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Properties of Certain Algebras between L^∞ and H^∞

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The main result of this paper is that if F is a closed subset of the unit circle, then $(H^\infty + L_F^\infty)/H^\infty$ is an M-ideal of L^∞/H^∞. Consequently, if $f \in L^\infty$ then f has a closest element in $H^\infty + L_F^\infty$. Furthermore, if $|F| > 0$ then $L^\infty/(H^\infty + L_F^\infty)$ is not the dual of any Banach space.

1. INTRODUCTION

Let L^∞ denote the usual Lebesgue space of functions on the unit circle. Let H^∞ denote the subalgebra of boundary values of bounded analytic functions in $|z| < 1$ and $H^\infty + C$ denote the closed linear span of H^∞ and C, where C is the set of continuous complex-valued functions on $|z| = 1$.

In [2], Axler *et al.* prove the following theorem: If f is in L^∞, then $\operatorname{dist}(f, H^\infty + C) = \|f - h\|_\infty$, for some h in $H^\infty + C$. Here $\|\cdot\|_\infty$ denotes the essential supremum norm and the distance is measured in this norm. In [9], Luecking, proves the following theorem: $(H^\infty + C)/H^\infty$ is an M-ideal of L^∞/H^∞.

In this paper, we prove Theorem 3.1 and Corollary 3.2 which yield the above results as special cases. To be more specific, let F be a closed subset of the unit circle. We prove that $(H^\infty + L_F^\infty)/H^\infty$ is an M-ideal of L^∞/H^∞. As a consequence, we get: If $f \in L^\infty$ then dist $(f, H^\infty + L_F^\infty) = \|f - h\|_\infty$, for some h in $H^\infty + L_F^\infty$. The preceding result is a contribution to an open question [2] which asks whether any L^∞ function has a closest element to an arbitrary Douglas algebra. Also, we prove (Theorem 3.3) that if $|F| > 0$ then $L^\infty/(H^\infty + L_F^\infty)$ is not the dual of any Banach space, which is a generalization of the known result [2] that $L^\infty/(H^\infty + C)$ is not the dual of any Banach space. In Section 4, we present some applications concerning supports of extreme points of the unit ball of $(H^\infty)^\perp$.

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2. Preliminaries

Let X be a compact Hausdorff space. We denote by $C(X)$ the space of all continuous complex-valued functions on X. Let A be a closed subalgebra of $C(X)$. A closed subset, S, of X is called a peak set for A if there exists f in A such that $f=1$ on S and $|f|<1$ off S. A weak peak set for A is an arbitrary intersection of peak sets for A. A closed subset, E, of X is called the essential set for A if E is the zero set of the largest closed ideal in $C(X)$ which lies in A. The following properties of the essential set E of an algebra A are needed in this paper.

(i) E is a weak peak set for A [3, p. 145].

(ii) If $\mu \perp A$, then the support of μ lies in E [10, p. 64].

Let Y be a Banach space. A subspace J of Y is called an L-ideal if there is a projection p of Y onto J such that $\|y\| = \|py\| + \|y - py\|$, $y \in Y$. Such a projection is called an L-projection. A subspace K of a Banach space X is called an M-ideal if the annihilator K is an L-ideal of X^* (see [1] for these concepts). The following properties of an M-ideal of X are needed in this paper.

Lemma A [1, Corollary 5.6]. *If M is an M-ideal of X and if $x \in X$, then there exists $m \in M$ such that* $\operatorname{dist}(x, M) = \|x - m\|$.

Lemma B [7, Theorem 3]. *If M is an M-ideal of X and if $x \in X \backslash M$ then the span* $\{m : m \in M, \operatorname{dist}(x, M) = \|x - m\|\} = M$.

We identify L^∞ with $C(M(L^\infty))$, via the Gelfand transform, where $M(L^\infty)$ denotes the maximal ideal space of L^∞. Thus H^∞ can be considered as a function algebra on $M(L^\infty)$. No notational distinction will be made between f in L^∞ viewed as a function on the unit circle and its Gelfand transform $\hat{f}$ viewed as a continuous function on $M(L^\infty)$.

Let z denote the identity function on the unit circle, T. For $\alpha \in T$ the fiber X_α of $M(L^\infty)$ over α is the set $X_\alpha = \{\phi \in M(L^\infty): \phi(z) = \alpha\}$. The fiber Y_α of $M(H^\infty)$ over α is the set $Y_\alpha = \{\phi \in M(H^\infty): \phi(z) = \alpha\}$.

3. The Main Results

Let F be a closed subset of the unit circle T, and let L_F^∞ be the set of those L^∞ functions which are continuous at each point of F. Davie *et al.* [5, p. 66] have shown that $H^\infty + L_F^\infty$ is a closed subalgebra of L^∞, and furthermore that it is a Douglas algebra. A closed subalgebra A of L^∞ which contains H^∞ is called a Douglas algebra if A is the closed algebra generated by H^∞

and $\{\bar{b} \in A: b$ is a Blaschke product$\}$. Chang and Marshall [4] have shown that every closed subalgebra of L^∞ which contains H^∞ is a Douglas algebra.

THEOREM 3.1. $(H^\infty + L_F^\infty)/H^\infty$ is an M-ideal in the space L^∞/H^∞.

The case in which $F = T$ has been proved by Luecking [9].

COROLLARY 3.2. *If $f \in L^\infty$, then* $\operatorname{dist}(f, H^\infty + L_F^\infty) = \|f - h\|_\infty$, *for some* $h \in H^\infty + L_F^\infty$.

In case $F = T$, this corollary is Theorem 4 of [2], where the proof given there uses an operator theory technique.

Proof of Corollary 3.2. By Theorem 3.1 and Lemma A we have for any f in L^∞, there exists g in $H^\infty + L_F^\infty$ such that $\operatorname{dist}(f, H^\infty + L_F^\infty) = \operatorname{dist}(f - g, H^\infty)$. It is well-known (see, e.g., [8, p. 197]) that there exists g_0 in H^∞ such that $\operatorname{dist}(f - g, H^\infty) = \|f - g - g_0\|_\infty$. Set $h = g + g_0$; then $\operatorname{dist}(f, H^\infty + L_F^\infty) = \|f - h\|_\infty$.

Remark. If $f \in L^\infty$, $f \notin H^\infty + L_F^\infty$ then the best approximation h in Corollary 3.2 is not unique. Indeed, if $f \notin H^\infty + L_F^\infty$ then by Theorem 3.1 and Lemma B, h is never unique.

Proof of Theorem 3.1. Let $E = \bigcup_{\alpha \in F} X_\alpha$, then E is closed in $M(L^\infty)$. We claim that E is the essential set for $H^\infty + L_F^\infty$. By Theorem 11.6 [5, p. 62], $H^\infty + L_F^\infty = \bigcup_{\alpha \in F} H_{X_\alpha}^\infty$, where $H_{X_\alpha}^\infty = \{f \in L^\infty;\ f_{|X_\alpha} \in H_{|X_\alpha}^\infty\}$. The essential set of $H_{X_\alpha}^\infty$ is X_α. To see this, note that $H_{X_\alpha}^\infty = H^\infty + J_{X_\alpha}$, where $J_{X_\alpha} = \{f \in L^\infty : f(x_\alpha) = 0\}$. Thus the essential set S for $H_{X_\alpha}^\infty$ lies in X_α. If S is proper, let $x \in X_\alpha \backslash S$ and choose a clopen set W containing x such that $W \cap S = \phi$. Since $X_w(S) = 0$, the function $f = X_w$ is in $H_{X_\alpha}^\infty$. Thus $f_{|X_\alpha} \in H_{|X_\alpha}^\infty$, and moreover $f_{|X_\alpha} = \{0, 1\}$. This is a contradiction, because the maximal ideal space of $H_{|X_\alpha}^\infty$ is Y_α, which is connected [6, p. 188]. Thus X_α is the essential set for $H_{X_\alpha}^\infty$. Consequently, $\bigcup_{\alpha \in F} X_\alpha$ lies in the essential set E' of $H^\infty + L_F^\infty$. Now let $f \in L^\infty$ be such that $f(\bigcup_{\alpha \in F} X_\alpha) = 0$. Then $f \in \bigcap_{\alpha \in F} H_{X_\alpha}^\infty$. Since E' is the essential set of $H^\infty + L_F^\infty$ and $\bigcup_{\alpha \in F} X_\alpha$ is closed, we get $E' = E$. Thus $E = \bigcup_{\alpha \in F} X_\alpha$ is the essential set of $H^\infty + L_F^\infty$.

The dual space $(L^\infty/H^\infty)^*$ is identified with the space $(H^\infty)^\perp = \{\mu \in C(X)^*: \int f d\mu = 0$ for all $f \in H^\infty\}$. We identify $((H^\infty + L_F^\infty)/H^\infty)^\perp$ with $(H^\infty + L_F^\infty)^\perp = \{\mu \in C(X)^*: \int f d\mu = 0$ for all $f \in H^\infty + L_F^\infty\}$. Thus to prove the theorem we have to produce an L-projection p of $(H^\infty)^\perp$ onto $(H^\infty + L_F^\infty)^\perp$.

Let m denote the Lifting Lebesgue measure from T to $X = M(L^\infty)$. That is, $\int_X \hat{f} dm = (1/2\pi) \int_0^{2\pi} f(e^{i\phi})\, d\phi$, for $f \in L^\infty$. Let $\mu \perp H^\infty$. Write $\mu = \mu_a + \mu_s$, where $\mu_a \ll m$ and $\mu_s \perp m$. By [6, p. 186], the measure μ_s is $\perp$ to $H^\infty + C$. Define $P\mu = X_E \mu_s$, where X_E is the characteristic function of E.

It is easy to see that $\|\mu\| = \|p\mu\| + \|\mu - p\mu\|$ and $p^2\mu = p\mu$. Note that $H^\infty + L_F^\infty|_E = H^\infty + C|_E$ and that E is a weak peak set for $H^\infty + C$. By [3, p. 106], $X_E\mu_s \perp H^\infty + C$. Consequently $X_E\mu_s \perp H^\infty + L_F^\infty$.

Finally, we have to show that P is onto. Let $\mu \perp H^\infty + L_F^\infty$. By (i) of Section 2 we have support $\mu \subset E$. Thus $\mu = X_E\mu$. By [9], $\mu \perp m$. Hence $p\mu = \mu = X_E\mu$. This ends the proof of Theorem 3.1.

The authors of [2] proved that $L^\infty/(H^\infty + C)$ is not the dual of any Banach space (in contrast to L^∞/H^∞, which is the dual of the space of functions in H^1 which have mean value zero). The following theorem is a generalization of their result.

THEOREM 3.3. *Let F be a closed subset of T such that $|F| > 0$. Then $L^\infty/(H^\infty + L_F^\infty)$ has no extreme points.*

Proof. Let $f + (H^\infty + L_F^\infty)$ be an extreme point of $L^\infty/(H^\infty + L_F^\infty)$. By Corollary 3.2, we can assume that $\|f\|_\infty = 1$. We claim that there exists h in $H^\infty + L_F^\infty$ such that $\|f + h\|_\infty = 1$ and h is not identically zero on E, where E is the essential set of $H^\infty + L_F^\infty$. By Lemma B, $\text{span}\{h + H^\infty\colon h \in H^\infty + L_F^\infty, \text{dist}(f, H^\infty + L_F^\infty) = \|f + h\|_\infty = 1\} = (H^\infty + L_F^\infty)/H^\infty$.

If $h = 0$ on E for every h in the above identity, then we would have the following contradiction: Let $g \in H^\infty + L_F^\infty$. Then there exists $h_1, h_2, \ldots, h_n$ in $H^\infty + L_F^\infty$, $h_i = 0$ on E, $i = 1, \ldots, n$; such that $g \in (h_1 + \cdots + h_n) + H^\infty$. Thus $H^\infty + L_F^\infty|_E = H^\infty|_E$. By (ii) of Section 2, E is a weak peak set for $H^\infty + L_F^\infty$. This forces $H^\infty + L_F^\infty|_E$ to be closed in $L^\infty|_E$. Since H^∞ is strongly logmodular on X, we get E is a weak peak set for H^∞. Let f be in H^∞ such that $f = 1$ on E and $|f| \neq 1$ on X. Thus $f = 1$ on F. Since $|F| > 0$ we have $f = 1$ a.e. This contradiction shows that there exists h in $H^\infty + L_F^\infty$ such that $\|f + h\|_\infty = 1$ and $h(x_0) \neq 0$ for some x_0 in E. Pick X_α such that $x_0 \in X_\alpha$ for some α in F. Note that $|f(x_0) + \frac{1}{2}h(x_0)| \neq 1$. For x in $M(L^\infty)$:

$$|f(x) + \tfrac{1}{2}h(x)| \leqslant \tfrac{1}{2}|f(x)| + \tfrac{1}{2}|f(x) + h(x)| \leqslant 1.$$

Define g in L^∞:

$$g(x) = 1 - |f(x) + \tfrac{1}{2}h(x)|, x \text{ in } M(L^\infty).$$

Thus $g \geqslant 0$ and moreover g is not identically zero. Let $S = \{x \in M(L^\infty)\colon g(x) = 0\}$, and let x_1 be in X_α such that $x_0 \neq x_1$. Choose a clopen set W such that $x_0 \in W$ and $W \cap [S \cup \{x_1\}] = \phi$. Let $a = \min\{g(x)\colon x \in W\}$, then $a > 0$. Hence $g \geqslant aX_W$. The function $X_W \notin H_{X_\alpha}^\infty$ because the maximal ideal space of $H^\infty|_{X_\alpha}$ is connected. Thus $X_W \notin H^\infty + L_F^\infty$, and hence $f \pm aX_W + (H^\infty + L_F^\infty) \neq f + (H^\infty + L_F^\infty)$. Furthermore, $\|f \pm aX_W + (H^\infty + L_F^\infty)\| \leqslant \|f \pm aX_W + \frac{1}{2}h\| \leqslant \sup_{x \in M(L^\infty)} \{|f(x) + \frac{1}{2}h(x)| + g(x)\} = 1$. Since $f + (H^\infty + L_F^\infty) = \frac{1}{2}[f + aX_W + (H^\infty + L_F^\infty)] + \frac{1}{2}[f - aX_W + (H^\infty + L_F^\infty)]$, we conclude that $L^\infty/(H^\infty + L_F^\infty)$ has no extreme points. This completes the proof of Theorem 3.3.

Remark. The condition $|F| > 0$ is essential in the proof of Theorem 3.3. The author does not know how to settle down the case $|F| = 0$.

4. Further Results

The proof of Theorem 3.1 allows us to state the following general result (*): If A is a closed subalgebra of L^∞ which contains H^∞, and if $A|_E = H^\infty + C|_E$, where E is the essential set of A, then A/H^∞ is an M-ideal of L^∞/H^∞.

THEOREM 4.1. *Let μ and v be any two extreme points of the unit ball of (H^∞). Then one of the following three conditions must hold.*

(1) $\operatorname{supp}\mu \subset \sup v$,

(2) $\operatorname{supp} v \subset \operatorname{supp}\mu$,

(3) $\operatorname{supp} v \cap \sup \mu = \phi$.

The author wishes to give a proof of the above result using an M-ideal approach, although one could perhaps give a proof following Hoffman's unpublished notes. In any case, we need the following results from these notes.

THEOREM C. *Let S be a closed subset of $M(L^\infty)$ which is an antisymmetric set for H^∞. Then S is a weak peak set for H^∞.*

Proof. Let $\bigcup$ be a neighborhood of S. Choose $f \in (H^\infty)^{-1}$ such that $\|f\|_\infty = 1$, $|f| = 1$ on S and $|f| < 1$ off $\bigcup$. Then $f + 1/f$ is real valued on S, and hence is constant on S. Thus $\operatorname{Re} f$ is constant on S. Similarly, $\operatorname{Im} f$ is constant on S, and so f is constant on S. Multiplying f by a constant of modulus one, we can assume $f = 1$ on S. The result now follows.

THEOREM D. *Let m and μ be probability measures on $M(L^\infty)$. If $m(\operatorname{supp}\mu) = \mu(\operatorname{supp} m) = 0$, them* $\operatorname{supp} m \cap \operatorname{supp}\mu = \phi$.

Proof. From the hypothesis: $m(\operatorname{supp} m \backslash \operatorname{supp}\mu) = 1$ and $\mu(\operatorname{supp}\mu \backslash \operatorname{supp} m) = 1$. Thus $\operatorname{supp} m \subset \overline{(\operatorname{supp} m \backslash \operatorname{supp}\mu)}$ and $\operatorname{supp}\mu \subset \overline{(\operatorname{supp}\mu \backslash \operatorname{supp} m)}$. But $\operatorname{supp} m \backslash \operatorname{supp}\mu$ and $\operatorname{supp}\mu \backslash \operatorname{supp} m$ are disjoint open sets and thus they have disjoint closures. Hence $\operatorname{supp} m \cap \operatorname{supp}\mu = \phi$.

Proof of Theorem 4.1. Let $s_1 = \operatorname{supp}\mu$ and $s_2 = \operatorname{supp} v$, and suppose that S_1 intersects S_2. The sets s_1 and s_2 are closed antisymmetric sets for H^∞

[3, p. 138]. Hence by Theorem C, s_1 and s_2 are weak peak sets for H^∞. By (*), $H^\infty_{S_1}/H^\infty$ and $H^\infty_{S_2}/H^\infty$ are M-ideals in L^∞/H^∞, where $H^\infty_{S_i}\{f\in L^\infty: f|_{S_i}\in H^\infty|_{S_i}\}, i=1,2$.

CLAIM. $X_{S_1}v=0$ and $X_{S_2}\mu=0$. Assume the claim for a moment, then $|v|(s_1)=|\mu|(s_2)=0$. Hence by Theorem D, we have $s_1\cap s_2=\phi$. This contradiction completes the proof of Theorem 4.1.

Proof of the claim. Let $p: (H^\infty)^\perp\to(H^\infty_{S_1})^\perp$ be an L-projection. If $v\perp H^\infty_{S_1}$, then by (ii) of Section 2 we get $s_2\subset s_1$. Thus $v\notin(H^\infty_{S_1})^\perp$. We show that $pv=0$. If not, then we can write

$$v=\|pv\|\frac{pv}{\|pv\|}+\|v-pv\|\frac{v-pv}{\|v-pv\|}.$$

Since $\|pv\|+\|v-pv\|=1$ and v is an extreme point, we have $v=pv/\|pv\|$. Hence $\|pv\|\,pv=pv$. Thus we have either $pv=0$ or $\|pv\|=1$. By assumption $pv\neq 0$, so we conclude that $\|v-pv\|=0$. That is, $pv=v$. This is a contradiction because $v\notin(H^\infty_{S_1})^\perp$, while $pv\perp H^\infty_{S_1}$. Thus we conclude that $pv=0$. Since $\chi_{S_1}v\perp H^\infty_{S_1}$, we get $p(v-X_{S_1}v)=-X_{S_1}v$. Now,

$$\|V-X_{S_1}v\|=\|X_{S_1}v\|+\|v\|=\|X_{S_1}v\|+\|v-X_{S_1}v\|+\|X_{S_1}v\|.$$

Thus $\|X_{S_1}v\|=0$. Hence $X_{S_1}v=0$. Similarly $X_{S_2}\mu=0$. This ends the proof of the claim, and consequently the proof of Theorem 4.1.

Remark 1. The above theorem is true if H^∞ is replaced by an arbitrary Douglas algebra. We omit the details.

Remark 2. Let μ and v be as in Theorem 4.1. Then the following are equivalent:

(1) $\operatorname{supp}\mu\subseteq\operatorname{supp}v$,

(2) For every Blaschke product B, which is constant on $\operatorname{supp}v$, implies B is constant on $\operatorname{supp}u$.

Proof. Clearly (1) implies (2). Conversely, let $A_1=\{f\in L^\infty: f|_{\operatorname{supp}v}\in H^\infty|_{\operatorname{supp}v}\}$ and $A=\{f\in L^\infty; f|_{\operatorname{supp}\mu}\in H^\infty|_{\operatorname{supp}\mu}\}$. It is easy to see that A_1 and A are closed subalgebras of L^∞. We claim that $A_1\subseteq A$. Let B be any Blaschke product such that $\bar{B}\in A_1$. Then B is constant on $\operatorname{supp}v$. By condition (2), B is constant on $\operatorname{supp}\mu$. Thus $\bar{B}\in A$. By the Chang–Marshall Theorem [4] we get $A_1\subseteq A$. Thus the essential set of A lies in the essential set of A_1. That is, $\operatorname{supp}\mu\subseteq\operatorname{supp}v$.

Finally, we end the paper with the following open question: What are the M-ideals of L^∞/H^∞? Perhaps the question is related to the condition in (*).

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