The Early Mathematics of Leonhard Euler

The year 2007 marked the tercentenary of the birth of Leonhard Euler. During the Enlightenment, this Swiss-born genius was preeminent across the mathematical sciences. The tercentenary jubilees, especially in Basel, his birthplace, and St. Petersburg and Berlin, the cities in which he spent his adult life, together with the efforts of the Euler Society, founded in 2002, have prompted a number of commemorative events around the world. In the United States, the Mathematical Association of America and the American Mathematical Society promoted Euler studies at their joint meetings in New Orleans, Louisiana, and at the annual Mathfest in San Jose, California, while the 2007 meeting of the History of Science Society in Crystal City, Virginia held a session supported by the International Commission on the History of Mathematics on his life and research. The anniversary has also been marked by the publication of several Euler-related books. Sandifer’s *The Early Mathematics of Leonhard Euler*, one of five celebratory works published by the Mathematical Association of America, is a skillful study of 41 of Euler’s first 50 articles, covering his Basel years and his first St. Petersburg period up to 1741.

While later landmark books written by Euler, such as the *Introductio in analysin infinitorum* of 1748, have been examined in detail, the first stage of his mathematical research is lesser known. Since the development of infinitary analysis and its applications were the Enlightenment sciences *par excellence*, this was a notable gap in the history of 18th-century mathematics. Sandifer begins to remove it. He presents Euler’s memoirs in chronological, rather than thematic, order, following Gustav Eneström’s index. However, Sandifer’s brief descriptions of each paper do provide guidance for making an organization by theme. Most memoirs are in Latin, which Euler like his teacher Johann I Bernoulli accepted as the language of learning. Sandifer selects only those papers in pure mathematics, mainly in infinitary analysis, origins of the calculus of variations, and number theory, excluding all work in applied mathematics.

Sandifer makes several fascinating and important discoveries. Among them he finds that Euler’s first memoir contains a rare error, in solving the wrong differential equation, a mistake that he corrected two years later. Sandifer takes the reader through Euler’s initial contribution to number theory, showing the five theorems by which he demonstrated for the first time that the fifth Fermat number, $2^{32} + 1$, is not prime. Sandifer also finds earlier than previously-known
origins of the constant \( \gamma \) (the limit as \( n \) approaches infinity of \( 1 + 1/2 + 1/3 + \cdots + 1/n - \log n \), or numerically the decimal fraction \( 0.57721566\ldots \)) in a memoir of 1729. Although no proof yet exists as to whether or not this number is irrational, Julian Havil’s engaging book \( \textit{Gamma} \), published in 2003, has described its significance in the 19th and 20th centuries.

Euler’s foremost early achievement in pure mathematics was his exact solution in 1734 of the longstanding Basel problem, in which he found that the sum of the reciprocals of the square integers, \( \zeta(2) \), is \( \pi^2/6 \). Sandifer reveals that Euler had not just one but four successful computations, and illuminates gaps in the proof of each, including a proof in an article written in 1741, which Sandifer describes as “complete and correct” but not his most elegant for the problem (p. 367). He finds Euler’s earliest application of the technique of mathematical induction in his first proof of Fermat’s little theorem: If \( p \) is a prime number relatively prime to \( a \), then \( p \) divides the expression \( a^{p-1} - 1 \). Sandifer also reviews Euler’s first solutions of the brachistochrone problem, concerning the curve of quickest descent, which were the best of the time, and of the Königsberg Bridges problem, a foundational work in the geometry of position and graph theory. His analysis of the four memoirs leading to Euler’s invention of a semi-geometric calculus of variations in the \( \textit{Methodus inveniendi} \) of 1744 complements related studies by Craig Fraser and Rüdiger Thiele.

But there was far more to Euler’s early work than just this. Sandifer describes Euler’s creation of new ways of solving differential equations and his research on continued fractions with connections to integration, product–sum formulas, trigonometric and generating functions, and the Euler–Maclaurin series. Prolific in research and writing and depending upon his near unerring intuition, Euler is known to have published his findings quickly. Sandifer brings out Euler’s close attention to summing reciprocals of powers, believing that this was related to approximating better the value of \( \pi \). Euler’s efforts to improve the approximations of \( \pi \) and to compute \( e \) by interpolating progressions of factorial numbers and proving it irrational in 1737, long before its rich applications were known, have elsewhere been likened to anchors of the old world of mathematics and the new, with \( e \) being described as the America of mathematics.

In addition to connecting Euler’s research to that of others, especially the Bernoullis, Christian Goldbach, and Jacopo Riccati, and probing the spirit and impetus for mathematical studies of the time, Sandifer notes major events in Euler’s life, such as his publication of the \( \textit{Mechanica} \) in 1736 and the sharing of the prestigious prize of the Paris Academy in 1738. He also adds a wider historical context by providing a brief annual list of world events, masterfully selecting most to relate directly to Euler’s life. All of this enriches his text, although a few items do not seem to merit inclusion, such as the births of Peter III, the French mathematician Jean-Charles Borda, the French chess master François André Philidor, or Paul Revere. Possible alternative events from the Enlightenment might include Voltaire’s \( \textit{Philosophical Letters} \) of 1733, or the births of Immanuel Kant in 1724 and Franz Joseph Haydn in 1732, while the citation of Maupertuis’ \( \textit{Sur la figure de la terre} \) might add that it referred to the Paris Academy’s Lapland expedition in 1736–1737, which Euler followed closely. In addition, a few errors appear in the events and life sections, such as the birth of Euler’s second son Karl Johann, which he gives as 26 July 1740 rather than the correct 15 July.

Sandifer’s text is a valuable contribution to the study of Euler’s mathematics, which is itself central to the history of science in the Enlightenment. Replete with mathematical equations and notations, the book principally addresses a readership with knowledge of mathematics up to differential equations. Sandifer does not provide translations of the memoirs, except for the last, but refers readers to the internet Euler Archive or the four-series Euler \( \textit{Opera omnia} \), currently from Birkhäuser. A critical analysis of selections from Euler’s more than 800 post-1741 memoirs separated according to his Berlin and second St. Petersburg periods, as well as his posthumous record, is even more daunting but also badly needed.

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