Discrete Mathematics 44 (1983) 45–54
North-Holland Publishing Company

PARTITIONABLE GRAPHS, CIRCLE GRAPHS, AND THE BERGE STRONG PERFECT GRAPH CONJECTURE*

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Received 20 April 1981
Revised 15 March 1982

Partitionable graphs have been studied by a number of authors in conjunction with attempts at proving the Berge Strong Perfect Graph Conjecture (SPGC). We give some new properties of partitionable graphs which can be used to give a new proof that the SPGC holds for $K_{1,3}$-free graphs. Finally, we will show that the SPGC also holds for the class of circle graphs.

1. Introduction

Given a graph G, let $\alpha(G)$ be the size of a maximum stable set, $\omega(G)$ be the size of a maximum clique, $\chi(G)$ be the size of a minimum stable set cover, i.e., a coloring, and $\theta(G)$ be the size of a minimum clique cover. A graph G is *perfect* if every induced subgraph H of G satisfies $\omega(H) = \chi(H)$. Lovász [12, 13] gave the following important characterization of perfect graphs.

Theorem 1 (The Perfect Graph Theorem, Lovász [12, 13]). *The following are equivalent:*

(i) *G is perfect.*
(ii) *The complement of G is perfect.*
(iii) *$\alpha(H)\omega(H) \geq |H|$ for all induced subgraphs H of G.*

A graph G is *p-critical* if it is minimally imperfect, that is, G itself is not perfect and yet every proper induced subgraph of G is perfect. The only known p-critical graphs are the odd chordless cycles of size ≥ 5 and their complements. Berge [1] has conjectured this to be the only case:

The Strong Perfect Graph Conjecture (SPGC). A graph is p-critical if and only if it is an odd chordless cycle of size ≥ 5 or the complement of one.

* The research for this paper was supported in part by the National Science Foundation under Grant No. MCS78-03820.

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An equivalent form which we will use is the following:

A graph is imperfect if and only if it contains an odd chordless cycle of size ≥ 5 or the complement of one.

The Berge SPGC has been shown to hold for a number of classes of graphs, including, planar graphs [19], circular arc graphs [20], $K_{1,3}$-free graphs [16], 3-chromatic graphs [21], toroidal graphs [11], and $(K_4$-$e)$-free graphs [17]. Padberg [14, 15] demonstrated the following important properties of p-critical graphs. Let the term *ω-clique* denote a clique of size ω and *α-stable set* a stable set of size α.

Theorem 2 (Padberg [14, 15]). *Let* $\alpha = \alpha(G)$, $\omega = \omega(G)$ *and* $n = |G|$. *If* G *is* p*-critical, then the following properties hold:*

(i) $n = \alpha\omega + 1$.

(ii) *G contains exactly n ω-cliques and n α-stable sets.*

(iii) *Every vertex of G is contained in exactly ω ω-cliques and α α-stable sets.*

(iv) *Each ω-clique intersects all but one α-stable set, and vice versa.*

Bland, Huang, and Trotter [2] introduced a notion that is closely related to p-critical graphs. We give a seemingly weaker, but equivalent, formulation. A graph G having n vertices is *partitionable* if there exist integers $\alpha, \omega \geq 2$ such that $n = \alpha\omega + 1$, and $\alpha \geq \theta(G-x)$ and $\omega \geq \chi(G-x)$ for all vertices x. Buckingham [3] has shown that this definition implies that, for all vertices x, $G-x$ can be partitioned into α ω-cliques and also into ω α-stable sets, which was the definition for partitionable graphs originally given in [2]. It is an immediate result of the Perfect Graph Theorem that *every p-critical graph is partitionable* where $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Examples of graphs which are partitionable but no p-critical have been given in [2], [5], and [6]. Bland, Huang, and Trotter showed that for partitionable graphs $\alpha(G) = \alpha$ and $\omega(G) = \omega$, and, more importantly, that partitionable graphs also satisfy the properties stated in Theorem 2. (See also Golumbic [10, Ch. 3] and Buckingham [3].) From these properties one may deduce the following four properties as given in [2], [3], and [21].

Let $\theta(G)$ denote a minimum clique cover of G and $X(G)$ denote a minimum stable set cover. Let Adj(x) be the set of vertices adjacent to x, and let $N(x)$ denote the neighborhood of a vertex x, that is, the union of x and Adj(x). If G is partitionable, then the following hold:

Property 1. *$\Theta(G-x)$ and $X(G-x)$ are unique for each vertex x. $\Theta(G-x)$ consists of α ω-cliques, and $X(G-x)$ consists of ω α-stable sets.*

Property 2. *The α α-stable sets of G that have an empty intersection with one of the ω-cliques of $\Theta(G-x)$ are precisely those α α-stable sets that contain x. Similarly, the ω ω-cliques of G that have an empty intersection with one of the α-stable set of $X(G-x)$ are precisely those ω ω-cliques that contain x.*

Property 3. *The induced subgraph formed by the symmetric difference of any two α-stable sets of G is a connected graph. In particular, the union of any two α-stable sets of $X(G-x)$ induces a connected subgraph.*

Property 4. *$G-N(x)$ is connected for each vertex x.*

In Section 2 we show that the properties of Theorem 2 completely characterize partitionable graphs. In Section 3 we give two important new properties of partitionable graphs, and we utilize these new properties in Section 4 to give a simpler proof than that in [16] of the Berge Strong Perfect Graph Conjecture for $K_{1,3}$-free graphs. Recently, in [9], a different short proof has been given of the SPGC for a class of graphs which properly include all $K_{1,3}$-free graphs. Finally, in Section 5 we show that the SPGC holds for the class of circle graphs, (see also [3, 4]).

2. A characterization of partitionable graphs

We now proceed to completely characterize partionable graphs.

Theorem 3. *Let $\alpha, \omega \geqslant 2$ be arbitrary integers and let G be a graph on n vertices. Then G is partitionable if and only if conditions* (i)–(iv) *of Theorem* 2 *are satisfied.*

Proof. ($\Rightarrow$) Bland, Huang, and Trotter [2].

($\Leftarrow$) We are given that $\alpha, \omega \geqslant 2$ and by (i) $n = \alpha\omega + 1$. So we must show that $\alpha \geqslant \Theta(G-x)$ and $\omega \geqslant \chi(G-x)$ for all vertices x. Construct two n-by-n (0, 1)-matrices A and B whose rows are the characteristic vectors of the n ω-cliques and n α-stable sets, respectively. By (iv) there is an arrangement of the rows of B such that $AB^{\mathrm{T}} = J - I$, where J is the n-by-n matrix of all ones and I the n-by-n identity matrix. Conditions (ii) and (iii) imply

$$AJ = JA = \omega J \quad \text{and} \quad BJ = JB = \alpha J.$$

Since $J-I$ is nonsingular, $(J-I)^{-1} = [1/(n-1)]J - I$, we conclude that A and B are nonsingular. Therefore,

$$A^{\mathrm{T}}B = B^{-1}BA^{\mathrm{T}}B = B^{-1}(J-I)B = J - I.$$

Choose a vertex x of G, and let $\boldsymbol{e}$ be the characteristic vector of $G-x$. Since $\boldsymbol{e}$ is a column of $A^{\mathrm{T}}B$, the same column of B designates α columns of A^{T} whose vector sum is $\boldsymbol{e}$, that is, the α ω-cliques corresponding to these α columns cover $G-x$. Hence $\Theta(G-x) \leqslant \alpha$. Similarly, $\boldsymbol{e}$ is also a row of $A^{\mathrm{T}}B$, and the corresponding for of A^{T} designates ω α-stable sets that cover $G-x$. Hence $\chi(G-x) \leqslant \omega$ $\square$

3. Additional properties of partitionable graphs

We now present an important new property of partitionable graphs that will be crucial in the following section. First, we establish a preliminary result.

Lemma 4. *If x is a vertex of a partitionable graph G, then any ω-clique of G that contains a vertex not adjacent to x consists of a vertex from each α-stable set of $X(G-x)$, and any α-stable set of G that contains a vertex adjacent to x consists of a vertex from each ω-clique of $\Theta(G-x)$.*

Proof. Any ω-clique C of G, that contains a vertex not adjacent to x, cannot contain x. Thus, the ω-clique C is covered by the ω α-stable sets of $X(G-x)$. Therefore, C consists of a vertex from each α-stable set of $X(G-x)$.

Similarly, any α-stable set of G that contains a vertex adjacent to x consists of a vertex from each ω-clique of $\Theta(G-x)$. □

For simplicity we will say that a subset of vertices is *connected* (resp., *biconnected*) if it induces a connected (resp., biconnected) subgraph. Furthermore, we will refer to a subset of vertices and its induced subgraph interchangeably whenever no confusion would arise.

We now present the main result of this section.

Theorem 5. *Let x be a vertex of a partitionable graph G. If S_1 and S_2 are distinct sets of $X(G-x)$, then $S_1 \cup S_2 \cup x$ is biconnected.*

Proof. Assume there is an S_1, S_2 and x such that $S_1 \cup S_2 \cup x$ is not biconnected, and let s be an articulation point. Hence $(S_1 \cup S_2 \cup x)-s$ can be partitioned into disjoint nonempty subgraphs G_1 and G_2. By Property 3, $S_1 \cup S_2$ is connected; thus s is not x. Without loss of generality assume s in S_1 and x is in G_2.

Observe $\Theta(G-s)$, which covers G_1. Since no vertex of G_1 is adjacent to x, by Lemma 4, each ω-clique of $\Theta(G-s)$ that contains a vertex of G_1 contains one vertex of S_1 and one of S_2, and this within G_1. Therefore, $|S_1 \cap G_1| = |S_2 \cap G_1|$.

Choose b in $S_2 \cap G_1$, and observe $\Theta(G-b)$, which covers $G_1 - b$. Since no vertex of $G_1 - b$ is adjacent to x, by Lemma 4, each ω-clique of $\Theta(G-b)$ that contains a vertex of $S_1 \cap G_1$ contains a vertex of $(S_2 \cap G_1)-b$. Therefore, $|S_1 \cap G_1| \leq |(S_2 \cap G_1)-b|$, a contradiction which proves the result. □

The next lemma generalizes a result proved by Sachs [18] for p-critical graphs.

Lemma 6. *If G is partitionable, then $2\omega - 2 \leq |\mathrm{Adj}(x)| \leq n - 2\alpha + 1$ for all vertices x, and these bounds are tight.*

Proof. Choose a vertex x and a vertex y not adjacent to x, and choose S_1, S_2 in

$X(G-y)$ such that x is in S_2. Thus $S_1 \cap \mathrm{Adj}(x) \neq \emptyset$. Choose z in $S_1 \cup \mathrm{Adj}(x)$, and observe $\Theta(G-z)$. One of the ω-cliques C of $\Theta(G-z)$ contains x. By Lemma 4, C contains a vertex of S_1, but this must be different from z. Thus, x is adjacent to some other vertex in S_1. Therefore, $|S_1 \cap \mathrm{Adj}(x)| \geq 2$.

There are $\omega - 1$ distinct such S_1's in $X(G-y)$ where x is not in S_1. Thus, by the above, $|\mathrm{Adj}(x)| \geq 2(\omega-1) = 2\omega - 2$. The second inequality follows from the fact that the complement of G is partitionable for α and ω reversed:

$$|\mathrm{Adj}(x)| \leq (n-1)-(2\alpha-2) = n-2\alpha+1.$$

To show that the bounds are tight consider the graph C_n^k having n vertices $v_0, v_1, \ldots, v_{n-1}$ and edges (v_i, v_j) where $|i-j| \leq k \pmod n$ (see Fig. 1). If $\alpha, \omega \geq 2$ are arbitrary integers, then $C_{\alpha\omega+1}^{\omega-1}$ is partitionable and $|\mathrm{Adj}(x)| = 2\omega - 2$ for all vertices x. Also the complement of $C_{\alpha\omega+1}^{\alpha-1}$ is partitionable with $|\mathrm{Adj}(x)| = n - 2\alpha + 1$ for all vertices x. □

4. The SPGC and $K_{1,3}$-free graphs

A graph is called $K_{1,3}$-free if it contains no vertex which is adjacent to a 3-stable set. Using Theorem 5 we give a new proof that the SPGC holds for $K_{1,3}$-free graphs. But first we give some preparatory lemmas.

Lemma 7. *If G is a partitionable $K_{1,3}$-free graph, then $|\mathrm{Adj}(x)| = 2\omega - 2$ for all vertices x.*

Proof. Choose a vertex x and a vertex y not adjacent to x. Since G is $K_{1,3}$-free, $|S \cap \mathrm{Adj}(x)| \leq 2$ for each S in $X(G-y)$. And since $\omega - 1$ S's of $X(G-y)$ cover all of $\mathrm{Adj}(x)$, $|\mathrm{Adj}(x)| \leq 2\omega - 2$. By Lemma 6, $|\mathrm{Adj}(x)| \geq 2\omega - 2$. Therefore, $|\mathrm{Adj}(x)| = 2\omega - 2$. □

An immediate consequence of Lemma 7 is the following.

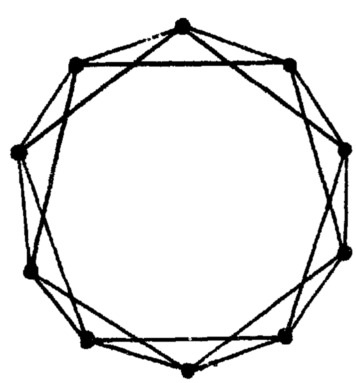

Fig. 1. The graph C_{10}^2 for which $\alpha = 3$ and $\omega = 3$.

Corollary 8. *If G is a partitionable $K_{1,3}$-free graph, then for any vertex x, $X(G-x)$ consists of the following:*

(i) *two α-stable sets S_1, S_2 such that $|S_1 \cap \mathrm{Adj}(x)| = 1$ and $|S_2 \cap \mathrm{Adj}(x)| = 1$,*

(ii) *$\omega - 2$ α-stable sets S_i $(i = 3, \ldots, \omega)$ such that $|S_i \cap \mathrm{Adj}(x)| = 2$.*

Furthermore, if y is a vertex adjacent to x, then $X(G-x)$ contains

(iii) *one α-stable set S_{i_1} such that $|S_{i_1} \cap \mathrm{Adj}(y)| = 0$,*

(iv) *one α-stable set S_{i_2} such that $|S_{i_2} \cap \mathrm{Adj}(y)| = 1$, and*

(v) *$\omega - 2$ α-stable sets S_{i_j} $(j = 3, \ldots, \omega)$ such that $|S_{i_j} \cap \mathrm{Adj}(y)| = 2$.*

Lemma 9. *If S_1 and S_2 are two stable sets of a $K_{1,3}$-free graph such that $S_1 \cup S_2$ is connected, then $S_1 \cup S_2$ is a chordless path or a chordless cycle.*

Proof. Since $S_1 \cup S_2$ is $K_{1,3}$-free and S_1 and S_2 are stable sets, the degree of any vertex within $S_1 \cup S_2$ is at most 2. Thus, for $S_1 \cup S_2$ to be connected, it must be a chordless path or a chordless cycle. □

We now give a new proof of the theorem.

Theorem 10 (Parthasarathy and Ravindra [16]). *If G is a partitionable $K_{1,3}$-free graph, then G contains an odd chordless cycle of size ≥ 5 or the complement of one.*

Proof. Every partitionable graph having $\omega = 2$ is a chordless cycle of size ≥ 5. And every partitionable graph having $\alpha = 2$ is the complement of a chordless cycle of size ≥ 5. Thus, assume $\alpha, \omega \geq 3$.

Choose a vertex x in G. By Corollary 8, there exist S_1, S_2 in $X(G-x)$ such that $|S_1 \cap \mathrm{Adj}(x)| = |S_2 \cap \mathrm{Adj}(x)| = 1$. By Property 3, $S_1 \cup S_2$ is connected. Hence, by Lemma 9, there are two cases to be considered for $S_1 \cup S_2$.

Case (i): *$S_1 \cup S_2$ is a chordless path.* By Theorem 5, $S_1 \cup S_2 \cup x$ is biconnected, hence x must be adjacent to the endpoints of the path. Since $|(S_1 \cup S_2) \cap \mathrm{Adj}(x)| = 2$, these are the only adjacencies of x in $S_1 \cup S_2 \cup x$. Thus, $S_1 \cup S_2 \cup x$ is an odd chordless cycle of size $2\alpha + 1$, which is ≥ 7. Therefore, G contains an odd chordless cycle of size ≥ 5.

Case (ii): *$S_1 \cup S_2$ is a chordless cycle.* The vertices of this cycle alternate between S_1 and S_2. Let $\{u\} = S_1 \cup \mathrm{Adj}(x)$ and $\{v\} = S_2 \cap \mathrm{Adj}(x)$. If u and v are not adjacent, then in the cycle either path connecting u and v together with x would form an odd chordless cycle of size ≥ 5, since u and v are of opposite parity.

Assume u and v are adjacent. By Theorem 3, there exists a unique ω-clique C_1 such that $C_1 \cap S_1 = \emptyset$. Thus, u is not in C_1 and $C_1 \cap S_2 \neq \emptyset$. By Property 2, x is in C_1, thus $C_1 \cap S_2 = \{v\}$, since v is the only adjacency of x in S_2. Choose z_1 in C_1 not adjacent to u (z_1 exists otherwise $C_1 \cup u$ would be a clique of size $\omega + 1$). Thus, z_1 is adjacent to x and v, but not to u (see Fig. 2).

Since G is $K_{1,3}$-free and z_1 is not adjacent to u, $|(x \cup S_1) \cap \mathrm{Adj}(z_1)| \leq 2$. But $|S_1 \cap \mathrm{Adj}(z_1)| \geq 1$ since z_1 is not in S_1, and thus $|S_1 \cap \mathrm{Adj}(z_1)| = 1$. Therefore, by

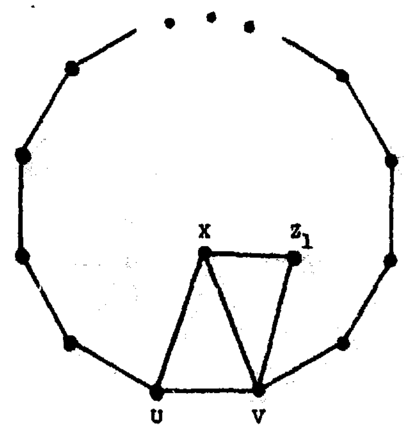

Fig. 2. The adjacencies between x, u, v, and z_1.

Corollary 8(iii)–(v), $|S_2 \cap \text{Adj}(z_1)| = 2$. Let $\{u_1\} = S_1 \cap \text{Adj}(z_1)$ and $\{v, v_1\} = S_2 \cap \text{Adj}(z_1)$.

If v and u_1 are not adjacent, then in the cycle there is a path from v to u_1 avoiding v_1, which joined with z_1 would form an odd chordless cycle of size ≥ 5. Similarly, if u_1 and v_1 are not adjacent, then the cycle contains a path from u_1 to v_1 avoiding v, which joined with z_1 would form an odd chordless cycle of size ≥ 5. Therefore, we may assume that v and u_1 are adjacent, and also that u_1 and v_1 are adjacent as shown in Fig. 3.

In similar manner, there exists a unique ω-clique C_2 such that $C_2 \cap S_2 = \emptyset$, and a vertex z_2 in C_2 which is not adjacent to v. Hence, z_2 and z_1 are distinct. Denote the remaining adjacencies of z_2 by u_2 in S_1 and v_2 in S_2. The analogous argument holds, and either an odd chordless cycle has already been located or we have the graph shown in Fig. 4, where u_2, v_2, u, v, u_1, and v_1 are distinct (since $\alpha \geq 3$), and u_2 and v_1 are adjacent if and only if $\alpha = 3$. Moreover, z_1 and z_2 are not adjacent since $u_2 \cup u \cup z_1$ is a 3-stable set and G is $K_{1,3}$-free.

Finally, the path in $S_1 \cup S_2$ from v_1 to u_2 avoiding v together with z_2, x, and z_1 forms an odd chordless cycle of size $2\alpha - 1$, which is ≥ 5. Therefore, G contains an odd chordless cycle of size ≥ 5 or the complement of one. □

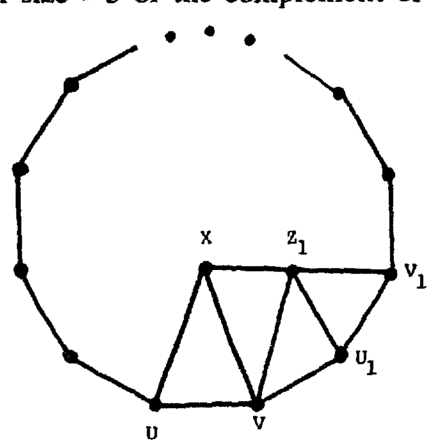

Fig. 3. $S_1 \cup S_2 \cup \{x, z_1\}$.

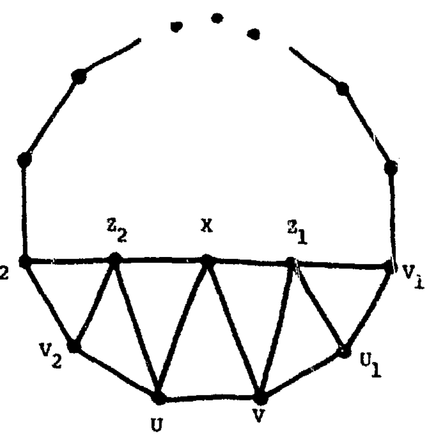

Fig. 4. $S_1 \cup S_2 \cup \{x, z_1, z_2\}$.

Thus, the Berge Strong Perfect Graph Conjecture holds for the class of $K_{1,3}$-free graphs.

5. The SPGC and circle graphs

In this section we show that the SPGC holds for the class of circle graphs. A *circle graph* is a graph derived from the intersecting chords of a circle: each vertex corresponds to a chord and each edge corresponds to two chords intersecting. Fig. 5 shows a circle graph and a collection of intersecting chords which represents it. Circle graphs are equivalent to the union graphs of Even and Itai [7] and to the overlap graphs of Gavril [8]. For a more extended study see Golumbic [10, Ch. 11] and Buckingham [3].

Lemma 11. *If G is a partitionable circle graph, then no circle with chords that represents G contains three parallel chords (see Fig. 6).*

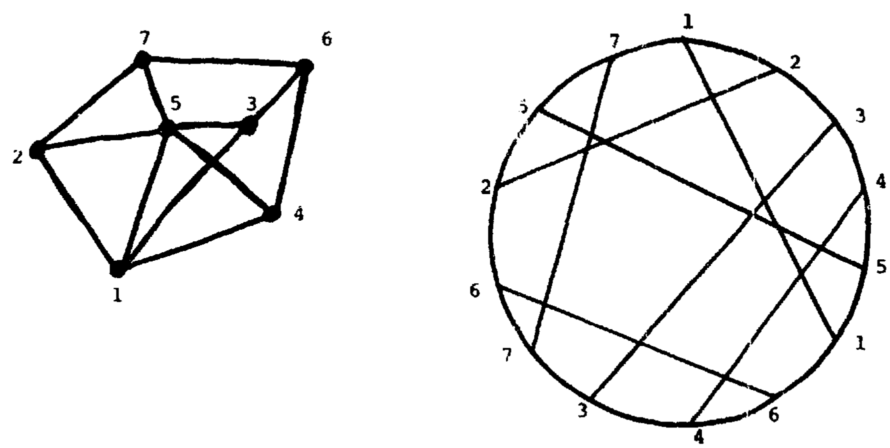

Fig. 5. A circle graph G and a collection of intersecting chords which represent it.

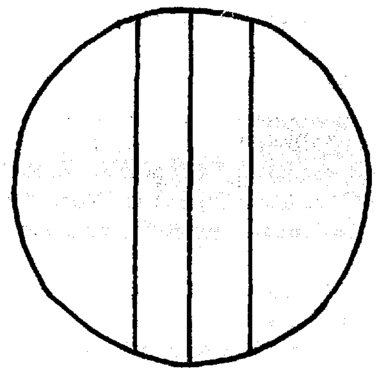

Fig. 6. Three parallel chords.

The term *three parallel chords* simply means three nonintersecting chords with one sandwiched between the other two, but not necessarily parallel in the Euclidean sense.

Proof. Assume G is partitionable and there is a circle with chords representing G that contains three parallel chords. Removing the center chord of the three parallel chords, and all chords intersecting it, results in a circle with chords that represents $G-N(x)$, where x corresponds to the center chord. Furthermore, the two vertices corresponding to the other two parallel chords will be in different connected components of $G-N(x)$. However, by Property 4, $G-N(x)$ must be connected, a contradiction. □

Theorem 12. *If G is a partitionable circle graph, then G contains an odd chordless cycle of size $\geqslant 5$ or the complement of one.*

Proof. By Lemma 11, G must be $K_{1,3}$-free since containing such a subgraph would imply that every circle with chords that represents G must contain three parallel chords. Therefore, by Theorem 10, G contains an odd chordless cycle of size $\geqslant 5$ or the complement of one. □

Thus, the Berge Strong Perfect Graph Conjecture holds for the class of circle graphs.

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