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Locating and total dominating sets in trees

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Abstract

A set S of vertices in a graph $G=(V,E)$ is a total dominating set of G if every vertex of V is adjacent to a vertex in S. We consider total dominating sets of minimum cardinality which have the additional property that distinct vertices of V are totally dominated by distinct subsets of the total dominating set.

MSC: 05C69

Keywords: Differentiating-total dominating set; Locating-total dominating set; Metric-locating-total dominating set

1. Introduction

The location of monitoring devices, such as surveillance cameras or fire alarms, to safeguard a system serves as a motivation for this work. The problem of placing monitoring devices in a system in such a way that every site in the system (including the monitors themselves) is adjacent to monitor site can be modelled by total domination in graphs. Applications where it is also important that if there is a "problem" at a facility, its location can be uniquely identified by the set of monitors can be modelled by the combination of two concepts, namely, total domination and locating in graphs. We consider three different variations of this combination.

Let $S=\{v_1, v_2, \ldots, v_k\}$ be a set of vertices in a connected graph $G=(V,E)$, and let $v \in V$. The k-vector (ordered k-tuple) $c_S(v)$ of v with respect to S is defined by

$$c_S(v) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k)),$$

where $d(v, v_i)$ is the distance between v and v_i $(1 \leqslant i \leqslant k)$. The set S is called a *locating set* if the k-vectors $c_S(v)$, $v \in V$, are distinct. This concept is studied in [1,5,10,12].

A set S of vertices of a graph $G=(V,E)$ is a *dominating set* (DS) of G if every vertex in $V-S$ is adjacent to a vertex of S, and S is a *total dominating set* (TDS) of G if every vertex in V is adjacent to a vertex in S. The minimum cardinality of a TDS is the *total domination number* $\gamma_t(G)$. A TDS of cardinality $\gamma_t(G)$ we call a $\gamma_t(G)$-set. Domination

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and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books [6,7].

In this paper we merge the concepts of a locating set and a total dominating set by defining three new sets, namely a locating-total dominating set, differentiating-total dominating set, and a metric-locating-total dominating set. We then establish bounds on these parameters in a tree and investigate the ratio of any two of these parameters in trees.

1.1. Notation

For notation and graph theory terminology we in general follow [6]. Specifically, let $G=(V,E)$ be a graph with vertex set V of order n and edge set E. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A path on n vertices is denoted by P_n. A *leaf* of G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A support vertex that is adjacent to at least two leaves we call a *strong support vertex*. A k-support vertex is a support vertex that is adjacent to exactly k leaves. If $v \in S$ and $w \in V-S$, then the vertex w is an *external private neighbor of* v (with respect to S) if $N(w) \cap S = \{v\}$.

The open neighborhood of vertex $v \in V$ is denoted by $N(v)=\{u \in V \mid uv \in E\}$ while its closed neighborhood is given by $N[v]=N(v) \cup \{v\}$. For a set $S \subseteq V$, $N(S)=\bigcup_{v \in S} N(v)$ and $N[S]=N(S) \cup S$. Hence the set S is a DS if $N[S]=V$, and a TDS if $N(S)=V$. The set S is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G, i.e., if $u,v \in S$, then $d_G(u,v) \geqslant 3$.

For $k \geqslant 1$, the *k-corona* of a graph H is the graph of order $(k+1)|V(H)|$ obtained from H by attaching a path of length k to each vertex of H so that the resulting paths are vertex disjoint. In particular, the 1-corona of H, also called the *corona* of H and denoted by $H \circ K_1$, is obtained from H by adding a pendant edge to each vertex of H.

1.2. Total domination and locating in graphs

In this section, we consider three different variations of total domination and locating in graphs. We begin by mentioning variations of domination and locating in graphs that occur in the literature.

Slater [11,12] defined a *locating-dominating set* in a connected graph G to be a dominating set S of G such that for every two vertices u and v in $V-S$, $N(u) \cap S \neq N(v) \cap S$. This concept is studied in [2,3,9,11–13] and elsewhere.

Gimbel et al. [4] defined a set S to be a *differentiating dominating set* (DDS) if S is a DS and for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The *differentiating domination number*, denoted $\gamma_D(G)$, of G is the minimum cardinality of a DDS of G.

In [8], the concepts of a locating set and a dominating set are merged by defining the *metric-locating-dominating set* in a connected graph G to be a set of vertices of G that is both a dominating set and a locating set in G.

We now consider analogous extensions to total dominating sets. Let S be a TDS in a connected graph G. We call the set S a

- *locating-total dominating set* (LTDS) if for every pair of distinct vertices u and v in $V-S$, $N(u) \cap S \neq N(v) \cap S$;
- *differentiating-total dominating set* (DTDS) if for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$;
- *metric-locating-total dominating set* (MLTDS) if S is also a locating set in G.

The *locating-total domination number*, denoted $\gamma_t^L(G)$, of G is the minimum cardinality of a LTDS of G. A LTDS of cardinality $\gamma_t^L(G)$ we call a $\gamma_t^L(G)$-set. The location-domination number is defined for every graph G with no isolated vertex, since V is such a set.

The *differentiating-total domination number*, denoted $\gamma_t^D(G)$, of G is the minimum cardinality of a DTDS of G. A DTDS of cardinality $\gamma_t^D(G)$ we call a $\gamma_t^D(G)$-set. We observe that although the differentiating-total domination number is not defined for every graph, it is defined for every tree of order at least three.

We define the *metric-location-total domination number* $\gamma_t^M(G)$ of G to be the minimum cardinality of a MLTDS in G. A MLTDS in G of cardinality $\gamma_t^M(G)$ we call a $\gamma_t^M(G)$-set. The metric-location-total domination number is defined for every graph G with no isolated vertex, since V is such a set.

2. Locating-total domination in trees

Every LTDS of a graph is also a TDS of the graph, and so $\gamma_t^L(G) \geqslant \gamma_t(G)$ for every graph G. In the special case when G is a path, every TDS of G is also a LTDS of G. Thus the locating-total domination number of a path is precisely its total domination number.

Theorem 1. *For $n \geqslant 2$, $\gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Next we present a lower bound on the locating-total domination number of a tree in terms of its order. Let $\mathscr{T}_1$ be the family of trees that can be obtained from k disjoint copies of P_4 by first adding $k-1$ edges in such a manner that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 2. *If T is a tree of order $n \geqslant 2$, then*

$$\gamma_t^L(T) \geqslant \tfrac{2}{5}(n+1),$$

with equality if and only if $T \in \mathscr{T}_1$.

Proof. Let T be a tree of order n, and let S be a $\gamma_t^L(T)$-set. Let $T_1, \ldots, T_k$ be the components of $T[S]$. Since every component of $T[S]$ has at least two vertices, $|S| \geqslant 2k$, and so $k \leqslant |S|/2$. Let P be the set of all external private neighbors of vertices in S. Since each vertex of S has at most one external private neighbor, $|P| \leqslant |S|$. Let $R = V - S - P$, and let $|R| = r$. Let K be a set of k vertices corresponding to the k components of $T[S]$. Let F be a forest of order $k + r$ with $V(F) = K \cup R$ where $E(F)$ consists of all edges of T that join two vertices of R and where $u \in K$ is joined to $v \in R$ in F if and only if the vertex v is adjacent in T to a vertex in the component of $T[S]$ corresponding to the vertex u. Then, $|E(F)| \geqslant 2|R| = 2r$, and so, $k + r = |V(F)| \geqslant |E(F)| + 1 \geqslant 2r + 1$. Thus, $r \leqslant k - 1$. Hence, $n - |S| = |V - S| = |P| + |R| \leqslant |S| + (k-1) \leqslant 3|S|/2 - 1$, and so $n \leqslant 5|S|/2 - 1$. Consequently, $\gamma_t^D(T) = |S| \geqslant 2(n+1)/5$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $k = |S|/2$ implying that each component of $T[S]$ is a K_2. Also, $V - S - P = R$ and $r = k - 1$. It follows that $T[R \cup S]$ is a tree in which each vertex in R has degree two. Moreover, $|P| = |S|$, and so, since $T[R \cup S]$ is a tree, $T[P \cup S]$ is the union of k disjoint paths P_4 where each vertex of P is a leaf of T. Hence, $T \in \mathscr{T}_1$. $\square$

The next result establishes a lower bound on the locating-total domination number of a tree in terms of its order and its number of leaves and support vertices. Let $\mathscr{T}_2$ be the family of trees T that can be obtained from any tree T' by attaching at least two leaves to each vertex of T' and, if T' is nontrivial, subdividing each edge of T' exactly once.

Theorem 3. *If T is a tree of order $n \geqslant 3$ with ℓ leaves and s support vertices, then*

$$\gamma_t^L(T) \geqslant \frac{n + 2(\ell - s) + 1}{3},$$

with equality if and only if $T \in \mathscr{T}_2$.

Proof. Let T be a tree of order n. If $n=3$, then $\gamma_t^L(T) = 2 = (n + 2(\ell - s) + 1)/3$ and $T \in \mathscr{T}_2$. If $n=4$, then either $T = K_{1,3}$, in which case $\gamma_t^L(T) = 3 = (n + 2(\ell - s) + 1)/3$ and $T \in \mathscr{T}_2$, or $T = P_4$, in which case $\gamma_t^L(T) = 2 > (n + 2(\ell - s) + 1)/3$. Suppose then that $n \geqslant 5$.

Let S be a $\gamma_t^L(T)$-set that contains a minimum number of leaves. At most one leaf neighbor of every support vertex is not in S. Assume that for some support vertex v, every leaf neighbor of v is in S. If v has a (non-leaf) neighbor x such that $N[x] \cap S = \{v\}$, then adding x to the set S and removing a leaf neighbor of v from S produces a new $\gamma_t^L(T)$-set containing fewer leaves than does S, a contradiction. Hence, every neighbor of v in $V - S$ has another neighbor in S. If v has two or more neighbors in S, then removing a leaf neighbor of v from S produces a LTDS with cardinality less than $\gamma_t^L(T)$, a contradiction. Hence, v has exactly one leaf neighbor u and $N[v] \cap S = \{u, v\} \subseteq S$. Then $(S - \{u\}) \cup \{x\}$, where x is a non-leaf neighbor of v, is a new $\gamma_t^L(T)$-set containing fewer leaves than does S, a contradiction. Hence for every support vertex v, exactly one leaf neighbor of v is not in S.

Let $T_1, \ldots, T_k$ be the components of $T[S]$. Notice that any support vertex and its leaves that are in S are in the same component of $T[S]$. Hence the number of components of $T[S]$ is bounded above by the number of vertices in S that are not leaves of T. Thus our choice of S implies that $k \leqslant |S| - \ell + s$.

Let P be the set of all external private neighbors of vertices in S. Thus, if $w \in P$, then $|N(w) \cap S| = 1$. Since no leaf of T in the set S has any external private neighbors, and since each vertex of S has at most one external private neighbor, $|P| \leqslant |S| - \ell + s$.

Let $R = V - S - P$ and let $|R| = r$. Note that each vertex in R is adjacent to at least two vertices in S. Let F be the forest of order $k + r$ as defined in the proof of Theorem 2. Then, as before, $|E(F)| \geqslant 2|R| = 2r$ and $r \leqslant k - 1 \leqslant |S| - \ell + s - 1$. Hence, $n - |S| = |V - S| = |P| + |R| \leqslant (|S| - \ell + s) + (|S| - \ell + s - 1)$, and so $n \leqslant 3|S| - 2(\ell - s) - 1$. Consequently, $\gamma_t^L(T) = |S| \geqslant (n + 2(\ell - s) + 1)/3$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $k = |S| - \ell + s$ implying that $k = s$ and that every vertex of S that is not a leaf of T is a support vertex of T that is adjacent to at least two leaves. Hence, $T[S]$ is a disjoint union of stars of order at least 2 every leaf of which is a leaf of T and every center of which is a support vertex of T. Moreover, $r = k - 1$ implying that each vertex in R has degree exactly two. Thus, T can be obtained from a tree T' of order k by adding at least two leaves adjacent to each vertex in T' and subdividing each edge of T' exactly once. Hence, $T \in \mathscr{T}_2$. $\square$

3. Differentiating-total domination in trees

Although not every graph is distinguishable, every tree with order at least three is distinguishable. Hence in this section we investigate the differentiating-total domination number in trees. We begin the simplest of trees, namely paths.

Theorem 4. *For $n \geqslant 3$,*

$$\gamma_t^D(P_n) = \begin{cases} \left\lceil \dfrac{3n}{5} \right\rceil & \textit{if } n \not\equiv 3 \pmod 5, \\ \left\lceil \dfrac{3n}{5} \right\rceil + 1 & \textit{if } n \equiv 3 \pmod 5. \end{cases}$$

Proof. We proceed by induction on n. Clearly, the result can be verified for small values of n, $3 \leqslant n \leqslant 7$. Let $n \geqslant 8$ and suppose the result holds for all paths of order n' where $3 \leqslant n' < n$. Let $T: v_1, v_2, \ldots, v_n$ be a path of order n. Let S be a $\gamma_t^D(T)$-set. Notice we can choose S so that $v_1 \notin S$. For if $v_1 \in S$, let v_j be the vertex of smallest subscript that is not in S, and replace v_1 in S by v_j to get a new $\gamma_t^D(T)$-set. Since $v_1 \notin S$, it follows that $\{v_2, v_3, v_4\} \subseteq S$. Similarly, we can choose S so that $v_5 \notin S$. For if $v_5 \in S$, we can replace v_5 with v_k in S, where v_k is the vertex of smallest subscript such that $k > 5$ and $v_k \notin S$, to form a new $\gamma_t^D(T)$-set. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$. Then, T' is a path of order $n' = n - 5 \geqslant 3$ and $S - \{v_2, v_3, v_4\}$ is a DTDS of T'. Thus, $\gamma_t^D(T') \leqslant |S| - 3 = \gamma_t^D(T) - 3$, or, equivalently, $\gamma_t^D(T) \geqslant \gamma_t^D(T') + 3$. Let

$$D = \bigcup_{i=0}^{\lfloor (n-4)/5 \rfloor} \{v_{5i+2}, v_{5i+3}, v_{5i+4}\}.$$

We consider two cases.

Case 1: $n \equiv 3 \pmod 5$. Then, $n' \equiv 3 \pmod 5$. Applying the inductive hypothesis to T', $\gamma_t^D(T') = \lceil 3n'/5 \rceil + 1 = \lceil 3n/5 \rceil - 2$. Hence, $\gamma_t^D(T) \geqslant \gamma_t^D(T') + 3 = \lceil 3n/5 \rceil + 1$. On the other hand the set $S = D \cup \{v_{n-2}, v_{n-1}, v_n\}$ is a DTDS of T, and so $\gamma_t^D(T) \leqslant |S| = \lceil 3n/5 \rceil + 1$. Consequently, $\gamma_t^D(T) = \lceil 3n/5 \rceil + 1$.

Case 2: $n \not\equiv 3 \pmod 5$. Then, $n' \not\equiv 3 \pmod 5$. Applying the inductive hypothesis to T', $\gamma_t^D(T') = \lceil 3n'/5 \rceil = \lceil 3n/5 \rceil - 3$. Hence, $\gamma_t^D(T) \geqslant \gamma_t^D(T') + 3 = \lceil 3n/5 \rceil$. If $n \equiv 0$ or $4 \pmod 5$, let $S = D$. If $n \equiv 1 \pmod 5$, let $S = D \cup \{v_{n-1}\}$. If $n \equiv 2 \pmod 5$, let $S = D \cup \{v_{n-2}, v_{n-1}\}$. Then, S is a DTDS of T, and so $\gamma_t^D(T) \leqslant |S| = \lceil 3n/5 \rceil$. Consequently, $\gamma_t^D(T) = \lceil 3n/5 \rceil$. $\square$

The following result provides an upper bound on the differentiating-total domination number of a tree in terms of its order and number of support vertices.

Theorem 5. *If $T \neq P_4$ is a tree of order $n \geqslant 4$ with s support vertices, then $\gamma_t^D(T) \leqslant n - s$.*

Proof. Let S be a packing in T consisting of precisely s leaves each a neighbor of a different support vertex. Since $T \neq P_4$ and $n \geqslant 4$, $T[V - S]$ is a tree of order at least 3. It follows that $V - S$ is a DTDS of T, and so $\gamma_t^D(T) \leqslant |S| = n - s$. □

As an immediate consequence of Theorem 5, we have the following result.

Corollary 6. *If T is a tree of order $n \geqslant 4$, then $\gamma_t^D(T) \leqslant n - 1$ with equality if and only if $T = P_4$ or T is a star.*

Next we present a lower bound on the differentiating-total domination number of a tree in terms of its order. Let $\mathscr{T}_3$ be the family of trees that can be obtained from k disjoint copies of a corona $P_3 \circ K_1$ by first adding $k-1$ edges in such a manner that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge with a single vertex. Since $\gamma_t^D(G) \geqslant \gamma_D(G)$ for all distinguishable graphs with no isolated vertex, our next result is a consequence of Theorem 16 in [4]. (We remark that a similar proof to that of Theorem 2 can be used to prove Theorem 7.)

Theorem 7. *If T is a tree of order $n \geqslant 3$, then*

$$\gamma_t^D(T) \geqslant \tfrac{3}{7}(n+1),$$

with equality if and only if $T \in \mathscr{T}_3$.

Since $\gamma_t^L(T) \leqslant \gamma_t^D(T)$ for trees T, the next bound follows directly from Theorem 3. The proof for sharpness is essentially the same as the proof of Theorem 3, so it is omitted. Let $\mathscr{T}_4$ be the family of trees T that can be obtained from any tree T' by attaching at least three leaves to each vertex of T' and, if T' is nontrivial, subdividing each edge of T' exactly once.

Theorem 8. *If T is a tree of order $n \geqslant 3$ with ℓ leaves and s support vertices, then*

$$\gamma_t^D(T) \geqslant \frac{n + 2(\ell - s) + 1}{3},$$

with equality if and only if $T \in \mathscr{T}_4$.

We show next that the ratio $\gamma_t^D(T)/\gamma_t^L(T)$ is bounded below by 1 and above by 3/2 when T is a tree.

Theorem 9. *For any tree T,*

$$\gamma_t^L(T) \leqslant \gamma_t^D(T) \leqslant \tfrac{3}{2}\gamma_t^L(T),$$

and these bounds are sharp.

Proof. The lower bound on $\gamma_t^D(T)$ follows from the observation that every DTDS of a tree is also a LTDS of the tree. To establish the upper bound on $\gamma_t^D(T)$, let S be a $\gamma_t^L(T)$-set. Suppose $T[S]$ has k components. Since every component of $T[S]$ has at least two vertices, $|S| \geqslant 2k$, and so $k \leqslant |S|/2$. For each 2-component of $T[S]$, add to the set S a vertex in $V - S$ that is adjacent to a vertex in that component. Then the resulting set S' is a DTDS of T, and so $\gamma_t^D(T) \leqslant |S'| \leqslant |S| + k \leqslant 3|S|/2 = 3\gamma_t^L(T)/2$.

Equality is achieved in the lower bound by taking, for example, T to be the corona of a tree of order at least 3, while equality is achieved in the upper bound by taking, for example, T to be the 3-corona of any tree. □

4. Metric-locating-total domination in trees

Every MLTDS of a graph is also a TDS of the graph, and so $\gamma_t(G) \leqslant \gamma_t^M(G)$ for every graph G. Our first aim in this section is to characterize the trees T for which $\gamma_t(T) = \gamma_t^M(T)$.

Theorem 10. *Let $T=(V,E)$ be a nontrivial tree and let S be the set of 2-support vertices of T. Then, $\gamma_t(T)=\gamma_t^M(T)$ if and only if*

(i) *T contains no k-support vertex for $k\geqslant 3$,*
(ii) *the set S is a packing in T, and*
(iii) *$N[S]=V$ or $F=T-N[S]$ is a forest with no isolates and with $\gamma_t(T)=\gamma_t(F)+2|S|$.*

Proof. First we consider the necessity. Suppose $\gamma_t(T)=\gamma_t^M(T)$. Let D be a $\gamma_t^M(T)$-set. Then, D is also a $\gamma_t(T)$-set. The set D contains all except possibly one leaf adjacent to every support vertex. If T contains a k-support vertex v for some $k\geqslant 3$, then deleting one leaf adjacent to v from the set D produces a TDS of T of cardinality less than that of D, a contradiction. Hence, (i) holds.

Let $v\in S$ and let v' be a leaf neighbor of v in D, and so $\{v,v'\}\subseteq D$. If $|D\cap N[v]|\geqslant 3$, then $D-\{v'\}$ is a TDS of T, a contradiction. Hence, $D\cap N[v]=\{v,v'\}$. It follows that S is an independent set. Suppose $u\in S-\{v\}$ and u' is the leaf neighbor of u in D, and so $D\cap N[u]=\{u,u'\}$. If w is a common neighbor of u and v, then $(D-\{u',v'\})\cup\{w\}$ is a TDS of T, a contradiction. Hence, $d(u,v)\geqslant 3$ and (ii) holds.

Suppose $N[S]\neq V$. Let $F=T-N[S]$. Then, $D_F=D\cap V(F)$ is a TDS of F and $|D_F|=|D|-2|S|$. It follows that the forest F has no isolated vertex and $\gamma_t(F)\leqslant|D_F|$. Any TDS of F can be extended to a TDS of T by adding to it the set S and one leaf neighbor adjacent to every vertex in S, and so $|D_F|+2|S|=\gamma_t^M(T)=\gamma_t(T)\leqslant\gamma_t(F)+2|S|\leqslant|D_F|+2|S|$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma_t(T)=\gamma_t(F)+2|S|$ and (iii) holds.

Next we consider the sufficiency. Suppose that conditions (i), (ii) and (iii) hold. Assume $S=\emptyset$, i.e., T has no strong support vertex. Let D be a TDS of T. Let $u,v\in V(T)-D$. If $N(u)\cap D\neq N(v)\cap D$, then $c_D(u)\neq c_D(v)$. Suppose, then, that $N(u)\cap D=N(v)\cap D$. Then, since T is a tree, there is a unique vertex $w\in D$ such that $N(u)\cap D=N(v)\cap D=\{w\}$. Since w is not a strong support vertex, at least one of u and v cannot be a leaf. We may assume that $\deg v\geqslant 2$. Let $x\in N(v)-\{w\}$. Since $N(v)\cap D=\{w\}$, $x\notin S$. Thus, x is adjacent to a vertex $y\in D$. Since T is a tree, $w\neq y$. Therefore, $d(v,y)=2$, while $d(u,y)=4$. Thus, once again, $c_D(u)\neq c_D(v)$. Hence, D is a MLTDS of T, and so $\gamma_t^M(T)\leqslant\gamma_t(T)$. Consequently, $\gamma_t(T)=\gamma_t^M(T)$.

Assume that $S\neq\emptyset$ and $N[S]=V$. Then adding one leaf neighbor adjacent to every vertex in S to the set S produces a MLTDS of T, and so $\gamma_t^M(T)\leqslant 2|S|$. By (ii), the set S is a packing in T, and so $\gamma_t(T)\geqslant 2|S|$. Thus, $\gamma_t^M(T)\leqslant\gamma_t(T)$. Consequently, $\gamma_t(T)=\gamma_t^M(T)$.

Finally, assume that $S\neq\emptyset$ and $N[S]\neq V$. Let $F=T-N[S]$. By (iii), $\delta(F)=1$ and $\gamma_t(T)=\gamma_t(F)+2|S|$. Let D_F be a $\gamma_t(F)$-set. Let D be the set obtained from D_F by adding to it the set S and one leaf neighbor adjacent to every vertex in S. Then, D is a TDS of T of cardinality $|D_F|+2|S|=\gamma_t(F)+2|S|=\gamma_t(T)$. Thus, D is a $\gamma_t(T)$-set. Further D is a MTLDS of T, and so $\gamma_t^M(T)\leqslant\gamma_t(T)$. Consequently, $\gamma_t(T)=\gamma_t^M(T)$. $\square$

As an immediate consequence of Theorem 10 and its proof, we have the following result.

Corollary 11. *If a nontrivial tree T contains no strong support vertex, then every TDS of T is a MLTDS of T. In particular, $\gamma_t^M(T)=\gamma_t(T)$.*

We establish next a relationship between the total domination number and the metric-location-total domination number of a tree in terms of the number of leaves and the number of support vertices.

Theorem 12. *If T is a tree with ℓ leaves and s support vertices, then*

$$\gamma_t(T)+\ell-2s\leqslant\gamma_t^M(T)\leqslant\gamma_t(T)+\ell-s,$$

and these bounds are sharp.

Proof. Let D be a $\gamma_t^M(T)$-set that contains as few leaves as possible. Then the set D contains all except one leaf adjacent to every support vertex as well as the set of support vertices. Let D' be the set obtained from D by removing all $\ell-s$ leaves of T that belong to the set D and adding to the set D all the original s leaves that were not in D. Then, D' is a TDS of T, and so $\gamma_t(T)\leqslant|D'|=|D|-\ell+2s=\gamma_t^M(T)-\ell+2s$. This establishes the lower bound.

Next we prove the upper bound. If T contains no strong support vertex, then, by Corollary 11, $\gamma_t^M(T) = \gamma_t(T)$. Thus since $\ell = s$ in this case, $\gamma_t^M(T) = \gamma_t(T) + \ell - s$. Hence we may assume that T has at least one strong support vertex for otherwise the result follows. Let T' be the tree obtained from T by deleting all except one leaf-neighbor of every strong support vertex of T. Then, $\gamma_t(T') = \gamma_t(T)$. Let S' be a $\gamma_t(T')$-set. Since T' is a tree with no strong support vertex, by Corollary 11, S' is a MLTDS of T'. Hence S' can be extended to a MLTDS of T by adding to it the set of $\ell - s$ leaves that were deleted from T when constructing T', and so $\gamma_t^M(T) \leqslant |S'| + \ell - s = \gamma_t(T) + \ell - s$.

As observed earlier, equality in the upper bound is achieved, for example, if T has no strong support vertex. Equality is achieved in the lower bound by taking, for example, a tree T constructed as follows: Take any nontrivial tree T'. For every vertex v' of T', add to T' a star of order at least 3 with center vertex v and join v and v'. Finally, subdivide every edge of T' exactly twice and let T be the resulting tree. Then, $\gamma_t(T) = 2|V(T')|$ (adding the support vertices of T to the set $V(T')$ forms the unique $\gamma_t(T)$-set) while $\gamma_t^M(T) = 2|V(T')| + \ell - 2s$ (for example, adding all but one leaf adjacent to every support vertex of T to the unique $\gamma_t(T)$-set forms a minimum MTDS of T). Hence, $\gamma_t^M(T) = \gamma_t(T) + \ell - 2s$. $\square$

We show next that the ratio $\gamma_t^L(T)/\gamma_t^M(T)$ is strictly less than 3/2 when T is a tree.

Theorem 13. *For any tree T,*

$$\gamma_t^M(T) \leqslant \gamma_t^L(T) \leqslant \tfrac{3}{2}\gamma_t^M(T) - 1,$$

and these bounds are sharp.

Proof. Since every LTDS is also a MLTDS, $\gamma_t^M(G) \leqslant \gamma_t^L(G)$ for all graphs G. This establishes the lower bound. To prove the upper bound, let $T = (V, E)$ and let D be a $\gamma_t^M(T)$-set that contains as few leaves as possible. Then the set D contains all except one leaf adjacent to every support vertex as well as the set of support vertices. Suppose $T[D]$ has x components. Since every component of $T[D]$ has at least two vertices, $|D| \geqslant 2x$, and so $x \leqslant |D|/2$.

Let R denote the set of vertices of degree at least two in T that do not belong to the set D. Since D is a TDS, each vertex of R is adjacent to at least one vertex of D. Let $|R| = r$. Suppose $T[R]$ has y components.

Let X be a set of x vertices corresponding to the x components of $T[D]$, and let Y be a set of y vertices corresponding to the y components of $T[R]$.

We show first that $y \leqslant x - 1$. Let F be the forest of order $x + y$ with $V(F) = X \cup Y$ where $u \in X$ is joined to $v \in Y$ if and only if there is an edge in T joining the component of $T[D]$ corresponding to the vertex u and the component of $T[R]$ corresponding to the vertex v. If a component of $T[R]$ consists of a single vertex v, then v is adjacent to at least two vertices of D (necessarily from different components of $T[D]$) in T and therefore the vertex of Y corresponding to the vertex v is adjacent to at least two vertices of X in F. On the other hand, since every vertex of R is adjacent to at least one vertex of D in T, every vertex of Y corresponding to a nontrivial component of $T[R]$ is adjacent to at least two vertices of X in F. Hence each vertex of Y in the forest F is adjacent to at least two vertices of X. Hence, $|E(F)| \geqslant 2|Y| = 2y$, and so $x + y = |V(F)| \geqslant |E(F)| + 1 \geqslant 2y + 1$. Thus, $y \leqslant x - 1$.

We show next that $r \leqslant 2(x - 1)$. Let H be the forest of order $x + r$ with $V(H) = X \cup R$ where $E(H)$ consists of all edges of T that join two vertices of R and where $u \in X$ is joined to $v \in R$ in H if and only if the vertex v is adjacent in T to a vertex in the component of $T[D]$ corresponding to the vertex u. Thus the degree of each vertex of R is the same in the tree T and the forest H. In particular, each vertex of R has degree at least two in H and is adjacent to at least one vertex of X in H. Hence the number of edges of H joining X and R is at least r. Since $H[R]$ is a forest of order r with y components, $H[R]$ has size $r - y$. Since every edge of H joins two vertices of R or joins a vertex of R with a vertex of X, it follows that H has size at least $2r - y$. However H is a forest of order $x + r$ and therefore has size at most $x + r - 1$, whence $r \leqslant x + y - 1 \leqslant 2(x - 1)$.

If a component of $T[R]$ consists of a single vertex v, then such a vertex v is adjacent to at least two vertices of D, and so $N(u) \cap D \neq N(v) \cap D$ for all other vertices $u \in V(T) - D$. In particular, if each component of $T[R]$ consists of a single vertex (i.e., if R is an independent set), then D is a LTDS of T, and so $\gamma_t^L(T) \leqslant |D| = \gamma_t^M(T)$, whence $\gamma_t^L(T) = \gamma_t^M(T)$. Hence we may assume that R is not an independent set.

Let $T_1, \ldots, T_t$ denote the nontrivial components of $T[R]$. For $i = 1, \ldots, t$, let T_i have order r_i. Then, $\sum_{i=1}^{t} r_i \leqslant r$. Let D_i be a minimum DS in T_i. Since the domination number of a graph with no isolated vertex is at most half its order, $|D_i| \leqslant r_i/2$ for each i. Let $D^* = \bigcup_{i=1}^{t} D_i$. Then, $|D^*| \leqslant r/2 \leqslant x - 1 \leqslant |D|/2 - 1$. Since each vertex of R is dominated by the set D, and since each vertex of $R - D^*$ is adjacent to at least two vertices of $D \cup D^*$, it follows that $D \cup D^*$ is a LTDS of T, and so $\gamma_t^L(T) \leqslant |D| + |D^*| \leqslant |D| + |D|/2 - 1 = 3|D|/2 - 1 = 3\gamma_t^M(T)/2 - 1$. This establishes the upper bound.

Equality is achieved in the lower bound by taking, for example, T to be the corona of a nontrivial tree. That the upper bound is sharp may be seen as follows. Let T be the tree obtained from the disjoint union of k paths P_4 by joining a support vertex from one of these paths to a support vertex from each of the other $k-1$ paths and then subdividing each new edge twice. Then, $\gamma_t^M(T) = 2k$ (the set of $2k$ support vertices of T form a minimum MLTDS of T) while $\gamma_t^L(T) = 3k - 1$ (for example, adding an independent set consisting of $k-1$ subdivided vertices to the set of $2k$ support vertices forms a minimum LTDS of T). Hence, $\gamma_t^L(T) = 3\gamma_t^M(T)/2 - 1$. □

As an immediate consequence of Corollary 11 and Theorem 13, we have the following result.

Corollary 14. *If a nontrivial tree T contains no strong support vertex, then*

$$\gamma_t(T) \leqslant \gamma_t^L(T) \leqslant \tfrac{3}{2}\gamma_t(T) - 1.$$

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