on the direction $\hat{y}^a = V^a(\sigma)$ one chooses to approach $M2$ at each point σ, and $K|_{M2}$ is ill-defined. Notice however, that if one performs the limit along an arbitrary but fixed vector field $V^a(\sigma)$, then the resulting four-form $(K|_{M2})^V$ can be shown to be closed. But such a definition saves the identity (3.1) *only formally*, because $K|_{M2}$, and hence $K J_8$, acquires a dependence on an unphysical vector field, and it cannot be used in the WZ-action.

References

[1] R. Bott, L.W. Tu, Differential Forms in Algebraic Geometry, Springer, Berlin, 1978.
[2] Y.K.E. Cheung, Z. Yin, Nucl. Phys. B 517 (1998) 69.
[3] J.M. Izquierdo, N.D. Lambert, G. Papadopoulos, P.K. Townsend, Nucl. Phys. B 460 (1996) 560.
[4] A.A. Tseytlin, Nucl. Phys. B 475 (1996) 149.
[5] A.A. Tseytlin, Class. Quantum Grav. 14 (1997) 2085.
[6] J.P. Gauntlett, D.A. Kastor, J. Traschen, Nucl. Phys. B 478 (1996) 544.
[7] M.S. Costa, Nucl. Phys. B 490 (1997) 202.
[8] N. Ohta, J.G. Zhou, Int. J. Mod. Phys. A 13 (1998) 2013.
[9] D. Youm, hep-th/9902208;
A. Gomberoff, D. Kastor, D. Marolf, J. Traschen, Phys. Rev. D 61 (2000) 024012;
S. Arapoglu, N.S. Deger, A. Kaya, Phys. Lett. B 578 (2004) 203.
[10] R.A. Brandt, F. Neri, D. Zwanziger, Phys. Rev. D 19 (4) (1979) 1153.
[11] K. Lechner, P.A. Marchetti, Nucl. Phys. B 672 (2003) 264.
[12] B. Julia, A. Zee, Phys. Rev. D 11 (1975) 2227.
[13] J. Froehlich, P.A. Marchetti, Nucl. Phys. B 551 (1999) 770.
[14] S. Chern, Ann. Math. 45 (1944) 747;
S. Chern, Ann. Math. 46 (1945) 674.
[15] K. Lechner, P.A. Marchetti, M. Tonin, Phys. Lett. B 524 (2002) 199.
[16] P. Brax, J. Mourad, Phys. Lett. B 416 (1998) 295.
[17] M.J. Duff, J.T. Liu, R. Minasian, Nucl. Phys. B 452 (1995) 261.
[18] D. Freed, J.A. Harvey, R. Minasian, G. Moore, Adv. Theor. Math. Phys. 2 (1998) 601.
[19] E. Witten, J. Geom. Phys. 22 (1997) 103.
[20] K. Lechner, P.A. Marchetti, JHEP 0101 (2001) 003.
[21] M. Cariglia, K. Lechner, Phys. Rev. D 66 (2002) 045003.