Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Computational and Applied Mathematics 189 (2006) 513–525
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Data visualization using rational spline interpolation

M. Sarfraz[a,*], Malik Zawwar Hussain[b]

[a] *Department of Information and Computer Science, King Fahd University of Petroleum and Minerals, KFUPM # 1510, Dhahran 31261, Saudi Arabia*

[b] *Department of Mathematics, University of The Punjab, Lahore, Pakistan*

Received 30 September 2004; received in revised form 14 March 2005

Abstract

A smooth curve interpolation scheme for positive, monotonic, and convex data has been developed. This scheme uses piecewise rational cubic functions. The two families of parameters, in the description of the rational interpolant, have been constrained to preserve the shape of the data. The rational spline scheme has a unique representation. The degree of smoothness attained is C^1.

Keywords: Rational spline; Interpolation; Approximation; Visualization; Data

1. Introduction

Interpolation is a fundamental process in scientific visualization. Smooth curve representation, to visualize the scientific data, is of great significance in various areas of scientific research including scientific visualization, computer graphics, geometric modeling, numerical analysis, approximation theory, etc. Specially, when the data are arising from some complex function or from some scientific phenomena, it becomes crucial to incorporate the inherited features of the data. It gives an insight and guide to understand some physical phenomenon pertaining to the data which one would otherwise only have partial information about. It is an effective way of communication as it helps to reflect the numeric data to a quickly understandable pictorial display.

* Corresponding author. Tel./fax: +966 3860 2763.
E-mail addresses: sarfraz@kfupm.edu.sa (M. Sarfraz), malikzawwar@math.pu.edu.pk (M.Z. Hussain).

0377-0427/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2005.04.039

If smoothness is one of the very important requirements for pleasing visual display of the data on one hand, the computational efficiency and accuracy are not less significant on the other hand. Ordinary spline schemes, although smoother, are not helpful for the interpolation of the shaped data. Severely misguided results, violating the inherited features of the data, are seen when undesired oscillations occur. Thus, unwanted oscillations, which may completely destroy the data features must be controlled.

This paper examines the problem of shape preservation of data (x_i, f_i), $i = 1, 2, \ldots, n$, where x_i represents the data site and f_i the data value at site x_i. Positivity, monotony and convexity are the basic and fundamental shapes, which normally arise in everyday scientific phenomena. These shapes are the targeted features here. As a first step, it is planned to generate some empirical model of the data to be visualized. As a second step, a model curve will be constructed which matches the data values at the location allowing no deviations. Afterwards, the model curve will be constrained to reflect a continuous visual display of the data.

Various authors have worked in the area of shape preservation [1–18]. In this paper, the shape preserving interpolation has been studied for positive, monotonic and convex data, using a rational cubic spline. The motivation of this work is due to the past work of many authors, e.g. quadratic interpolation methodology has been adopted in [11,10] for the shape preserving curves. Fritsch and Carlson [7] and Fritsch and Butland [6] have discussed the piecewise cubic interpolation to monotonic data. Also, Passow and Roulier [13] considered the piecewise polynomial interpolation to monotonic and convex data. In particular, an algorithm for quadratic spline interpolation is given in [11]. An alternative to the use of polynomials for the interpolation of monotonic and convex data is the application of piecewise rational quadratic and cubic functions by Gregory [8]. Rational functions have been discussed by Sarfraz [14] in a parametric context.

The theory of methods, in this paper, has a number of advantageous features. It produces C^1 interpolant. No additional points (knots) are needed. In contrast, the quadratic spline methods of Schumaker [18] and the cubic interpolation method of Brodlie and Butt [2] require the introduction of additional knots when used as shape preserving methods. The interpolant is not concerned with an arbitrary degree as in [8]. It is a rational cubic with cubic numerator and cubic denominator. The rational spline curve representation is bounded and unique in its solution.

The paper begins with a definition of the rational function in Section 2 where the description of rational cubic spline, which does not preserve the shape of positive and/or monotone data, is given. Although this rational spline was discussed in [15], it was in the parametric context which was useful for the designing applications. This section reviews it for the scalar representation so that it can be utilized to preserve the scalar valued data. The positivity problem is discussed in Section 3 for the generation of a C^1 spline which can preserve the shape of a positive data. The sufficient constraints on the shape parameters have been derived to preserve and control the positive interpolant. The monotonicity problem is discussed in Section 4 for the generation of a C^1 spline which can preserve the shape of a monotonic data. The sufficient constraints, in this section, lead to a monotonic spline solution. Section 5 discusses the scheme when a data set has convexity features. Section 6 concludes the paper.

2. Rational cubic spline with shape control

Let (x_i, f_i), $i = 1, 2, \ldots, n$, be a given set of data points, where $x_1 < x_2 < \cdots < x_n$. Let

$$h_i = x_{i+1} - x_i, \quad \Delta_i = \frac{f_{i+1} - f_i}{h_i}, \quad i = 1, 2, \ldots, n-1. \tag{2.1}$$

Consider the following piecewise rational cubic function:

$$s(x) \equiv s_i(x) = \frac{U_i(1-\theta)^3 + v_i V_i \theta(1-\theta)^2 + w_i W_i \theta^2(1-\theta) + Z_i\theta^3}{(1-\theta)^3 + v_i\theta(1-\theta)^2 + w_i\theta^2(1-\theta) + \theta^3}, \tag{2.2}$$

where

$$\theta = \frac{x - x_i}{h_i}. \tag{2.3}$$

To make the rational function (2.2) C^1, one needs to impose the following interpolatory properties:

$$\begin{aligned} &s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1},\\ &s^{(1)}(x_i) = d_i, \quad s^{(1)}(x_{i+1}) = d_{i+1}, \end{aligned} \tag{2.4}$$

which provide the following manipulations:

$$\begin{aligned} &U_i = f_i, \quad Z_i = f_{i+1},\\ &V_i = f_i + \frac{h_i d_i}{v_i}, \quad W_i = f_{i+1} - \frac{h_i d_{i+1}}{w_i}, \end{aligned} \tag{2.5}$$

where $s^{(1)}$ denotes the derivative with respect to x and d_i denotes derivative value given at knot x_i. This leads the piecewise rational cubic (2.2) to the following piecewise Hermite interpolant $s \in C^1[x_1, x_n]$:

$$s(x) \equiv s_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{2.6}$$

where

$$\begin{aligned} &P_i(\theta) = f_i(1-\theta)^3 + v_i V_i\theta(1-\theta)^2 + w_i W_i\theta^2(1-\theta) + f_{i+1}\theta^3,\\ &Q_i(\theta) = (1-\theta)^3 + v_i\theta(1-\theta)^2 + w_i\theta^2(1-\theta) + \theta^3. \end{aligned}$$

One can note that when $v_i = w_i = 3$, the rational function obviously becomes the standard cubic Hermite polynomial. Variations for the values of v_i's and w_i's control (tighten or loosen) the curve in different pieces of the curve. This behaviour can be seen in the following subsection.

2.1. Shape control analysis

The parameters v_i's and w_i's can be utilized properly to modify the shape of the curve according to the desire of the user. Their effectiveness, for the shape control at knot points, can be seen that if $v_i, w_{i-1} \to \infty$, then the curve is pulled towards the point (x_i, f_i) in the neighbourhood of the knot position x_i. This shape behaviour can be observed by looking at $s_i(x)$ in Eq. (2.6). This form is similar to that of a Bernstein–Bezier formulation. One can observe that when $v_i, w_{i-1} \to \infty$, then V_i and $W_{i-1} \to f_i$.

The interval shape control behaviour can be observed by rewriting $s_i(x)$ in Eq. (2.6) in the following simplified form:

$$s(x) = f_i(1-\theta) + f_{i+1}\theta + \frac{[(1-\theta)(d_i - \Delta_i) + \theta(\Delta_i - d_{i+1}) + \theta(1-\theta)\Delta_i(w_i - v_i)]h_i\theta(1-\theta)}{Q_i(\theta)}. \tag{2.7}$$

When both v_i and $w_i \to \infty$, it is simple to see the convergence to the following linear interpolant:

$$s(x) = f_i(1-\theta) + f_{i+1}\theta. \tag{2.8}$$

It should be noted that the shape control analysis is valid only if the bounded derivative values are assumed. A description of appropriate choices for such derivative values is made in the following subsection.

2.2. Determination of derivatives

In most applications, the derivative parameters $\{d_i\}$ are not given and hence must be determined either from the given data (x_i, f_i), $i=1, 2, \ldots, n$, or by some other means. In this article, they are computed from the given data in such a way that the C^1 smoothness of the interpolant (2.6) is maintained. These methods are the approximations based on various mathematical theories. The descriptions of such approximations are as follows:

2.2.1. Arithmetic mean method

- This is the three-point difference approximation with

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ (h_i \Delta_{i-1} + h_{i-1}\Delta_i)/(h_i + h_{i-1}) & \text{otherwise}, \; i = 2, 3, \ldots, n-1, \end{cases} \tag{2.9}$$

and the end conditions are given as

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } \operatorname{sgn}(d_1^*) \neq \operatorname{sgn}(\Delta_1), \\ d_1^* = \Delta_1 + (\Delta_1 - \Delta_2)h_1/(h_1 + h_2) & \text{otherwise}. \end{cases} \tag{2.10}$$

$$d_n = \begin{cases} 0 & \text{if } \Delta_{n-1} = 0 \text{ or } \operatorname{sgn}(d_n^*) \neq \operatorname{sgn}(\Delta_{n-1}), \\ d_n^* = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}/ \\ \qquad (h_{n-1} + h_{n-2}) & \text{otherwise}. \end{cases} \tag{2.11}$$

2.2.2. Geometric mean method

- These are the non-linear approximations which are defined as follows:

$$d_i = \begin{cases} 0 & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ \Delta_{i-1}^{h_i/(h_{i-1}+h_i)} \Delta_i^{h_{i-1}/(h_{i-1}+h_i)} & \text{otherwise}, \; i = 2, 3, \ldots, n-1, \end{cases} \tag{2.12}$$

and the end conditions are given as

$$d_1 = \begin{cases} 0 & \text{if } \Delta_1 = 0 \text{ or } \Delta_{3,1} = 0, \\ \Delta_1\{\Delta_1/\Delta_{3,1}\}^{h_1/h_2} & \text{otherwise}. \end{cases} \tag{2.13}$$

$$d_n = \begin{cases} 0 & \text{if } \Delta_{n-1} = 0 \text{ or } \Delta_{n,n-2} = 0, \\ \Delta_{n-1}\{\Delta_{n-1}/\Delta_{n,n-2}\}^{h_{n-1}/h_{n-2}} & \text{otherwise}, \end{cases} \tag{2.14}$$

where

$$\Delta_{3,1} = (f_3 - f_1)/(x_3 - x_1),$$
$$\Delta_{n,n-2} = (f_n - f_{n-2})/(x_n - x_{n-2}). \quad (2.15)$$

For given bounded data, the derivative approximations in Sections 2.2.1 and 2.2.2 are bounded. Hence, for bounded values of the appropriate shape parameters,

$$v_i, w_i, \quad i = 1, 2, \ldots, n-1, \quad (2.16)$$

the interpolant is bounded and unique. Therefore, we can conclude the above discussion in the following:

Theorem 1. *For bounded $v_i, w_i, \forall i$ and the derivative approximations in Sections* 2.2.1 *and* 2.2.2, *the spline solution of the interpolant* (2.6) *exists and is unique*.

3. Positive spline interpolation

The rational spline method, described in the previous section, has deficiencies as far as positivity preserving issue is concerned. For example, the rational cubic in Section 2 does not guarantee to preserve the shape of the positive data. It is required to assign appropriate values to the shape parameters so that it generates a data preserved shape. Thus, some further treatment is required to achieve a shape preserving spline for positive data.

One way to achieve the positivity preserving interpolant for the above spline method is to change the values of shape parameters v_i's and w_i's, on a trial-and-error basis, in those regions of the curve where the shape violations are found. This strategy may result in a required interpolant but this is not a comfortable and accurate way to manipulate the desired shape preserving curve. Another way, which is more effective, useful and is the objective of this article, is the automated generation of positivity preserving curve. This requires an automated computation of suitable shape parameters and derivative values. To proceed with this strategy, some mathematical treatment is required which will be explained in the following paragraphs.

For simplicity of presentation, let us assume the data set to be positive:

$$(x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n),$$

so that

$$x_1 < x_2 < \cdots < x_n, \quad (3.17)$$

and

$$f_1 > 0, \quad f_2 > 0, \ldots, f_n > 0. \quad (3.18)$$

In this paper we shall develop sufficient conditions on piecewise rational cubics under which C^1 positive interpolation is preserved. The key idea, to preserve positivity using $s(x)$, is to assign suitable automated values to v_i, w_i in each interval.

As $v_i, w_i > 0$ guarantee strictly positive denominator $Q_i(\theta)$, initial conditions on v_i, w_i are

$$v_i > 0, \ \ w_i > 0 \quad i = 1, 2, \ldots, n-1. \tag{3.19}$$

Since $Q_i(\theta) > 0$ for all $v_i, w_i > 0$, the positivity of the interpolant (2.6) depends on the positivity of cubic polynomial $P_i(\theta)$. Thus, the problem reduces to the determination of appropriate values of v_i, w_i for which the polynomial $P_i(\theta)$ is positive. Now, $P_i(\theta)$ can be expressed as follows:

$$P_i(t) = \alpha_i \theta^3 + \beta_i \theta^2 + \gamma_i \theta + \delta_i, \tag{3.20}$$

where

$$\left.\begin{aligned} \alpha_i &= (1-w_i) f_{i+1} - (1-v_i) f_i + (d_{i+1} + d_i) h_i, \\ \beta_i &= w_i f_{i+1} - (3 - 2v_i) f_i - (d_{i+1} + d_i) h_i, \\ \gamma_i &= d_i h_i - (3 - v_i) f_i, \\ \delta_i &= f_i. \end{aligned}\right\} \tag{3.21}$$

For the strict inequality (for positive data) in (2.6), according to Butt and Brodlie [3], $P_i(\theta) > 0$ if and only if

$$(P_i'(0), P_i'(1)) \in R_1 U R_2, \tag{3.22}$$

where

$$R_1 = \left\{ (a, b) : a > \frac{-3 f_i}{h_i}, \ b < \frac{3 f_{i+1}}{h_i} \right\}, \tag{3.23}$$

$$\begin{aligned} R_2 = \{(a, b) : \, & 36 f_i f_{i+1} (a^2 + b^2 + ab - 3\Delta_i (a + b) + 3\Delta_i^2) \\ & + 3(f_{i+1} a - f_i b)(2 h_i ab - 3 f_{i+1} a + 3 f_i b) \\ & + 4 h_i (f_{i+1} a^3 - f_i b^3) - h_i^2 a^2 b^2 > 0\}. \end{aligned} \tag{3.24}$$

We have

$$P_i'(0) = \frac{f_i}{h_i}(v_i - 3) + d_i,$$

$$P_i'(1) = d_{i+1} - \frac{f_{i+1}}{h_i}(w_i - 3).$$

Now (3.22) is true when

$$(P_i'(0), P_i'(1)) \in R_1,$$

$$P_i'(0) > \frac{-3 f_i}{h_i}, \qquad P_i'(1) < \frac{3 f_{i+1}}{h_i}.$$

This leads to the following constraints:

$$v_i > m_i, \qquad w_i > M_i, \tag{3.25}$$

where

$$m_i = \mathrm{Max}\left\{0, \frac{-h_i d_i}{f_i}\right\}, \quad M_i = \mathrm{Max}\left\{0, \frac{h_i d_{i+1}}{f_{i+1}}\right\}. \tag{3.26}$$

Further,

$$(P_i'(0), P_i'(1)) \in R_2$$

if

$$\begin{aligned} &36 f_i f_{i+1}[\phi_1^2(r_i, u_i) + \phi_2^2(w_i) + \phi_1(v_i)\phi_2(w_i) - 3\Delta_i(\phi_1(v_i) + \phi_2(w_i)) + 3\Delta_i^2] \\ &\quad + 3[f_{i+1}\phi_1(v_i) - y_i\phi_2(w_i)][2h_i\phi_1(v_i)\phi_2(w_i) - 3f_{i+1}\phi_1(v_i) + 3f_i\phi_2(w_i)] \\ &\quad + 4h_i[f_{i+1}\phi_1^3(v_i) - y_i\phi_2^3(w_i)] - h_i^2\phi_1^2(v_i)\phi_2^2(w_i) > 0, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} &\phi_1(v_i) = P_i'(0), \\ &\phi_2(w_i) = P_i'(1). \end{aligned} \tag{3.28}$$

This leads to the following:

Theorem 2. *For a strictly positive data, the rational cubic interpolant* (2.6) *preserves positivity if and only if either* (3.25) *or* (3.27) *is satisfied.*

Remark 1. The constraints (3.26) can be further modified to incorporate both shape preserving and shape control features. Without loss of generality, one can find parameters r_i and q_i satisfying

$$r_i, q_i \geqslant 1, \tag{3.29}$$

such that the constraints (3.25) and (3.26) lead to the following sufficient conditions for the freedom over the choice of r_i and q_i:

$$v_i = (1 + m_i)r_i, \quad w_i = (1 + M_i)q_i. \tag{3.30}$$

One can make the choice of r_i and q_i to be the greatest lower bound as follows:

$$r_i = 1, \quad q_i = 1. \tag{3.31}$$

This choice satisfies (3.25) and will be considered as a default choice. Some more practical sufficient conditions, which satisfy (3.25) too, are the following:

$$v_i = w_i = 1 + \max(m_i r_i, M_i q_i). \tag{3.32}$$

Remark 2. v_i and w_i satisfying (3.27) can be determined but it requires a lot of computations. Hence, the alternate choice, in Remark 1, can be considered for practical results.

Remark 3. This curve approximation method can be used in both cases when either d_i's are particularly specified or estimated by some method like in Section 2.2.

4. Monotone spline interpolation

The rational cubic in Section 2 does not preserve the shape of the monotonic data. Thus, it looks as if ordinary spline schemes do not provide the desired shape features and hence some further treatment is required to achieve a shape preserving spline for monotonic data. This requires an automated computation of suitable shape parameters and derivative values. To proceed with this strategy, some mathematical treatment is required which will be explained in the following paragraphs.

For simplicity of presentation, let us assume a monotonic increasing set of data so that

$$f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n, \tag{4.33}$$

or equivalently

$$\Delta_i \geqslant 0, \quad i = 1, 2, \ldots, n-1. \tag{4.34}$$

(In a similar fashion one can deal with a monotonic decreasing data.) For a monotonic interpolant $s(x)$, it is then necessary that the derivative parameters follow:

$$d_i \geqslant 0 \quad (d_i \leqslant 0, \text{ for monotonic decreasing data}), \quad i = 1, 2, \ldots, n. \tag{4.35}$$

Now $s(x)$ is monotonic increasing if and only if

$$s^{(1)}(x) \geqslant 0 \tag{4.36}$$

for all $x \in [x_1, x_n]$. For $x \in [x_i, x_{i+1}]$ it can be shown, after some simplification, that

$$s^{(1)}(x) = \sum_{j=1}^{6} \frac{A_{j,i}\theta^{j-1}(1-\theta)^{6-j}}{[Q_i(x)]^2}, \tag{4.37}$$

where

$$\left.\begin{aligned}
A_{1,i} &= d_i,\\
A_{2,i} &= 2w_i\left(\Delta_i - \tfrac{1}{w_i}d_{i+1}\right) + d_i,\\
A_{3,i} &= 3\Delta_i + 2w_i\left(\Delta_i - \tfrac{1}{w_i}d_{i+1}\right) + v_i w_i\left(\Delta_i - \tfrac{1}{v_i}d_i - \tfrac{1}{w_i}d_{i+1}\right),\\
A_{4,i} &= 3\Delta_i + 2v_i\left(\Delta_i - \tfrac{1}{v_i}d_i\right) + v_i w_i\left(\Delta_i - \tfrac{1}{v_i}d_i - \tfrac{1}{w_i}d_{i+1}\right),\\
A_{5,i} &= 2v_i\left(\Delta_i - \tfrac{1}{v_i}d_i\right) + d_{i+1},\\
A_{6,i} &= d_{i+1}.
\end{aligned}\right\} \tag{4.38}$$

The denominator in (4.37), being a squared quantity, is positive; therefore the sufficient conditions for monotonicity on $[x_i, x_{i+1}]$ are

$$A_{j,i} \geqslant 0, \quad j = 1, 2, \ldots, 6, \tag{4.39}$$

where the necessary conditions

$$d_i \geqslant 0 \quad \text{and} \quad d_{i+1} \geqslant 0 \tag{4.40}$$

are assumed. If $\Delta_i > 0$ (strict inequality), then the following are sufficient conditions for (4.39):

$$\begin{aligned} &\Delta_i - \frac{1}{v_i} d_i \geqslant 0, \\ &\Delta_i - \frac{1}{w_i} d_{i+1} \geqslant 0, \\ &\Delta_i - \frac{1}{v_i} d_i - \frac{1}{w_i} d_{i+1} \geqslant 0, \end{aligned} \tag{4.41}$$

which lead to the following constraints:

$$v_i = \frac{l_i d_i}{\Delta_i}, \quad w_i = \frac{k_i d_{i+1}}{\Delta_i}, \tag{4.42}$$

where l_i and k_i are positive quantities satisfying

$$\frac{1}{l_i} + \frac{1}{k_i} \leqslant 1. \tag{4.43}$$

This, together with (4.42), leads to the following sufficient conditions for the freedom over the choice of l_i and k_i:

$$l_i \geqslant 1 + \frac{d_{i+1}}{d_i}, \quad k_i \geqslant 1 + \frac{d_i}{d_{i+1}}. \tag{4.44}$$

One can make the choice of l_i and k_i to be the greatest lower bound as follows:

$$l_i = 1 + \frac{d_{i+1}}{d_i}, \quad k_i = 1 + \frac{d_i}{d_{i+1}}. \tag{4.45}$$

This choice satisfies (4.43). Further simplification of (4.42) and (4.45) leads to the following sufficient conditions for monotonicity:

$$v_i = \frac{d_i + d_{i+1}}{\Delta_i}, \quad w_i = \frac{d_i + d_{i+1}}{\Delta_i}. \tag{4.46}$$

This choice satisfies (4.39) and it also provides acceptable results. It should be noted that if $\Delta_i = 0$, then it is necessary to set $d_i = d_{i+1} = 0$, and thus

$$s(x) = f_i = f_{i+1} \tag{4.47}$$

is a constant on $[x_i, x_{i+1}]$. Hence the interpolant (2.6) is monotonic increasing together with conditions (4.40) and (4.46). For the case where the data are monotonic but not strictly monotonic (i.e., when some $\Delta_i = 0$) it would be necessary to divide the data into strictly monotonic parts. If we set $d_i = d_{i+1} = 0$

whenever $\Delta_i = 0$, then the resulting interpolant will be C^0 at break points. The above discussion can be summarized as:

Theorem 3. *Given conditions* (4.35) *on the derivative parameters*, *the conditions for monotonicity in* (4.46) *are the sufficient conditions for the interpolant* (2.6) *to be monotonic increasing*.

5. Convex spline interpolation

The rational cubic, in Section 2, does not preserve the shape of the convex data. Thus, it looks as if ordinary spline schemes do not provide the desired shape features and hence some further treatment is required to achieve a shape preserving spline for convex data. This requires an automated computation of suitable shape parameters and derivative values. To proceed with this strategy, some mathematical treatment is required which will be explained in the following paragraphs.

For simplicity of presentation, let us assume a strictly convex set of data so that

$$\Delta_1 < \Delta_2 < \cdots < \Delta_{n-1}. \tag{5.48}$$

In a similar fashion, one can deal with a concave data so that

$$\Delta_1 > \Delta_2 > \cdots > \Delta_{n-1}. \tag{5.49}$$

For a convex interpolant $s(x)$, it is then necessary that the derivative parameters should be such that

$$d_1 < \Delta_1 < \cdots \Delta_{i-1} < d_i < \Delta_i \cdots < \Delta_{n-1} < d_n, \tag{5.50}$$

and

$$(d_1 > \Delta_1 > \cdots \Delta_{i-1} > d_i > \Delta_i \cdots > \Delta_{n-1} > d_n, \text{ for concave data}).$$

Now $s(x)$ is convex if and only if

$$s^{(2)}(x) \geqslant 0, \tag{5.51}$$

for all $x \in [x_1, x_n]$. This can be achieved by differentiating (4.37) as follows:

$$s^{(2)}(x) = \sum_{j=1}^{8} \frac{B_{j,i}\theta^{j-1}(1-\theta)^{8-j}}{h_i[Q_i(x)]^3}, \tag{5.52}$$

where

$$\left.\begin{array}{lcl}
B_{1,i} & = & A_{2,i} - A_{1,i}(2v_i - 1), \\
B_{2,i} & = & 2A_{3,i} - A_{2,i}(v_i - 2) - A_{1,i}(v_i + 4w_i), \\
B_{3,i} & = & 3A_{4,i} + 3A_{3,i} - 3w_i A_{2,i} - 3A_{1,i}(w_i + 2), \\
B_{4,i} & = & 4A_{5,i} + 4A_{4,i}(v_i + 1) + A_{3,i}(v_i - 2w_i) - A_{2,i}(2w_i + 5) - 5A_{1,i}, \\
B_{5,i} & = & 5A_{6,i} + A_{5,i}(2v_i + 5) + A_{4,i}(2v_i - w_i) - A_{3,i}(w_i - 4) - 4A_{2,i}, \\
B_{6,i} & = & 3A_{6,i}(v_i + 2) + 3v_i A_{5,i} - 3A_{4,i} - 3A_{3,i}, \\
B_{7,i} & = & A_{6,i}(4v_i + w_i) + A_{5,i}(w_i - 2) - 2A_{4,i}, \\
B_{8,i} & = & A_{6,i}(2w_i - 1) - A_{5,i}.
\end{array}\right\} \tag{5.53}$$

Since the denominator in (5.52), for the selection of $v_i, w_i > 0$, is positive, the sufficient conditions for convexity on $[x_i, x_{i+1}]$ are

$$v_i, w_i > 0, \quad B_{j,i} \geqslant 0, \quad j = 1, 2, \ldots, 8, \tag{5.54}$$

where the necessary conditions

$$\Delta_i - d_i \geqslant 0 \quad \text{and} \quad d_{i+1} - \Delta_i \geqslant 0 \tag{5.55}$$

are assumed. After some simplifications, one can rewrite the first and the last equations, from (5.53), as follows:

$$\begin{aligned} B_{1,i} &= 2\{(w_i - v_i)\Delta_i + v_i(\Delta_i - d_i) - (d_{i+1} - d_i)\}, \\ B_{8,i} &= 2\{(w_i - v_i)\Delta_i + w_i(d_{i+1} - \Delta_i) - (d_{i+1} - d_i)\}. \end{aligned} \tag{5.56}$$

If $\Delta_i - d_i > 0$ and $d_{i+1} - \Delta_i > 0$ (strict inequalities), then the following are sufficient conditions for (5.56):

$$\begin{aligned} &v_i = w_i, \\ &v_i(\Delta_i - d_i) - (d_{i+1} - d_i) \geqslant 0, \\ &w_i(d_{i+1} - \Delta_i) - (d_{i+1} - d_i) \geqslant 0. \end{aligned} \tag{5.57}$$

These are equivalent to the following constraints:

$$v_i = w_i = q_i + \max\left(\frac{d_{i+1} - d_i}{\Delta_i - d_i}, \frac{d_{i+1} - d_i}{d_{i+1} - \Delta_i}\right), \tag{5.58}$$

where q_i are non-negative quantities satisfying

$$q_i \geqslant 0. \tag{5.59}$$

After some manipulations, it is trivial to show that the sufficient conditions (5.58), for (5.56), are also sufficient for (5.54).

Since

$$\frac{d_{i+1} - \Delta_i}{\Delta_i - d_i} + \frac{\Delta_i - d_i}{d_{i+1} - \Delta_i} \geqslant \max\left(\frac{d_{i+1} - d_i}{\Delta_i - d_i}, \frac{d_{i+1} - d_i}{d_{i+1} - \Delta_i}\right), \tag{5.60}$$

therefore, the sufficient conditions (5.58) for convexity take the following form:

$$v_i = w_i = q_i + \frac{d_{i+1} - \Delta_i}{\Delta_i - d_i} + \frac{\Delta_i - d_i}{d_{i+1} - \Delta_i}, \quad q_i \geqslant 0. \tag{5.61}$$

However, the following choice of parameters

$$v_i = w_i = q_i + \max\left(\frac{d_{i+1} - d_i}{\Delta_i - d_i}, \frac{d_{i+1} - d_i}{d_{i+1} - \Delta_i}\right), \quad q_i = 0, \tag{5.62}$$

will be considered for practical implementation of default curve design. This choice satisfies (5.58) and it also provides acceptable results.

Remark 4. The default value of the parameters q_i, being taken as zero, provides visually pleasing results and produces automated curve interpolation. Further modification is achieved by taking other positive values in various intervals.

Remark 5. It should be noted that if $\Delta_i - d_i = 0$ or $d_{i+1} - \Delta_i = 0$, then it is necessary to set $d_i = d_{i+1} = \Delta_i$. The interpolant then will be linear in that region, i.e.,

$$s(x) = (1 - \theta) f_i + \theta f_{i+1}. \tag{5.63}$$

It should be also noted that if $\Delta_i = 0$, then it is necessary to set $d_i = d_{i+1} = 0$, and thus

$$s(x) = f_i = f_{i+1} \tag{5.64}$$

is a constant on $[x_i, x_{i+1}]$. Hence the interpolant (2.6) is convex together with conditions (5.61). For the case, where the data are convex but not strictly convex, it would be necessary to divide the data into strictly convex parts. If we set $d_i = d_{i+1} = 0$ whenever $\Delta_i = 0$, then the resulting interpolant will be C^0 at break points.

The above discussion can be summarized as follows:

Theorem 4. *Given conditions* (5.50) *on the derivative parameters and the data*, *the constraints* (5.61) *are the sufficient conditions for the interpolant* (2.6) *to be convex.*

6. Concluding remarks

A rational cubic interpolant, with two families of shape parameters, has been utilized to obtain C^1 positivity, monotonicity, and convexity preserving interpolatory spline curves. The shape constraints are restricted on shape parameters to assure the shape preservation of the data. For the C^1 interpolant, the choices on the derivative parameters have been defined. The solution to the shape preserving spline exists and provides a unique solution. The scheme is automatic and the user does not have to worry about struggling and looking for some appropriate choice of parameters as in the case of ordinary rational spline.

Accuracy and computational efficiency are very important and the presented interpolation scheme is computationally efficient. However, more computational saving can be pursued. The rational spline scheme could have been implemented and it would have demonstrated nice looking visually pleasant results. This work is left due to the fear of the length of the paper. The authors intend to do it in a subsequent paper.

The work done, in this paper, can be extended for the data type having dual features of shape. The authors are in the process of completing it as a future work. A possible extension of this work, for future, may also be to act on the parameterization and relax the continuity conditions from C^1 to G^1. One can also think of generalizing the curve case to surface case as a future work; this research is also in progress with the authors.

Acknowledgements

The authors are thankful for the anonymous referee's useful comments. The first author acknowledges the support of King Fahd University of Petroleum and Minerals in the development of the paper.

References

[1] K.W. Brodlie, Methods for drawing curves, in: R.A. Earnshaw (Ed.), Fundamental Algorithm for Computer Graphics, Springer, Berlin/Heidelberg, 1985, pp. 303–323.
[2] K.W. Brodlie, S. Butt, Preserving convexity using piecewise cubic interpolation, Comput. & Graphics 15 (1991) 15–23.
[3] S. Butt, K.W. Brodlie, Preserving positivity using piecewise cubic interpolation, Comput. & Graphics 17 (1) (1993) 55–64.
[4] P. Constantini, Boundary-valued shape preserving interpolating splines, ACM Trans. Math. Software 23 (2) (1997) 229–251.
[5] A. DeVore, Z. Yan, Error analysis for piecewise quadratic curve fitting algorithms, Comput. Aided Geom. Design 3 (1986) 205–215.
[6] F.N. Fritsch, J. Butland, A method for constructing local monotone piecewise cubic interpolants, SIAM J. Sci. Statist. Comput. 5 (1984) 303–304.
[7] F.N. Fritsch, R.E. Carlson, Monotone piecewise cubic interpolation, SIAM J. Numer. Anal. 17 (1980) 238–246.
[8] J.A. Gregory, Shape preserving spline interpolation, Comput. Aided Design 18 (1) (1986) 53–57.
[9] K. Greiner, A survey on univariate data interpolation and approximation by splines of given shape, Math. Comput. Modelling 15 (1991) 97–l06.
[10] A. Lahtinen, Monotone interpolation with application to estimation of taper curves, Ann. Numer. Math. 3 (1996) 151–161.
[11] D.F. McAllister, J.A. Roulier, An algorithm for computing a shape preserving osculatory quadratic spline, ACM Trans. Math. Software 7 (1981) 331–347.
[12] H.P. Moreton, C.H. Sequin, Minimum variation curves and surfaces for computer-aided geometric design, designing fair curves and surfaces, Nick Sapidis (Ed.), Proc. of SIAM'94 Conference, 1995, pp. 123–159.
[13] E. Passow, J.A. Roulier, Monotone and convex spline interpolation, SIAM J. Numer. Anal. 14 (1977) 904–909.
[14] M. Sarfraz, Convexity preserving piecewise rational interpolation for planar curves, Bull. Korean Math. Soc. 29 (2) (1992) 193–200.
[15] M. Sarfraz, Interpolatory rational cubic spline with biased, point and interval tension, Comput. & Graphics 16 (4) (1992) 427–430.
[16] M. Sarfraz, Preserving monotone shape of the data using piecewise rational cubic functions, Comput. & Graphics 21 (1) (1997) 5–14.
[17] M. Sarfraz, S. Butt, M.Z. Hussain, Visualization of shaped data by a rational cubic spline interpolation, Internat. J. Comput. & Graphics 25 (5) (2001) 833–845.
[18] L.L. Schumaker, On shape preserving quadratic spline interpolation, SIAM J. Numer. Anal. 20 (1983) 854–864.