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On Deskins's conjecture concerning the supersolvability of a finite group[1]

Zhao Yaoqing *

Department of Mathematics, Guangxi University, Nanning, Guangxi 530004, People's Republic of China

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Abstract

For a maximal subgroup M of a finite group G, a θ-subgroup for M is any subgroup C of G such that $C \not\leq M$ and $core_G(M \cap C)$ is maximal among proper normal subgroups of G contained in C. The aim of this note is to give an answer to Deskins's conjecture on the supersolvability of a finite group by means of the θ-subgroup. © 1998 Elsevier Science B.V.

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1. Introduction and statement of result

All groups considered are finite. In [2], Deskins defines the index complex associated to a maximal subgroup of a finite group as follows: let M be a maximal subgroup of a group G, a subgroup C of G is said to be a completion for M in G if C is not contained in M while every proper subgroup of C that is normal in G is contained in M. The set $I(M)$ of all completions of M is called the index complex of M in G.

If C is a completion of M in G the product of all normal subgroups of G that are proper subgroups of C is itself a proper normal subgroup of C. Denote this subgroup by $k(C)$.

It is clear that $I(M)$ can be partially ordered by set-theoretic inclusion. The maximal elements of $I(M)$ are called maximal completions of M in G.

In [3], Deskins proved that a group G is solvable if and only if each maximal subgroup of G has a maximal completion C with $C/k(C)$ abelian. In the same paper, he shows that the supersolvability cannot be characterized in the same way. He conjectures

* Current address: Department of Mathematics and Physics, Wuyi University, Jiangmen, Guangdong 529020, People's Republic of China. E-mail: hyliang@wyu.edu.cn.

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that a group G is supersolvable if and only if for each maximal subgroup M of G, $I(M)$ contains a maximal completion C with $G = CM$ and $C/k(C)$ cyclic.

In [1], Ballester-Bolinches and Ezquerro pointed out that the answer to this conjecture is negative as the symmetric group S_4 shows. The authors proved in the same paper that a group G is supersolvable if and only if for each maximal subgroup M, $I(M)$ contains an element C which is subnormal in G such that $G=CM$ and $C/k(C)$ is cyclic.

In this note we try to give another answer to Deskins's conjecture.

For a subgroup H of a group G, the core of H in G, $core_G(H)$, is the largest G-normal subgroup of H. We introduce the definition of θ-subgroups as follows:

Definition. Given a maximal subgroup M of G, say that $C \leq G$ is a θ-subgroup for M if $C \nleq M$ and $core_G(M \cap C)$ is maximal among proper normal subgroups of G contained in C. Also, the set of all θ-subgroups for M is denoted by $\theta(M)$.

It is easy to see that for a maximal subgroup M of G, if $C \in I(M)$ then C is a θ-subgroup for M and $k(C) = core_G(M \cap C)$; therefore $I(M)$ is contained in $\theta(M)$.

It is clear that $\theta(M)$ can be partially ordered by set-theoretic inclusion; we call the maximal elements of $\theta(M)$ maximal θ- subgroups of M.

The main result of this note is the following:

Theorem. *Given a finite group G, suppose that for each maximal subgroup M of composite index in G there exists a maximal θ -subgroup C for M such that $G = CM$ and $C/core_G(M \cap C)$ is cyclic. Then either G is supersolvable or else it has a homomorphic image isomorphic to the symmetric group S_4.*

It is clear that if a group G has no homomorphic image isomorphic to S_4 then the condition given in the theorem is necessary and sufficient for G to be supersolvable.

2. Preliminaries

Lemma 1. *If C is a maximal θ-subgroup for a maximal subgroup M of G and $N \triangleleft G$, $N \leq core_G(M \cap C)$, then C/N is a maximal θ-subgroup for M/N. Conversely, if C/N is a maximal θ-subgroup for M/N, then C is a maximal θ-subgroup for M.*

Proof. Suppose that C is a maximal θ-subgroup for M. It follows that $C/N \in \theta(M/N)$. If C/N is not a maximal θ-subgroup in $\theta(M/N)$, then $C/N < H/N$, $H/N \in \theta(M/N)$, implying that $C < H$. Now we see that H is a θ-subgroup for M, violating the maximality of C in $\theta(M)$. □

Conversely, it is easy to see that if C/N is a maximal θ- subgroup for M/N, then C is a θ-subgroup for M. If C is not a maximal θ-subgroup, suppose that $C < H$, $H \in \theta(M)$. This implies that $C/N < H/N$. Since $N \leq core_G(M \cap C) \leq core_G(M \cap H)$, so $H/N \in \theta(M/N)$, violating the maximality of $C/N \in \theta(M/N)$.

Lemma 2. *Let A be an abelian group that acts on some group N. Suppose X is an A-invariant subgroup of N that contains every A- invariant proper subgroup of N. If X is abelian, then N is solvable.*

Proof. If X contains a nontrivial A-invariant normal subgroup M of N then M is abelian and N/M is solvable by induction. We can thus assume that X contains no nontrivial A-invariant normal subgroup of N.

It is no loss to assume that A acts faithfully on N. Suppose $X = 1$. Then no proper subgroup of N admits the action of A. If $|A|$ and $|N|$ are coprime, this forces N to be a p-group and we are done. So we may assume that $|A|$ and $|N|$ have a common prime divisor q. If Q is a Sylow q-subgroup of A, then Q acts on N and N has a Q-invariant Sylow q-subgroup R such that $C_R(Q) \neq 1$. So we have that $C_N(Q)$ is nontrivial and proper in N. Since A is abelian, $C_N(Q)$ admits A. This is a contradiction, and so we can assume that $X > 1$. Let P be a nontrivial Sylow p-subgroup of X. Then P admits A as does its normalizer in N. If the normalizer $N_N(P) = N$, then $P \lhd N$, an impossibility. So $N_N(P) < N$, this forces $N_N(P) = X$. It is easy to see that P is also a Sylow p-subgroup of N. Thus p does not divide $|N : X|$. Since $P \leq Z(N_N(P))$, by a well-known theorem of Burnside N has a normal p-complement which is proper in N and admits A and thus is contained in X. Thus $|N : X|$ is a p-power. This forces $N = X$ and N is abelian. $\square$

3. Proof of the main result

Proof of the theorem. Assume that G does not have a homomorphism onto S_4 and that it is not supersolvable. Let N be a minimal normal subgroup of G. We work for a contradiction by taking the following steps:

(i) G/N is supersolvable by induction.

First of all, we note that if M is a maximal subgroup of G, $L = core_G(M)$ and K/L is a chief factor of G, then it is easy to see that K is a maximal element of $\theta(M)$.

To show that G/N satisfies the hypothesis and so is supersolvable, let M/N be a maximal subgroup of composite index. From Lemma 1, we must find a maximal element A of $\theta(M)$ such that A contains N, $AM = G$ and $A/core_G(A \cap M)$ is cyclic. To do this, let C be a maximal element of $\theta(M)$ and suppose that $CM = G$ and $C/core_G(C \cap M)$ is cyclic. If C contains N, we are done by taking $A = C$. Otherwise, write $L = core_G(M)$ and note that L is not contained in C so that $C < LC$ and hence $LC \notin \theta(M)$. Note also that $L = core_G(LC \cap M)$ and so there exists a subgroup A which is normal in G with $L < A < LC$. We may choose A so that A/L is a chief factor of G. So, A is a maximal element of $\theta(M)$ and certainly A contains N. Since M is maximal and does not contain the normal subgroup A, we have $AM = G$. Finally, $L = core_G(A \cap M)$ and we need only show that A/L is cyclic. This follows because CL/L is cyclic since $C/(C \cap L)$ is a homomorphic image of $C/core_G(C \cap M)$, which is cyclic.

(ii) N is solvable.

We may assume that N is the unique minimal normal subgroup of G. Since G is not supersolvable and G/N is supersolvable, there exists a maximal subgroup M of composite index and we know that M does not contain N. It follows that

$$\theta(M) = \{N\} \cup \{X \subseteq G \mid X \nsubseteq M \quad \textit{and} \quad N \nsubseteq X\}.$$

Since $core_G(C \cap M) = 1$, by hypothesis, there exists a maximal θ-subgroup C of this set such that $CM = G$ and C is cyclic. If $C = N$, then certainly N is solvable, so we can assume that C does not contain N. By the maximality of C as an element of $\theta(M)$, we know that every subgroup of G strictly larger than C contains N. Suppose Y is any subgroup of N normalized by C but not contained in $N \cap C$. Then $C < YC$ and it follows that $N \subseteq YC$ and $N = Y(N \cap C)$. Thus Y is normal in N and N/Y is cyclic and so $N' \subseteq Y$. But $N' = N$, or else $N' = 1$ and we are done, and thus $Y = N$. This shows that every proper C-invariant subgroup of N is contained in $N \cap C$ and Lemma 2 yields that N is solvable, as desired.

(iii) a contradiction.

Now N is an elementary abelian p-group and $M \cap N = 1$. Hence M acts faithfully on N. Also $\mid G : M \mid = \mid N \mid$ is a p-power.

Denote by K^G the normal closure of subgroup K in G. We see that $(C \cap M)^G = (C \cap M)^M$ and this must be trivial since M contains no nontrivial normal subgroup of G. Thus $\mid C \mid = \mid G : M \mid$ and C is a p-group. Since C is maximal in the p-group CN and so has prime index. We see also that $\mid C \cap N \mid \leq p$, so we may assume that $\mid N \mid = p^2$. Thus $\mid M \mid$ is not divisible by p^2 since M is embedded in the automorphism group of N. Hence $H = CN$ is a full Sylow p-subgroup of G of order p^3 and exponent p^2. If $p > 2$, since we may assume that H is not cyclic, then $H = < a, b >$, $a^{p^2} = 1 = b^p$, $b^{-1}ab = a$ or $H = < a, b >$, $a^{p^2} = 1 = b^p$, $b^{-1}ab = a^{1+p}$. In both the cases N contains all elements of order p in H and hence in G, and this is a contradiction since p divides $\mid M \mid = \mid G : N \mid$. We thus have $p = 2$ and thus $\mid M \mid$ divides 6. By considering the permutation representation of G on 2^2 cosets of M, we see that G is isomorphic to a subgroup of S_4. Since $\mid M \mid \neq 3$ and we may assume that $\mid M \mid \neq 2$, it follows that $G \cong S_4$, a contradiction. $\square$

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