Engineering Science and Technology, an International Journal 18 (2015) 32–41

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journal homepage: http://www.elsevier.com/locate/jestch

Full length article

# A new hypercube variant: Fractal Cubic Network Graph

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ARTICLE INFO

Article history:
Received 18 April 2014
Received in revised form
6 August 2014
Accepted 2 September 2014
Available online 27 November 2014

Keywords:
Hamiltonian graph
Eulerian graph
Gray code
Interconnection network
Fractal Cubic Network Graph

ABSTRACT

Hypercube is a popular and more attractive interconnection networks. The attractive properties of hypercube caused the derivation of more variants of hypercube. In this paper, we have proposed two variants of hypercube which was called as “Fractal Cubic Network Graphs”, and we have investigated the Hamiltonian-like properties of Fractal Cubic Network Graphs $\mathrm{FCNG}_r(n)$. Firstly, Fractal Cubic Network Graphs $\mathrm{FCNG}_r(n)$ are defined by a fractal structure. Further, we show the construction and characteristics analyses of $\mathrm{FCNG}_r(n)$ where $r = 1$ or $r = 2$. Therefore, $\mathrm{FCNG}_r(n)$ is a Hamiltonian graph which is obtained by using Gray Code for $r = 2$ and $\mathrm{FCNG}_1(n)$ is not a Hamiltonian Graph. Furthermore, we have obtained a recursive algorithm which is used to label the nodes of $\mathrm{FCNG}_2(n)$. Finally, we get routing algorithms on $\mathrm{FCNG}_2(n)$ by utilizing routing algorithms on the hypercubes.


## 1. Introduction

In this paper, we considered a new perspective for network topologies. The most basic network topologies used in practice are trees, cycles, grids, tori, meshes and hypercubes. We used hypercube topology for construction of a hypercube variant – called Fractal Cubic Network Graph.

In recent years, hypercubes of dimension $n$ $H(n)$ have been studied extensively main properties (cf. the survey [16] and [7,9,12,13,23,24,26,28]). Also, hypercube variants have been derived to ensure the performance increasing of hypercubes. These are *the Folded Hypercube* ([2,8]), *the Crossed Cube* (*Twisted Cube*) ([11,14,22]) and *the Hierarchical Cubic Network* (cf. the survey [1] and [10,17,18]) which are the most popular hypercube variants. Aims of these approaches are just to reduce the diameter of the hypercube but reserve its advantages.

Rest of paper, we denote Fractal Cubic Network Graph ($\mathrm{FCNG}_r(n)$) where $r = 1$ or $r = 2$. Fractal Cubic Network Graph ($\mathrm{FCNG}_r(n)$) is defined and constructed as hyper-cubic networks. $\mathrm{FCNG}_r(n)$ is a recursive graph which is constructed by a fractal structure in Fig. 1. $\mathrm{FCNG}_2(n)$ is a Hamilton Graph and $\mathrm{FCNG}_1(n)$ is not a Hamiltonian Graph.

In this paper, we have investigated the Hamiltonian-like properties of $\mathrm{FCNG}_r(n)$. The rest of this paper is organized as follows. Section 2 informs about fractals. Section 3 describes the definitions of $\mathrm{FCNG}_r(n)$ and shows the characteristics analyses of $\mathrm{FCNG}_r(n)$. The recursive algorithm, which is to label to the nodes of mesh structure in Theorem 4, is given in Section 4. In Section 5, we get routing algorithms on $\mathrm{FCNG}_2(n)$ are similar to the routing algorithms on the hypercubes. In Section 6, we give deriving subcubes from faulty hypercubes. Finally, we give a conclusion.

## 2. Background and related work

In 1902, Jacques Hadamard introduced the idea of well-posed problem in differential equations theory. According to Hadamard, a well-posed problem must have three properties. These are existence, uniqueness and sensitively related to initial condition. First definition of Chaos is third properties. In the last century, one of the greatest revolutions in science, Chaos Theory was emerged by Edward Lorenz. In 1961, Lorenz was using a numerical computer model to rerun a weather prediction, when, as a shortcut on a number in the sequence, he entered the decimal .506 instead of entering the full .506127. The result was a completely different weather scenario. This study was a

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Peer review under responsibility of Karabuk University.

http://dx.doi.org/10.1016/j.jestch.2014.09.004

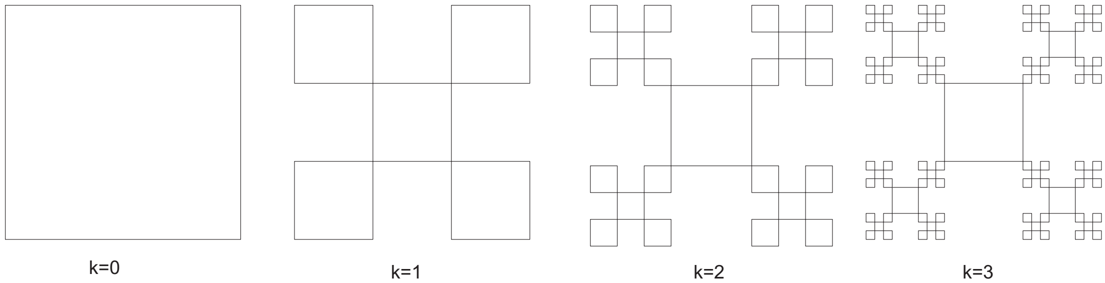


**Fig. 1.** Fractal structures.

spearhead study. With the increase in popularity of the theory of chaos, until then made theoretical studies was questioned and reexamined. It was effected different research areas and uncovered different approaches. One of these different approaches was fractal [6].

The fractal is a geometric pattern that is repeated at ever smaller scales to produce irregular shapes and surfaces that can't be represented by classical geometry. The term "*fractal*" was first used by mathematician Benoit Mandelbrot in 1975 [25]. Mandelbrot based it on the Latin *frāctus* meaning "*broken*" or "*fractured*". Mandelbrot, research noise on data transmission, came across fractal structure.

Fractal patterns with various degrees of self-similarity have been rendered or studied in images, structures and sounds and found in nature, technology and art [3,30]. Besides, in addition, Fractals emerge in connection with nonlinear and chaotic systems. So, there are many mathematical structures that are fractals; e.g. Sierpinski Triangle, Koch Snowflake, Peano Curve, Julia Set, Mandelbrot Set, and Lorenz Attractor [29].

An interconnection network is usually represented by a graph $G=(V,E)$, which is a pair of sets where $V$ represents the set of nodes and $E$ represents the set of edges. A hypercube graph $H(n)$ has been used for interconnection networks.

The hypercube graph $H(n)$ is a recursively definable graph. $H(n)$ of dimension $n$ connects up to $2^n$ nodes, each of which can be labeled by $n$-bit address uniquely, using a direct connection between two nodes if and only if their $n$-bit addresses differ in exactly one bit position. The reason for the popularity of the $H(n)$ can be attributed to its topological properties, the ability to use simple routing algorithms and the ability to permit the embedding of commonly-rewired interconnection patterns.

There are various types of graphs which are obtained from fractals [4,5,15,21,27]. In [19], Komjathy et al. demonstrated generation of hierarchical scale-free graphs from fractals. Here, we generate a network topology from the fractal structure in Fig. 1.

## 3. Construction and analytical properties of FCNG$_r(n)$

In this section, we give the definitions of FCNG$_r(n)$ and show the characteristics analyses of FCNG$_r(n)$.

### 3.1. Construction of FCNG$_1(k)$

$\mathrm{FCNG}_1(k)=(V_1(k),E_1(k))$ can be constructed from four $\mathrm{FCNG}_1(k-1)$ graphs $k=(1,2,3,\ldots)$. Throughout this paper, "$\|$" denotes the concatenation of two strings. The Hamming distance $\sum_{i=0}^{n-1}(a_i\oplus b_i)$ where summation is equal to summation of $a_i\oplus b_i$ (bitwise-XOR operation).

**Definition 1** ($\mathrm{FCNG}_1(k)$). Fractal Cubic Network Graphs can be defined as follow. Then, for $k>0$, $\mathrm{FCNG}_1(k)=(V_1(k),E_1(k))$ can be constructed as follows for $k>0$

$$\mathrm{FCNG}_1(k)=11\,\|\,\mathrm{FCNG}_1(k-1)\cup 01\,\|\,\mathrm{FCNG}_1(k-1)\cup \\ 10\,\|\,\mathrm{FCNG}_1(k-1)\cup 00\,\|\,\mathrm{FCNG}_1(k-1)$$

where
$V_1(k)=11\,\|\,V_1(k-1)\cup 01\,\|\,V_1(k-1)\cup 10\,\|\,V_1(k-1)\cup 00\,\|\,V_1(k-1)$
and
$E_1(k)=11\,\|\,E_1(k-1)\cup 01\,\|\,E_1(k-1)\cup 10\,\|\,E_1(k-1)\cup 00\,\|\,E_1(k-1)\cup E'$
where
$E'=\{(e_i,e_j)|e_i=\underbrace{\text{string of binary values}}_{m}\,ab\,\underbrace{\text{string of binary values}}_{2k-m}$ and $e_j=\underbrace{\text{string of binary values}}_{m}\,cd\,\underbrace{\text{string of binary values}}_{2k-m}$ and $\sum(ab)\oplus(cd)=1$, ($\oplus$ is and xor operator) the labels of $e_i$ and $e_j$ are same except $ab$ of $e_i$ and $cd$ of $e_j\}$ where $m=0,2,4,\ldots$. Assume that $v_1$ and $v_2$ are two nodes whose labels are $a_{2k+1}a_{2k}a_{2k-1}\ldots a_1a_0$ and $b_{2k+1}b_{2k}b_{2k-1}\ldots b_1b_0$ respectively. The labels $\overbrace{a_{2k+1}a_{2k}}^{c_k}\,\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\,\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\,\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\,\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_kc_{k-1}\ldots c_1c_0$ and $d_kd_{k-1}\ldots d_1d_0$, respectively. The edges $c_i\oplus d_i=1$ for $1\le i\le k$ are added to $E_1(k)$. The edges in the set $E'$ are called $k$th order external edges.

The edges in $\mathrm{FCNG}_1(k)$ can be considered as internal and external edges. Assume that the labels of nodes of $\mathrm{FCNG}_1(k)$ are $\overbrace{b_{2k+1}b_{2k}}^{d_k}\,\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\,\overbrace{b_1b_0}^{d_0}$. All edges in $H(2)$ are in $\mathrm{FCNG}_1(k)$ and the edges constituted by changing bits $d_{\mathrm{even}(i)}$, $0\le i\le k$, are also in $\mathrm{FCNG}_1(k)$. These edges are called internal edges and the edges constituted by changing bits $d_{\mathrm{odd}(i)}$, $0\le i\le k$ are called external edges.

For example, we obtain labeling of nodes of $\mathrm{FCNG}_1(k)$ as follow:

a) $\mathrm{FCNG}_1(0)$ is a network which is obtained for $k=0$ in Fig. 1. For example, mesh structure in Fig. 2 is $\mathrm{FCNG}_1(0)$. $\mathrm{FCNG}_1(0)$ is same as $H(2)$, and $E_1(0)=\{(00,01),(00,10),(10,11),(01,11)\}$.

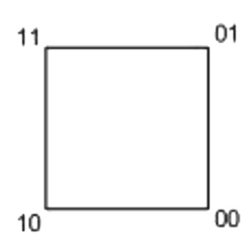


**Fig. 2.** Labeling of nodes of $\mathrm{FCNG}_1(0)$.

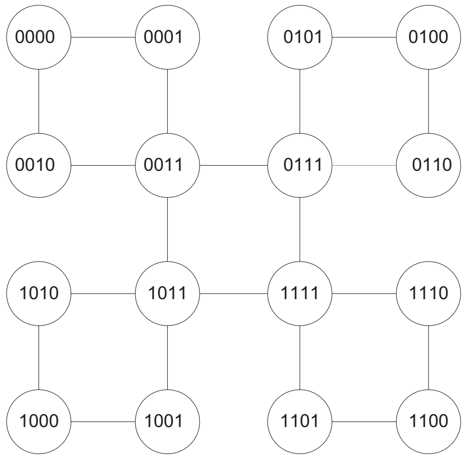


**Fig. 3.** Construction of $FCNG_1(1)$.

b) $FCNG_1(1)$ is a fractal network graph which is obtained by connecting the number of four $FCNG_1(0)$ graphs. For example, mesh structure in Fig. 3 is $FCNG_1(1)$ and it is a subgraph of $H(4)$. $FCNG_1(1)$ contains all edges in

$$\begin{aligned}E_1(1) &= 11 \parallel E_1(0) \cup 01 \parallel E_1(0) \cup 10 \parallel E_1(0) \cup 00 \parallel E_1(0) \cup E' \\ &= 11 \parallel \{(00,01),(00,10),(10,11),(01,11)\} \cup \\ &\quad 01 \parallel \{(00,01),(00,10),(10,11),(01,11)\} \cup \\ &\quad 10 \parallel \{(00,01),(00,10),(10,11),(01,11)\} \cup \\ &\quad 00 \parallel \{(00,01),(00,10),(10,11),(01,11)\} \cup E'\end{aligned}$$

$$\begin{aligned}&\{(1100,1101),(1100,1110),(1110,1111),(1101,1111)\} \cup \\ &\{(0100,0101),(0100,0110),(0110,0111),(0101,0111)\} \cup \\ &\{(1000,1001),(1000,1010),(1010,1011),(1001,1011)\} \cup \\ &\{(0000,0001),(0000,0010),(0010,0011),(0001,0011)\} \cup E'\end{aligned}$$

where the first order external edges are $E' = \{(0011,0111),(0011,1011),(1011,1111),(0111,1111)\}$.

c) $FCNG_1(k)$ is a fractal network graph which is obtained by connecting the number of four $FCNG_1(k-1)$ graphs. For example, mesh structure in Fig. 5 is $FCNG_1(3)$.

The following Fig. 1 is an example for a fractal structures.

**Example 1**. Assume that initial node is 11. Then, labeling of nodes of $FCNG_1(0)$ is same as labeling of nodes of $H(2)$.

**Example 2**. $FCNG_1(1)$ can be constructed by combining four $H(2)$ graphs. There will be 16 edges by concatenating $11, 01, 10, 00$ partial binary labels to all labels of nodes of $H(2)$. Then there will be 4 edges by. The labels $\overbrace{a_3a_2}^{c_1}$ $\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_3b_2}^{d_1}$ $\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_1c_0$ and $d_1d_0$, respectively. The edges $c_i \oplus d_i = 1$ for $1 \le i \le k$ are added to $E_1(1)$. Fig. 3 illustrates an example of $FCNG_1(1)$.

**Definition 2**. A subgraph of a graph $G$ is a graph whose vertex set is a subset of that of $G$, and whose adjacency relation is a subset of that of $G$ restricted to this subset. A subgraph $H$ is a spanning subgraph of a graph $G$ if it has the same vertex set as $G$.

**Lemma 1**. The total number of nodes in $FCNG_1(k)$ is $2^{2k+2}$ in the network topology which obtained from fractal structure.

**Proof**. Easily, the proof has accomplished by mathematical induction.

*Basic Step*: $FCNG_1(0)$ is same as $H(2)$. It is known that $H(2)$ contains 4 nodes.
*Hypothesis Step*: Assume that $FCNG_1(n-1)$ contains $2^{2n}$ nodes.
*Final Step*: $FCNG_1(n)$ contains four $FCNG_1(n-1)$ graphs. So, the number of nodes in the $FCNG_1(k)$ is $4 \times 2^{2n} = 2^{2n+2} = 2^{2(n+1)} = 2^{2(k+1)}$. ■

It is known that $FCNG_1(k)$ is a subgraph of $H(2k+2)$. The label of any node in $FCNG_1(k)$ can be divided into $k+1$ parts such that each part has length of two bits. The rightmost two bits determine $H(2)$ as subgraph of $FCNG_1(k)$, and these two bits can be called as the first double bits. The next double bits determine the combination of four $FCNG_1(0)$ and obtaining $FCNG_1(1)$ by adding the first order external edges to the set of edges of $FCNG_1(1)$. In order to construct $FCNG_1(k)$, four $FCNG_1(k-1)$ graphs are combined and the $k$th order external edges are added to the edges set of $FCNG_1(k)$.

**Theorem 1**. Assume that $k \in Z^+$, and $FCNG_1(k)$ is a subgraph of $H(2k+2)$.

**Proof**. Any node of $FCNG_1(k)$ has a label of length $2k+2$. Two nodes are adjacent, if their labels differ exactly in one bit position. In this case, all edges in $FCNG_1(k)$ are also in $H(2k+2)$. The edge between nodes $1000\overbrace{b_{2k-3}b_{2k-4}\ldots b_1b_0}^{2k-2}$ and $0000\overbrace{b_{2k-3}b_{2k-4}\ldots b_1b_0}^{2k-2}$ is an edge of $H(2k+2)$, however, it is not an edge of $FCNG_1(k)$. ■

### 3.2. Eulerian properties of $FCNG_1(k)$

A graph $G$ is an Euler graph if it has the following two properties:

a) $G$ has got an Eulerian circuit where it visits all edges of graph exactly once and returns to started node.
b) The node degrees of $G$ are all even.

The Eulerian properties of $FCNG_1(k)$ is investigated in this section. Firstly, we find an Eulerian circuit of $FCNG_1(k)$ can be obtained by removing an edge at each step. Since the degrees of all nodes are even. $FCNG_1(k)$ is not an Hamiltonian graph, and this can be seen in example 3.

**Example 3**. The Fig. 4 depicts $FCNG_1(1)$ which consists of five circuits. The nodes contained $C_3$ circuit are traversed more than once, so it is not an Hamiltonian graph, on contrary, it is an Eulerian graph. $FCNG_1(k)$ graphs have fractal structures.

**Example 4**. The Figs. 5 and 6 are examples of $FCNG_1(k)$ graphs. All of them are Eulerian graphs and not Hamiltonian graphs.

**Theorem 2**. The network topology which obtained from fractal structure for $k$ is an Eulerian graph, which produced by traveling all nodes of mesh structure, by using $2n+2$ bit labels. Total number of edges in $FCNG_1(k)$ is $E_1(k) = 4(4^{k+1}-1)/3$ and total number of nodes is $2^{2k+2}$.

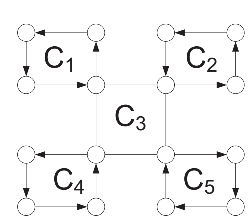


**Fig. 4.** The circuits of $FCNG_1(1)$.

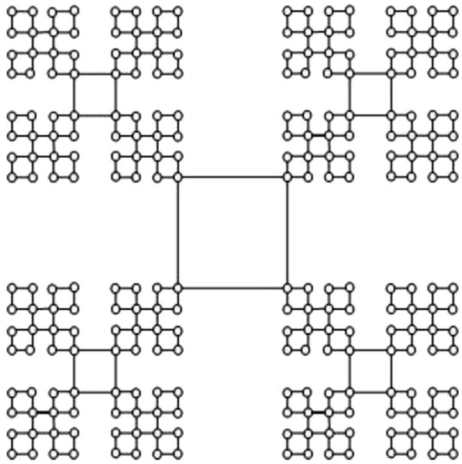

**Fig. 5.** $\mathrm{FCNG}_1(3)$

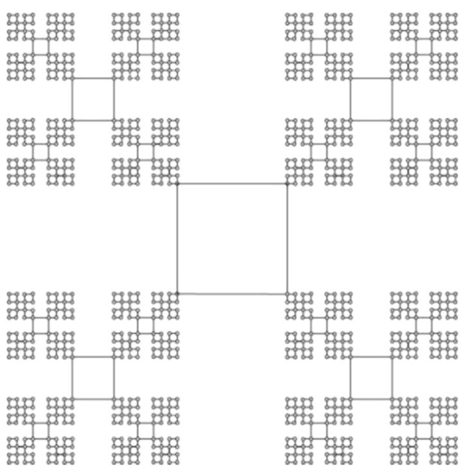

**Fig. 6.** $\mathrm{FCNG}_1(4)$

**Proof**. Firstly, total number of nodes is $2^{2k+2}$ by Lemma 1. The rest of the proof has accomplished by mathematical induction.

*Basic Step*: $\mathrm{FCNG}_1(0)$ is same as $H(2)$ and $H(2)$ has four edges. That is $4 = 2^{2k+2} = 2^2$ for $k = 0$. In order to construct $\mathrm{FCNG}_1(1)$, four $\mathrm{FCNG}_1(0)$s are connected with four the first order external edges. So, $\mathrm{FCNG}_1(1)$ consists of $4 \times 2^2 + 4 = 2^2 + 2^4 = \sum_{i=0}^{1} 2^{2i+2} = \frac{4(4^{k+1}-1)}{3} = 20$ edges for $k = 1$.

*Hypothesis Step*: Assume that $\mathrm{FCNG}_1(n-1)$ contains $E_1(n-1) = \sum_{i=0}^{n-1} 2^{2i+2} = 4(4^n - 1)/3$ edges for $k = n - 1$.

*Final Step*: For $k = n$, $\mathrm{FCNG}_1(n)$ consists of four $\mathrm{FCNG}_1(n-1)$s and four the second order external edges. This means that $\mathrm{FCNG}_1(n)$ consists of

$$\sum_{i=0}^{n} 2^{2i+2} = 4 \times \sum_{i=0}^{n-1} 2^{2i+2} + 4 = 4 \times [4(4^n - 1)/3] + 4$$
$$= 4\left(4^{k+1} - 1\right)/3$$

edges. So on, the number of edges in $\mathrm{FCNG}_1(k)$ is $\sum_{i=0}^{k} 2^{2i+2} = \frac{4(4^{k+1}-1)}{3}$. ■

**Theorem 3**. The node degrees of $\mathrm{FCNG}_1(k)$ are 2 and 4. The number of nodes whose degrees are 2, is $(2.4^{k+1} + 4)/3$ and the number of nodes whose degrees are 4, is $4(4^k - 1)/3$.

**Proof**. In the following table, by using the intuitive approach.

| Dimension $(2k+2)$ | Number of nodes | Number of nodes of degree 2 | Number of nodes of degree 4 |
|---|---|---|---|
| 0 | 4 | $2^2 = 4$ | 0 |
| 1 | 16 | $2^4 - 2^2 = 12$ | 4 |
| 2 | 64 | $2^6 - 2^4 - 2^2 = 44$ | $2^4 + 2^2 = 20$ |
| 3 | 256 | $2^8 - 2^6 - 2^4 - 2^2 = 172$ | $2^6 + 2^4 + 2^2 = 84$ |
| 4 | 1024 | $2^{10} - 2^8 - 2^6 - 2^4 - 2^2 = 684$ | $2^8 + 2^6 + 2^4 + 2^2 = 340$ |
| 5 | 4096 | $2^{12} - \sum_{i=1}^{5} 2^{2i} = 2732$ | $\sum_{i=1}^{5} 2^{2i} = 1364$ |
| … | … | … | … |
| $k$ | $2^{2k+2}$ | $2^{2k+2} - \sum_{i=1}^{k} 2^{2i} = (2.4^{k+1} + 4)/3$ | $\sum_{i=1}^{k} 2^{2i} = 4(4^k - 1)/3$ |

Easily, the proof has accomplished by mathematical induction. ■

### 3.3. *Construction of* $\mathrm{FCNG}_2(k)$

$\mathrm{FCNG}_2(k) = (V_2(k), E_2(k))$ can be constructed from four $\mathrm{FCNG}_2(k-1)$ graphs $(k = 1,2,3,\ldots)$.

**Definition 3** ($\mathrm{FCNG}_2(k)$). Fractal Cubic Network Graphs can be defined as follow. Then, for $k > 0$, $\mathrm{FCNG}_2(k) = (V_2(k), E_2(k))$ can be constructed as follows for. $k > 0$

$$\mathrm{FCNG}_2(k) = 11 \parallel \mathrm{FCNG}_2(k-1) \cup 01 \parallel \mathrm{FCNG}_2^{-1}(k-1) \cup$$
$$00 \parallel \mathrm{FCNG}_2\left(k-1\right) \cup 10 \parallel \mathrm{FCNG}_2^{-1}\left(k-1\right)$$

where $\mathrm{FCNG}_2^{-1}(k-1)$ is labeling of nodes of $\mathrm{FCNG}_2(k-1)$ which in reverse order $V_2(k) = 11 \parallel V_2(k-1) \cup 01 \parallel V_2(k-1) \cup 00 \parallel V_2(k-1) \cup 10 \parallel V_2(k-1)$ and $E_2(k) = 11 \parallel E_2(k-1) \cup 01 \parallel E_2(k-1) \cup 00 \parallel E_2(k-1) \cup 10 \parallel E_2(k-1) \cup E'$ where $E' = \{(e_i, e_j) \mid e_i = ab$ and $e_j = cd$ and $(ab) \oplus (cd) = 1$, the labels of $e_i$ and $e_j$ are same except $ab$ of $e_i$ and $cd$ of $e_j\}$ where $m = 0, 2, 4, \ldots$. Assume that $v_1$ and $v_2$ are two nodes whose labels are $a_{2k+1}a_{2k}a_{2k-1}\ldots a_1 a_0$ and $b_{2k+1}b_{2k}b_{2k-1}\ldots b_1 b_0$, respectively. The $i$th order external edges can be obtained by the any of the following conditions:

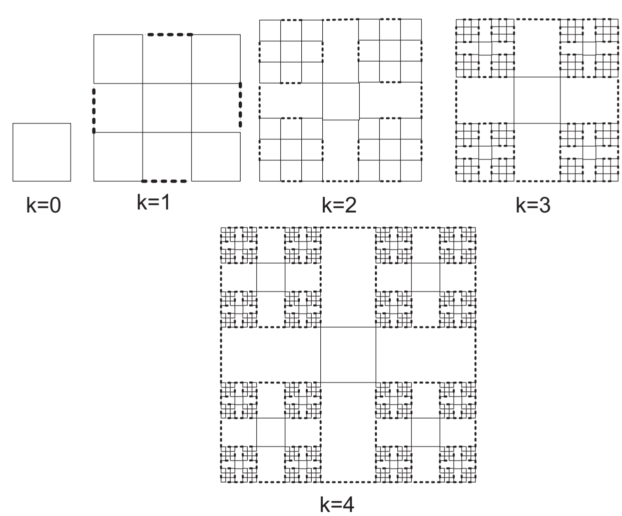


**Fig. 7.** Some examples of $\mathrm{FCNG}_2(k)$.

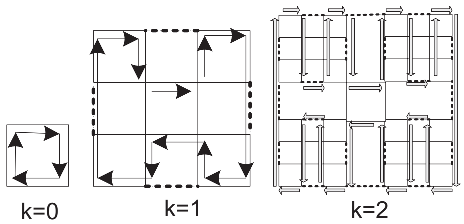


**Fig. 8.** Basic step of mathematical induction proof.

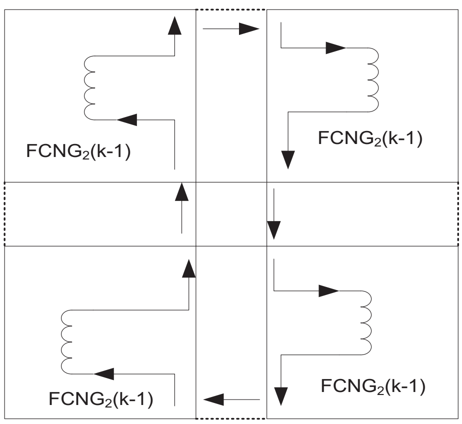


**Fig. 9.** Proof of Hamiltonian property of FCNG$_2(k)$.

a) $a_{2i} \oplus b_{2i} = 1$ and $a_{2i+1} \oplus b_{2i+1} = 0, 1 \le i \le k$
b) $a_{2i} \oplus b_{2i} = 0$ and $a_{2i+1} \oplus b_{2i+1} = 1, 1 \le i \le k$

Some examples of FCNG$_2(k)$s are seen in Fig. 7. The dashed edges are external edges. It can be seen that all FCNG$_2(k)$ graphs are not Eulerian graphs, since some nodes have odd degrees. All graphs in Fig. 7 have fractal structures as seen in the figure .

**Theorem 4.** FCNG$_2(k)$ graphs are Hamiltonian graphs.

**Proof.** The proof of the theorem can be done by mathematical induction.

*Basic Step*: $k = 0, k = 1$ or $k = 2$, as Fig. 8.

*Hypothesis Step*: Assume that FCNG$_2(k-1)$ is an Hamiltonian graph.

*Final Step*: FCNG$_2(k)$ is a Hamiltonian graph as seen in Fig. 9. The FCNG$_2(k)$ in Fig. 9 consists of four FCNG$_2(k-1)$ wherein there is a Hamiltonian path in each FCNG$_2(k-1)$. FCNG$_2(k)$ contains four Hamiltonian FCNG$_2(k-1)$ graphs and the Hamiltonian path in FCNG$_2(k)$ is illustrated in Fig. 9. ■

**Theorem 5.** The number of edges of FCNG$_2(k)$ is defined as a recurrence relation such as $|E_2(k)| = 4|E_2(k-1)| + 8$, and $|E_2(k)| = (5.4^{k+1} - 8)/3$.

**Proof.** Initially, for $k = 0$, FCNG$_2(0)$ is same as $H(2)$. $H(2)$ contains 4 edges and therefore FCNG$_2(0)$ contains 4 edges. FCNG$_2(1)$ contains the edges of four FCNG$_2(0)$ and 8 external edges. Then $|E_2(1)| = 4|E_2(0)| + 8$. FCNG$_2(2)$ contains the edges of four FCNG$_2(1)$ and 8 external edges. Then $|E_2(2)| = 4|E_2(1)| + 8$. So, FCNG$_2(k)$ contains edges of four FCNG$_2(k-1)$ graphs and 8 external edges. As results, $|E_2(k)| = 4|E_2(k-1)| + 8$. Also, we get

$$\begin{aligned}|E_2(k)| &= 4|E_2(k-1)| + 8 = 4^2|E_2(k-2)| + 4 \cdot 8 + 8 \\ &= 4^3|E_2(k-3)| + 4^2 \cdot 8 + 4^1 \cdot 8 + 4^0 \cdot 8 \\ &= 4^k|E_2(0)| + 8\sum_{i=0}^{k-1} 4^i = \left(5 \cdot 4^{k+1} - 8\right)/3.\end{aligned}$$

**Theorem 6.** The node degrees of FCNG$_2(k)$ are 2, 3 and 4. The number of nodes whose degrees are 2, is 4, the number of nodes whose degrees are 3, is $8(4^k - 1)/3$, and the number of nodes whose degrees are 4, is $4(4^k - 1)/3$.

**Proof.** In the following table, by using the intuitive approach.

| Dimension $(2k+2)$ | Number of nodes | Number of nodes of degree 2 | Number of nodes of degree 3 | Number of nodes of degree 4 |
|---|---|---|---|---|
| 0 | 4 | $2^2 = 4$ | 0 | 0 |
| 1 | 16 | $2^2 = 4$ | $2^3 = 8$ | $2^2 = 4$ |
| 2 | 64 | $2^2 = 4$ | $2^5 + 2^3 = 40$ | $2^4 + 2^2 = 20$ |
| 3 | 256 | $2^2 = 4$ | $2^7 + 2^5 + 2^3 = 168$ | $2^6 + 2^4 + 2^2 = 84$ |
| 4 | 1024 | $2^2 = 4$ | $2^9 + 2^7 + 2^5 + 2^3 = 680$ | $2^8 + 2^6 + 2^4 + 2^2 = 340$ |
| 5 | 4096 | $2^2 = 4$ | $\sum_{i=1}^{5} 2^{2i+1} = 2728$ | $\sum_{i=1}^{5} 2^{2i} = 1364$ |
| … | … | … | … | … |
| $k$ | $2^{2k+2}$ | $2^2 = 4$ | $\sum_{i=1}^{k} 2^{2i+1} = 8(4^k - 1)/3$ | $\sum_{i=1}^{k} 2^{2i} = 4(4^k - 1)/3$ |

Easily, the proof has accomplished by mathematical induction. ■

### 3.4. Hamiltonian properties of FCNG$_2(k)$

The Hamiltonian graph has got Hamiltonian cycle, Hamiltonian circuit that visits each node once (except for the vertex that is both the start and end, which is visited twice). A graph which contains a Hamiltonian cycle (closed path) is called Hamiltonian graph. The Hamiltonian properties of FCNG$_2(k)$ is investigated in this section.

## 4. Recursive algorithm

In this section, we give a recursive algorithm is used for labeling to the nodes of mesh structure in Theorem 4. Firstly, we give an example.

**Example 5**. Labeling of nodes $\mathrm{FCNG}_2(2)$ for initial node: 011100 as Example 2. Firstly, sub initial nodes, $b_6b_5:01, b_4b_3:11$ and $b_2b_1:00$. Let, we choose a Hamilton path for $b_6b_5:01$ which is $\{01,00,10,11\}$, a Hamilton path for $b_4b_3:11$ which is $\{11,01,00,10\}$ and a Hamilton path for $b_2b_1:00$ which is $\{00,01,11,10\}$. From definition 3.

$$\mathrm{FCNG}_2(2)=[01||(\mathrm{FCNG}_2(1))]\cup\left[00\middle|\middle|\mathrm{FCNG}_2^{-1}(1)\right]\cup \\ [10||(\mathrm{FCNG}_2(1))]\cup\left[11\middle|\middle|\mathrm{FCNG}_2^{-1}(1)\right].$$

where $\mathrm{FCNG}_2^{-1}(1)$ is labeling of nodes of $\mathrm{FCNG}_2(1)$ which reverse order. Similarly, from definition 3,

$$FCNG_2(1)=[11||(\mathrm{FCNG}_2(0))]\cup\left[01\middle|\middle|\mathrm{FCNG}_2^{-1}(0)\right]\cup \\ [00||(\mathrm{FCNG}_2(0))]\cup\left[10\middle|\middle|\mathrm{FCNG}_2^{-1}(0)\right]$$

where $\mathrm{FCNG}_2^{-1}(0)$ is labeling of nodes of $\mathrm{FCNG}_2(0)$ which reverse order. (Labeling of nodes of $\mathrm{FCNG}_2^{-1}(0)$is$\{10,11,01,00\}$). Thus, we obtain labeling of nodes of $\mathrm{FCNG}_2(1)$

$$\{11 \parallel \{00,01,11,10\}\}\cup\{01 \parallel \{10,11,01,00\}\}\cup \\ \{00 \parallel \{00,01,11,10\}\}\cup\{10 \parallel \{10,11,01,00\}\},$$

namely, we get

$$\left\{\begin{matrix}1100,1101,1111,1110,0110,0111,0101,0100,\\ 0000,0001,0011,0010,1010,1011,1001,1000\end{matrix}\right\}.$$

Thus, we obtain labeling of nodes of $\mathrm{FCNG}_2(2)$

$$\begin{matrix}01\left|\middle|\left\{\begin{matrix}1100,1101,1111,1110,0110,0111,0101,0100,\\ 0000,0001,0011,0010,1010,1011,1001,1000\end{matrix}\right\}\right.\cup\\ 00\left|\middle|\left\{\begin{matrix}1000,1001,1011,1010,0010,0011,0001,0000,\\ 0100,0101,0111,0110,1110,1111,1101,1100\end{matrix}\right\}\right.\cup\\ 10\left|\middle|\left\{\begin{matrix}1100,1101,1111,1110,0110,0111,0101,0100,\\ 0000,0001,0011,0010,1010,1011,1001,1000\end{matrix}\right\}\right.\cup\\ 11\left|\middle|\left\{\begin{matrix}1000,1001,1011,1010,0010,0011,0001,0000,\\ 0100,0101,0111,0110,1110,1111,1101,1100\end{matrix}\right\}\right.\end{matrix}$$

namely, we get

$$\begin{matrix}\{011100,011101,011111,011110,010110,010111,010101,010100\}\cup\\ \{010000,010001,010011,010010,011010,011011,011001,011000\}\cup\\ \{001000,001001,001011,001010,000010,000011,000001,000000\}\cup\\ \{000100,000101,000111,000110,001110,001111,001101,001100\}\cup\\ \{101100,101101,101111,101110,100110,100111,100101,100100\}\cup\\ \{100000,100001,100011,100010,101010,101011,101001,101000\}\cup\\ \{111000,111001,111011,111010,110010,110011,110001,110000\}\cup\\ \{110100,110101,110111,110110,111110,111111,111101,111100\}.\end{matrix}$$

Thus, $\mathrm{FCNG}_2(2)$ is a Hamilton graph by using 6 bit Gray Code.

**Theorem 7**. . The following recursive algorithm is used for labeling to the nodes of mesh structure in Theorem 4. The time complexity of the recursive algorithm is $O(|V|)$ where $|V|=2^{2k+2}$. (See the algorithm 1 in Appendix).

## 5. Routing on $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$

The routing algorithms on the $\mathrm{FCNG}_2(k)$ are similar to the routing algorithms on the hypercubes. $\mathrm{FCNG}_2(k)$ has $(2k+2)$-bit labels and routing algorithms are designed with respect to this length. The routing algorithms are one-to-all broadcasting, all-to-all broadcasting, and all-to-all personalized communication algorithms. All of the following algorithms are revised version of routing algorithms for hypercubes in [20]. (See the algorithms 2, 3 and 4 in Appendix).

There are three algorithms and their time complexities can be analyzed. The first algorithm is one-to-all broadcasting algorithm and its aim is to send a message from source to all remaining nodes. Assume that $t_s$ is start-up time and $t_w$ is a time of sending one word on a link (edge). Assume that the message sending strategy is store-and-forward. The upper bound for number of message sending steps is $2k+2$. The time complexity for sending message of size $mT_{\mathrm{one-to-all}}$ for $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are $T_{\mathrm{one-to-all}}\le 2(2k+2)(t_s+mt_w)$. The second algorithm is all-to-all broadcasting algorithm and its time complexity for sending messages of size $mT_{\mathrm{all-to-all}}\le(2k+2)(t_s+(2k+2)mt_w)+(2k)(t_s+3\mathrm{x}2^{2k}mt_w)$. The time complexity for all-to-all personalized communication for sending messages of sizes $m$ is $T_{\mathrm{all-to-all-per}}\le(2k)(t_s+2^{2k}mt_w)+2(t_s+3\mathrm{x}2^{2k}mt_w)+2(t_s+3\mathrm{x}\ 2^{2k}mt_w)$.

## 6. Deriving subcubes from faulty hypercubes

$\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are two special subcubes of $H(2k+2), k\in Z^+$.

**Theorem 8**. The hypercube $H(2k+2)$ has fault-tolerance property in case of faulted edges are in the following edges sets.

a) The node labels of $H(2k+2)$ are $\overbrace{a_{2k+1}a_{2k}}^{c_k}\ \overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\ \overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\ \overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\ \overbrace{b_1b_0}^{d_0}$ for $\mathrm{FCNG}_1(k)$. If the faulted edges are not in the set $E_1(k)=11||E_1(k-1)\cup 01||E_1(k-1)$

$\cup 10||E_1(k-1)\cup 00||E_1(k-1)\cup\{$The labels $\overbrace{a_{2k+1}a_{2k}}^{c_k}\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_kc_{k-1}\ldots c_1c_0$ and $d_kd_{k-1}\ldots d_1d_0$, respectively. The edges $c_i\oplus d_i=1$ for $1\le i\le k\}$, then $\mathrm{FCNG}_1(k)$ can be derived from faulted $H(2k+2)$ and it is fault-tolerance interconnection networks.

b) The node labels of $H(2k+2)$ are $\overbrace{a_{2k+1}a_{2k}}^{c_k}\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\overbrace{b_1b_0}^{d_0}$ for $\mathrm{FCNG}_2(k)$. If the faulted edges are not in the set $E_2(k)=11||E_2(k-1)\cup 01||E_2(k-1)\cup 10||E_2(k-1)\cup 00||E_2(k-1)\cup$ $\{$The labels $\overbrace{a_{2k+1}a_{2k}}^{c_k}\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_kc_{k-1}\ldots c_1c_0$ and $d_kd_{k-1}\ldots d_1d_0$, respectively. The edges $a_{2i}\oplus b_{2j}=1$ and $a_{2i+1}\oplus b_{2j+1}=0$, $1\le j\le k$ or $a_{2i}\oplus b_{2j}=0$ and $a_{2i+1}\oplus b_{2j+1}=1,1\le j\le k\}$, then $\mathrm{FCNG}_2(k)$ can be derived from faulted $H(2k+2)$ and it is fault-tolerance interconnection networks.

c) $H(2k+2)$ is fault-tolerance with respect to maximum $\frac{(3k-1)4^{k+1}+4}{3}$ faulted edges.

d) $H(2k+2)$ is fault-tolerance with respect to maximum $\frac{(3k-2)4^{k+1}+8}{3}$ faulted edges.

**Proof**: The proofs of $a$ and $b$ are straight forward.

The edges set $11||E_1(k-1)\cup 01||E_1(k-1)\cup 10||E_1(k-1)\cup 00||E_1(k-1)\cup$ $\{$The labels $\overbrace{a_{2k+1}a_{2k}}^{c_k}\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_kc_{k-1}\ldots c_1c_0$ and $d_kd_{k-1}\ldots d_1d_0$, respectively. The edges $c_i\oplus d_i=1$ for $1\le i\le k\}$ is the edges set of $\mathrm{FCNG}_1(k)$.

The edges set $11||E_1(k-1)\cup 01||E_1(k-1)\cup 10||E_1(k-1)\cup 00||E_1(k-1)\cup$ $\{$The labels $\overbrace{a_{2k+1}a_{2k}}^{c_k}\overbrace{a_{2k-1}a_{2k-2}}^{c_{k-1}}\ldots\overbrace{a_3a_2}^{c_1}\overbrace{a_1a_0}^{c_0}$ and $\overbrace{b_{2k+1}b_{2k}}^{d_k}\overbrace{b_{2k-1}b_{2k-2}}^{d_{k-1}}\ldots\overbrace{b_3b_2}^{d_1}\overbrace{b_1b_0}^{d_0}$ can be illustrated by $c_kc_{k-1}\ldots c_1c_0$ and $d_kd_{k-1}\ldots d_1d_0$, respectively. The edges $a_{2i}\oplus b_{2j}=1$ and $a_{2i+1}\oplus b_{2j+1}=0,1\le j\le k$ or $a_{2i}\oplus b_{2j}=0$ and $a_{2i+1}\oplus b_{2j+1}=1,1\le j\le k\}$ is the edges set of $\mathrm{FCNG}_2(k)$.

The number of edges in $H(2k+2)$ is $(k+1)4^{k+1}$, the number of edges in $\mathrm{FCNG}_1(k)$ is $\frac{4(4^{k+1}-1)}{3}$ and the number of edges in $\mathrm{FCNG}_2(k)$ is $\frac{5x4^{k+1}-8}{3}$. $\mathrm{FCNG}_1(k)$ can be derived from $H(2k+2)$ in case of the number of faulted edges is less than or equal to $(k+1)4^{k+1}-\frac{4(4^{k+1}-1)}{3}=\frac{(3k-1)4^{k+1}+4}{3}$, and $\mathrm{FCNG}_2(k)$ can be derived from $H(2k+2)$ in case of the number of faulted edges is less than or equal to $(k+1)4^{k+1}-\frac{5x4^{k+1}-8}{3}=\frac{(3k-2)4^{k+1}+8}{3}$.

## 7. Summary of results

$\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are variants of hypercubes. The constructions of $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are similar to the construction of hypercubes. Both graphs $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are scalable, since they are variants of hypercubes. The routing algorithms are similar to the routing algorithms on the hypercubes due to the revision of routing algorithms of hypercubes and they are subgraphs of hypercubes. Briefly, the properties of both $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ can be illustrated as follow.

a) Both graphs $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are scalable.
b) The routing algorithms on both graphs $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ are revised versions of the routing algorithms on the hypercubes.
c) $\mathrm{FCNG}_1(k)$ is an Eulerian graph.
d) $\mathrm{FCNG}_2(k)$ is a Hamiltonian graph.
e) The number of nodes of $\mathrm{FCNG}_1(k)$ is $2^{2k+2}$.
f) The number of nodes of $\mathrm{FCNG}_2(k)$ is $2^{2k+2}$.
g) The number of edges of $\mathrm{FCNG}_1(k)$ is $4(4^{k+1}-1)/3$.
h) The number of edges of $\mathrm{FCNG}_2(k)$ is $(5.4^{k+1}-8)/3$.
i) Both graphs $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ have fractal structures.
j) The diameter of graph $\mathrm{FCNG}_1(k)$ is $2\times$ diameter $(\mathrm{FCNG}_1(k-1))+2$ and the diameter of graph $\mathrm{FCNG}_2(k)$ is $2\times$ diameter $(\mathrm{FCNG}_2(k))+2$ where diameter $(\mathrm{FCNG}_1(0))=2$ and diameter $(\mathrm{FCNG}_2(0))=2$.
k) The number of nodes whose degrees are 2, is $(2.4^{k+1}+4)/3$ and the number of nodes whose degrees are 4, is $4(4^k-1)/3$.
l) The number of nodes whose degrees are 2, is 2, the number of nodes whose degrees are 3, is $8(4^k-1)/3$ and the number of nodes whose degrees are 4, is $4(4^k-1)/3$.
m) The cost of a graph is the summation of the multiplication of number of nodes and their degrees. The cost of $\mathrm{FCNG}_1(k)$ is $2(4^{k+2}-4)/3$ and the cost of $\mathrm{FCNG}_2(k)$ is $4.(4^k+1)$.

## 8. Conclusion and future work

In this paper, two subcubes of faulty hypercubes were derived and their properties were analyzed. These two subcubes $\mathrm{FCNG}_1(k)$ and $\mathrm{FCNG}_2(k)$ can be used as interconnection networks in case of faults occurred in the hypercube. New hypercube variants may be studied by using different fractal structures in nature, as a future work.

## Acknowledgment

The authors are very grateful to the referees for their valuable suggestions, which helped to improve the paper significantly.

## Appendix

**Algorithm-1:** Recursive Algorithm

```
Define k = n, m = 2n + 2,                              // int n > 0: iteration number
        M_0 = {p → q → r → s}, M_0^-1 = {s → r → q → p}
Begin
  G: // mesh structure that m = 2n + 2 bit Gray Code is labeling for k = n
  from Lemma 1 and Theorem 4
input (initial node);
outer trace (m, initial node(m)) {                     // m: initial node length
  if ((m − 2)! = 0) then {
    initial node = initial node(m − 2);
    m = m − 2;
    inner trace (m, initial node(m)){
 for i = 1 to n do {
   if i == 1 then {                    // construct of M_1
     if (initial node == 00) then {
       switch (M_1) {
         case 1: M_1 = {{00||[M_0]} → {01||[M_0^-1]} → {11||[M_0]} → {10||[M_0^-1]}};
         case 2: M_1 = {{00||[M_0]} → {10||[M_0^-1]} → {11||[M_0]} → {01||[M_0^-1]}};
         }
     } else if (initial node == 01) then{
       switch (M_1) {
         case 1: M_1 = {{01||[M_0]} → {11||[M_0^-1]} → {10||[M_0]} → {00||[M_0^-1]}};
         case 2: M_1 = {{01||[M_0]} → {00||[M_0^-1]} → {10||[M_0]} → {11||[M_0^-1]}};
         }
     } else if (initial node == 10) then {
       switch (M_1) {
         case 1: M_1 = {{10||[M_0]} → {00||[M_0^-1]} → {01||[M_0]} → {11||[M_0^-1]}};
         case 2: M_1 = {{10||[M_0]} → {11||[M_0^-1]} → {01||[M_0]} → {00||[M_0^-1]}};
         }
     } else {
       switch (M_1) {
         case 1: M_1 = {{11||[M_0]} → {10||[M_0^-1]} → {00||[M_0]} → {01||[M_0^-1]}};
         case 2: M_1 = {{11||[M_0]} → {01||[M_0^-1]} → {00||[M_0]} → {10||[M_0^-1]}};
         }
     }
     G ← M_1;
     return (G);
   }                                   // end if
   else {                       // construct of M_{i+1}, (for i = n, M_{n+1})
   if (initial node == 00) then {
     switch (M_{i+1}) {
       case 1: M_{i+1} = {{00||[M_i]} → {01||[M_i^-1]} → {11||[M_i]} → {10||[M_i^-1]}};
       case 2: M_{i+1} = {{00||[M_i]} → {10||[M_i^-1]} → {11||[M_i]} → {01||[M_i^-1]}};
       }
```

```
            } else if (initial node == 01) then {
              switch (M_{i+1}) {
                case 1: M_{i+1} = {{01||[M_i]} → {11||[M_i^{-1}]} → {10||[M_i]} → {00||[M_i^{-1}]}};
                case 2: M_{i+1} = {{01||[M_i]} → {00||[M_i^{-1}]} → {10||[M_i]} → {11||[M_i^{-1}]}};
                }
            } else if (initial node == 10) then {
              switch (M_{i+1}) {
                case 1: M_{i+1} = {{10||[M_i]} → {00||[M_i^{-1}]} → {01||[M_i]} → {11||[M_i^{-1}]}};
                case 2: M_{i+1} = {{10||[M_i]} → {11||[M_i^{-1}]} → {01||[M_i]} → {00||[M_i^{-1}]}};
                }
            } else {
              switch (M_{i+1}) {
                case 1: M_{i+1} = {{11||[M_i]} → {10||[M_i^{-1}]} → {00||[M_i]} → {01||[M_i^{-1}]}};
                case 2: M_{i+1} = {{11||[M_i]} → {01||[M_i^{-1}]} → {00||[M_i]} → {10||[M_i^{-1}]}};
                }
            }
            G ← M_{i+1};
            return (m − 2, initial_node(m − 2), G);
          }                                   // end else
        }                                     // end for
    }                                         // end of inner block
    else {                                       // construct of M_0, (n = 0, m = 2)
      if (initial_node == 00) then {
        switch (M_0) {
          case 1: M_0 = 00 → 01 → 11 → 10;
          case 2: M_0 = 00 → 10 → 11 → 01;
          }
      } else if (initial_node == 01) then {
        switch (M_0) {
          case 1: M_0 = 01 → 11 → 10 → 00;
          case 2: M_0 = 01 → 00 → 10 → 11;
          }
      } else if (initial_node == 10) then {
        switch (M_0) {
          case 1: M_0 = 10 → 00 → 01 → 11;
          case 2: M_0 = 10 → 11 → 01 → 00;
          }
      } else {
        switch (M_0) {
          case 1: M_0 = 11 → 10 → 00 → 01;
          case 2: M_0 = 11 → 01 → 00 → 10;
          }
      }
      G ← M_0;
      return (G);
      }                                          // end else
  }                                              // end of outer block
  end
```

**Algorithm-2:** One-to-All Broadcasting Algorithm

```
Routing(k,current_node_id,MSG)
  1. mask←2^(2k+2)-1
  2. for i←(2k+2)-1, …, 0
  3. mask ← mask XOR 2^i
  4. if (current_node_id AND mask ==0)
  5.      if current_node_id AND 2^i==0 and
              edge (current_node_id, current_node_id XOR 2^i)
              exist then
  6.            Destination← current_node_id XOR 2^i
  7.            Send MSG to Destination
                else if edge (current_node_id, current_node_id
                      XOR 2^i)exist then
  8.            Source← current_node_id XOR 2^i
  9.          Receive MSG from Source
```

**Algorithm-3:** All-to-All Broadcasting Algorithm

```
Routing(current_node_id, current_node_MSG, k, result)
  1. result←current_node_MSG
  2. for i←0, 1, …, (2k+2)-1
  3. Receiver ← current_node_id XOR 2^i
  4.      if edge (current_node_id,Receiver) exist then
  5.         Send result to Receiver
  6.    Receive MSG from Receiver
  7.    result←result ∪ MSG
```

**Algorithm-4:** All-to-All Personal Communication Algorithm

```
Routing(k,current_node_id)
  1. for i←1,2, …, 2^(2k+2)-1
  2.      Receiver ← current_node_id XOR i
  3.      if edge (current_node_id,Receiver) exist then
  4.           Send MSG_current_node_id to Receiver
  5.           Receive MSG_Receiver from Receiver
```

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