§ 1. We shall prove (in § 2) the $P$-adic analogue of Tijdeman's result [7, theorem 3]. A positive lower bound for $\sum_{k=1}^{3}|\alpha_k|_P$, $\alpha_k$, $\alpha$ algebraic, $t$ transcendental, is determined (in § 3) explicitly (except for an absolute constant) in terms of the maximum of the degrees and heights of $\alpha_1$, $\alpha_2$, $\alpha_3$. Theorem 1 is applied to prove theorem 2.

Let $P$ be a positive rational prime number. Denote by $\mathbb{Q}$ the field of rational numbers. Let $\mathbb{Q}_P$ be the completion of $\mathbb{Q}$ with respect to the $P$-adic valuation $|\cdot|_P$ on $\mathbb{Q}$. Let $\mathbb{T}_P$ be the completion of the algebraic closure of $\mathbb{Q}_P$ and denote its valuation by $|\cdot|_P$ also.

§ 2. Theorem 1. Let $m$, $n$ and $t$ be positive integers. Assume that $m, n > 2$ and set $r = mt$. Let $b_1, \ldots, b_m \in T_P$, $b_i$ distinct and $|b_i|_P < 1$, $i = 1, \ldots, m$. Let $a_1, \ldots, a_n \in T_P$. Let $w_1, \ldots, w_n \in T_P$, $w_i$ distinct and $|w_i|_P < P^{-(1/(P-1)+\varepsilon)}$ where $\varepsilon > 0$ is an arbitrary fixed constant. Set

$$a_0 = \min_{1 \leq i < j \leq n} |w_i - w_j|_P, \quad b_0 = \min_{1 \leq i < j \leq m} |b_i - b_j|_P, \quad A = \max_{1 \leq i \leq m} |a_i|_P.$$

$$g(z) = \sum_{k=1}^{m} a_k e^{w_k z} \quad \text{and} \quad E = \max_{0 \leq q < t} |g^{(q)}(b_0)|_P.$$

Assume that

$$r > \frac{30}{\varepsilon} \left( \left( \frac{1}{P-1} + \varepsilon \right) (n-1) + 1 \right).$$

Then

$$A < EP^{(1/3)} e^{P^{t-1} b_0 r} a_0^{-n} A^t.$$

The proof of theorem 1 depends on the following lemmas.

Lemma 1. Let $g(z)$ be defined as in theorem 1. Let $R_1, R_2 \in T_P$ satisfying

$$P^{(1/3)} < |R_1|_P < |R_2|_P < P^{(2/3)}.$$

1) I am very thankful to Professor K. Ramachandra for encouragement. My thanks are also due to Professor R. Tijdeman for going through an earlier draft of manuscript and for suggesting me some changes.
Then

\[
\max_{|z|_p \leq |R_1|_p} |g(z)|_P < \sqrt{2} P^{((1/P-1)+\varepsilon)(n-1)} \max_{|z|_p \leq |R_1|_p} |g(z)|_P.
\]

(See [6, appendix]). This is a $P$-adic analogue of a result of Balkema and Tijdeman [2, p. 10].

**Lemma 2.** Let $g(z)$ be defined as in theorem 1. Then

\[
\max_{0 < j < n} |g^{(j)}(0)|_P \geq \max_{1 < i < n} \left( |a_i|_P \prod_{i=1}^{n} |w_i - w_i|_P \right).
\]

(For proof, see [3, p. 296].)

**Proof of theorem 1. Case I: When $t = 1$.**

Choose $R_1, R_2 \in T_P$ such that

\[
P(\varepsilon^{1/5}) < |R_1|_P < P(\varepsilon^{1/6}), P(\varepsilon^{1/5}) < |R_2|_P < P(\varepsilon^{1/6}).
\]

(We shall make use of Schnirelmann integral and we refer the readers to the paper of Adams [1, Appendix] for its definition and properties).

For every $z$ such that $|z|_P = |R_1|_P$, consider the integral

\[
\int_{0, R_2} g(z; \xi) \prod_{i=1}^{m} \frac{(z - b_i)}{(\xi - b_i)} \ d\xi = g(z) + \sum_{j=1}^{n} \frac{g(b_j)}{(b_j - z)} \prod_{i=1}^{m} \frac{(z - b_i)}{(b_j - b_i)}.
\]

Let $R_3 = R_2 |R_1$. Hence $|R_3|_P = |R_2 R_1|_P > 1$. For the $p$-adic value $|f|_P$ of the L.H.S. of (3), we deduce from lemma 1, (1) and (2)

\[
|f|_P < |R_3|_P^{-m} \max_{|z|_P \leq |R_2|_P} |g(z)|_P < \sqrt{2} |R_3|_P^{-m} P^{((1/P-1)+\varepsilon)(n-1)} \max_{|z|_P \leq |R_1|_P} |g(z)|_P < \frac{1}{\sqrt{2}} P^{((1/P-1)+\varepsilon)(n-1)+1-1/(30)} \max_{|z|_P \leq |R_1|_P} |g(z)|_P
\]

(4)

Further notice that for every $z$ such that $|z|_P = |R_1|_P$, the $P$-adic value $|\sum_j f|_P$ on the R.H.S. of (3) does not exceed

\[
\left\{ \begin{array}{l}
E |R_1|_P b_0^{-m} \\
< EP(\varepsilon)^m b_0^{-m},
\end{array} \right.
\]

(5)

Combining (3), (4) and (5), we get

\[
\max_{|z|_P \leq |R_1|_P} |g(z)|_P < EP(\varepsilon)^m b_0^{-m}.
\]

(6)
Now for $0 < J < n$, we have

$$|g^{(J)}(0)|_p = \left| J! \int_{\mathbb{A}_1} \frac{g(z)}{z^{J+1}} \, dz \right|_p < \max_{|h_p| < |\mathbb{A}_1|_p} |g(z)|_p.$$

By lemma 2, we get

$$\max_{0 < J < n} |g^{(J)}(0)|_p > A \alpha^n.$$

Combining (8), (7) and (6), we get

$$A \leq EP^{(1/2)}b_{-m}a_{-n}.$$

**Case II. When $t > 1$.**

Write the sum on the R.H.S. of (3) as the sum of integrals. (See [3, p. 292, formula No. 241 and use $|1/n|_p < P^{1/2n}$] while majorising the sum. And proceed exactly as in case 1.

|§ 3. Theorem 2. Let $t$ be transcendental in $T_\alpha$ satisfying $|t|_p < 1$ and for each triple $(l, m, n) \neq (0, 0, 0)$ of rational integers with $|l|, |m|, |n| < q$,

$$|l + mt + nt^2|_p > \exp(-F(q))$$

where $F(q)$ is a positive real valued increasing function defined over all natural numbers and $F(q)$ tends to infinity with $q$. Let $\alpha \neq 1$ be an algebraic number in $T_\alpha$ satisfying

$$|\alpha - 1|_p < P^{-1/2n}.$$ Then

$$\min (\sum_{k=1}^{3} |\alpha^k - \alpha_k|_p) \exp(-c_{13} D^{14} \log(3DH) b'_{D, H})$$

with $b'_{D, H} = [c_{14} D^{14} \log(3DH)]^3$]

($c_{13}$ and $c_{14}$ are positive constants depending only on $\alpha$, $t$ and $P$) where the minimum is taken over all the triples $(\alpha_1, \alpha_2, \alpha_3)$ of algebraic numbers of $T_\alpha$ satisfying i) the height of $\alpha_1, \alpha_2, \alpha_3 < H(>1)$ ii) the degree of the field obtained by adjoining to $\mathbb{Q}$ the numbers $\alpha_1, \alpha_2, \alpha_3$ do not exceed $D$.

**Proof of theorem 2.** Let $K$ be a finite extension of $\mathbb{Q}$. Let $\|s, v \in S_\infty$ be the set of all the archimedian valuations on $K$. Let $a \in K$. Define

$$\|a\| = \max_{v \in S_\infty} |a|_v.$$

We shall require the following lemma which is a generalisation of a lemma due to Siegel.

**Lemma 3.** Let $K$ be a finite extension of degree $N$ over $\mathbb{Q}$ with ring of integers $I_K$. Consider the system of linear equations:

$$y_k = a_{k, 1} x_1 + \ldots + a_{k, q} x_q \quad (k = 1, \ldots, p)$$
where \( p < q \) and \( a_{i,j} \in I_k \) satisfying \( ||a_{i,j}|| < A(>1) \) for all \((i,j)\). Then the above linear forms have a non-trivial solution \( x_1, \ldots, x_q \) in rational integers satisfying
\[
|x_k| < 1 + (2qA)p^{N(N+1)/2q - p^N(N+1)} \quad k = 1, \ldots, q
\]
provided \( 2q > pN(N+1) \).

(See Ramachandra [4, p. 16].)

It is no loss of generality to assume that \( |x_i|_F < 1, \quad i = 1, 2, 3 \) since otherwise \( \sum_{i=1}^3 |x_i|^k - \alpha_k < 1 \). It is sufficient to prove the theorem when \( 0 < |A - 1|_F < P^{-(1/P-1+\epsilon)} \) where \( \epsilon > 0 \) is an arbitrary fixed constant. Assume that \( H \) is sufficiently large. (i.e. \( H > H_0 \) and \( H_0 \) independent of \( D \)). Denote by \( c_1, c_2, \ldots \) positive constants \( > 1 \) depending only on \( \alpha, \tau, p \) and \( \epsilon \). Set \( \alpha^k = \alpha_k + \epsilon_k \), \( k = 1, 2, 3 \) and assume that \( |\epsilon_k|_F < 1, \quad K = 1, 2, 3 \). Set
\[
|\epsilon_4|_F = \max \{ |\epsilon_1|_F, |\epsilon_2|_F, |\epsilon_3|_F \}.
\]

Consider the following auxiliary function
\[
\phi(z) = \sum_{\lambda_1=0}^L \sum_{\lambda_2=0}^L p(\lambda_1, \lambda_2) \alpha^{(\lambda_1+\lambda_2)z}, \quad |z|_F < P^\epsilon
\]
where \( p(\lambda_1, \lambda_2) \) are rational integers, not all zero, to be determined under the conditions
\[
q(l, m, n) = 0, \quad 1 < l < Q, \quad 1 < m < Q, \quad 1 < n < Q
\]
where
\[
q(l, m, n) = \sum_{\lambda_1=0}^L \sum_{\lambda_2=0}^L p(\lambda_1, \lambda_2) \alpha^{l_1} \alpha^{m_1} \alpha^{n_1} \alpha^{l_2} \alpha^{m_2} \alpha^{n_2}
\]
\((L \text{ and } Q \text{ are large integers to be suitably chosen})\).

Set
\[
K = Q(x, \alpha_1, \alpha_2, \alpha_3) \quad \text{and} \quad [K: Q] = N.
\]

(9) is a set of \( Q^3 \) equations in \((L+1)^2\) variables \( p(\lambda_1, \lambda_2) \). Assume that
\[
(L + 1)^2 > N(N+1) Q^3
\]
By lemma 3, there exist rational integers \( p(\lambda_1, \lambda_2) \), not all zero, satisfying (9) and
\[
|p(\lambda_1, \lambda_2)| < D^5 L Q H^{13/2} Q,
\]
Further notice that for all positive integers \( l, m, n \)
\[
|\phi(l + mt + nl^2) - q(l, m, n)|_F < |\epsilon_4|_F.
\]
Since \( p(\lambda_1, \lambda_2) \) are not all zero, assume that \( p(\lambda_1', \lambda_2') \neq 0 \) for some \((\lambda_1', \lambda_2')\). Define
\[
\phi_1(z) = \frac{1}{p(\lambda_1', \lambda_2')} \phi(z).
\]
Set
\[ c_1 = \frac{30}{\epsilon} \left( \frac{P}{P-1} + \epsilon \right), \quad Q^* = \lfloor c_1 L^{2/3} \rfloor + 1, \quad c_2 = 4 + 8c_1^2(1 + 4\epsilon \log P) - 4 \log(\log \alpha|p). \]

Assume that
\[ |\varepsilon_4|_p < D^{-5LQ} H^{-13LQ} \exp(-c_2 L^2 F(L)). \]

We claim that there exists a triple \((l, m, n)\), \(Q < \max (l, m, n) < Q^*\), such that
\[ q(l, m, n) \neq 0. \]

If not, then \(q(l, m, n) = 0\), \(1 < l, m, n < Q^*\) and so
\[ |\phi(l + mt + nt^2)|_p < |\varepsilon_4|_p, \quad 1 < l, m, n < Q^*. \]

Further
\[ |\phi(l + mt + nt^2)|_p < D^{5LQ} H^{13LQ}|\varepsilon_4|_p, \quad 1 < l, m, n < Q^*. \]

In the notation of theorem 1, set \(g(z) = \phi_1(z), n = (L + 1)^2, m = Q^3, t = 1, r = Q^3, \quad w_t = (\lambda_1 + \lambda_2 t) \log \alpha\) for some \(i_1, i_2, 0 < i_1, i_2 < L, \quad b_t = l_1 + m_1 t + n_1 t^2, 1 < l_1, m_1, n_1 < Q^*, a_0 > \exp(-F(L) \cdot (\log \alpha|p), b_0 > \exp(-F(Q^*)), A > 1\) and \(E < D^{8LQ} H^{13LQ}|\varepsilon_4|_p\). Notice that (1) is satisfied and so by theorem 1, we get
\[ 1 < A < 4P^{1/8}Q^3 \cdot \exp(Q^3 F(Q^*)) \cdot \exp(4L^2 F(L))D^{5LQ} H^{13LQ}|\varepsilon_4|_p|\log \alpha|_p - 4L^2 \]
\[ < \exp(c_2 L^2 F(L)) D^{5LQ} H^{13LQ}|\varepsilon_4|_p < 1 \]
which is a contradiction. This proves our claim.

Take a triple \((l_0, m_0, n_0)\) with \(\max (l_0, m_0, n_0) = Q_1\) subject to the following: i) \(Q_1 < Q^*\) ii) \(q(l_0, m_0, n_0) \neq 0\) iii) \(q(l, m, n) = 0\) for all triples \((l, m, n)\) such that \(\max (l, m, n) < Q_1\). Clearly \(Q_1 > Q\).

Notice that \(q(l_0, m_0, n_0)\) is a non-zero algebraic number whose denominator (say \(B\)) \(< H^{8LQ_1}\) and \(0 < ||Bq(l_0, m_0, n_0)|| < D^{10LQ_1} H^{26LQ_1}\).

By product formula for \(K\), we get
\[ 0 < |Bq(l_0, m_0, n_0)|_p < D^{-10NLQ_1} H^{-26NLQ_1}(DH)^{-26NLQ_1}. \]

Assume that
\[ |\varepsilon_4|_p < (DH)^{-26NLQ_1}. \]

From (11), (13) and (14), we get
\[ |\phi(l_0 + m_0 t + n_0 t^2)|_p > (DH)^{-26NLQ_1}. \]

Now we approximate \(\phi(l_0 + m_0 t + n_0 t^2)\) from above.

Choose \(R \in T_p\) such that \(1 < |R|_p < P^*\).
Consider the integral
\[
\int_{\partial I} \frac{\phi(z)z}{(z-l_0-m_0 t-n_0 t^2)} \prod_{(l,m,n) \leq Q_1 \text{ and } \max(l,m,n) < Q_1} \left( \frac{l_0 + m_0 t + n_0 t^2 - l - m t - n t^2}{z-l-m t-n t^2} \right) dz
\]
\[
= \phi(l_0 + m_0 t + n_0 t^2) + \sum_{(l_1, m_1, n_1) \leq Q_1} \frac{\phi(l_1 + m_1 t + n_1 t^2)}{(l_1 + m_1 t + n_1 t^2 - l_0 - m_0 t - n_0 t^2)} \times \prod_{(l,m,n) \leq Q_1 \text{ and } \max(l,m,n) < Q_1} \left( \frac{l_0 + m_0 t + n_0 t^2 - l_1 - m_1 t - n_1 t^2}{l_1 + m_1 t + n_1 t^2 - l - m t - n t^2} \right) (l_0 + m_0 t + n_0 t^2 - l_1 - m_1 t - n_1 t^2).
\]

For the \(p\)-adic value \(|\int_{\partial I} P|\) of the L.H.S. of (16), we deduce
\[
(|e_4|_P \exp(Q_1^3 F(Q_1)) < |R|_P^{-Q_1^3 / 8}.
\]

Further notice that the sum \(\sum |p|\) on the R.H.S. of (16) has \(p\)-adic valuation
\[
< |e_4|_P \exp(Q_1^3 F(Q_1)).
\]

Assume that
\[
|e_4|_P < (D_1)^{-26 NL Q_1} \exp(-Q_1^3 F(Q_1)).
\]

((14) follows from (19)).

From (15), (16), (17), (18) and (19), we get
\[
|\phi(l_0 + m_0 t + n_0 t^2)|_P < |R|_P^{-Q_1^3 / 8}.
\]

**Final Step.** We shall choose \(L\) and \(Q\) such that (15) and (20) are inconsistent, i.e.
\[
(DH)^{-26 NL Q_1} > |R|_P^{-Q_1^3 / 8}
\]
i.e.
\[
Q_1^3 \log |R|_P > 208 \ NL Q_1 \log(DH).
\]

As \(N < c_3 D\), it is sufficient to prove that
\[
Q_1^3 \log |R|_P > 208 \ c_3 DL Q_1 \log(DH)
\]
Set
\[
L = [c_4 D Q_1^{3/2}], \quad c_4 = 2^{1/2} c_3.
\]
(Notice that (10) is satisfied).

It is sufficient to prove that
\[
Q_1^{13} > c_3 D^3 \log(DH) \quad \text{with} \quad c_3 = 208 \ c_3 c_4 |\log |R|_P|.
\]

Set
\[
Q = [c_6 D^4 (\log(DH))^3], \quad c_6 = c_3^2.
\]
As $Q_1 > Q$ implies $Q_1 > Q + 1$ and so the above inequality is satisfied. Consequently from (12) and (19), either

$$|e_4|_P > D^{-3LQ} H^{-13LQ} \exp \left( -c_2 L^2 F(L) \right)$$

or

$$|e_4|_P > (DH)^{-36NLQ_1} \exp \left( -Q_1^2 F(Q_1) \right).$$

Notice that

$$Q_1^2 F(Q_1) < Q^2 F(Q^*) < 8c_2^2 L^2 F(L) < c_2 L^2 F(L).$$

So

$$|e_4|_P > (DH)^{-36NLQ_1} \exp \left( -c_2 L^2 F(L) \right).$$

Notice that

$$L^2 < c_4^2 D^2 Q^8 < c_5^2 c_9^2 D^{14} (\log(DH))^6 = c_7 D^{14} (\log(DH))^6.$$ 

So

$$\exp(-c_2 L^2 F(L)) > \exp \left( -c_9 D^{14} (\log(DH))^6 F(\beta_D, \mu) \right)$$

where

$$\beta_D, \mu = [c_9 D^7 (\log(3DH))^5].$$

Further notice that

$$LQ_1 < c_{10} D^{38/3} \log (DH)^5$$

and so

$$(DH)^{-36NLQ_1} > \exp \left( -c_11 D^{38/3} (\log(DH))^6 \right).$$

Hence for $H > H_0$, $H_0$ independent of $D$, we have

$$\sum_{k=-1}^9 |x^k - \alpha_k|^\mu > \exp(-c_{12} D^{14} (\log(DH))^6 F(\beta_D, \mu)), \beta_D, \mu = [c_9 D^7 (\log(3DH))^5].$$

And the theorem follows trivially from here.

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