For each finite simple group of Lie type there is a natural geometry called its building. It may be defined using certain special (parabolic) subgroups, and in this paper we mimic this construction for the twenty-six sporadic simple groups.

Earlier work on sporadic group geometries by Buekenhout [3] focused on some very interesting geometries which contained buildings of rank at least 2 as residues. Some further developments of such ideas appeared in [30] and [19]. In the present paper however, all the geometries are constructed in a consistent way using the group, and they do not necessarily contain buildings of rank 2 as residues. The idea is as follows. Let $G$ be a finite simple group. If $G$ is of Lie type let $p$ be the characteristic of the field over which $G$ is defined, and if $G$ is sporadic let $p$ be any prime. Let $B$ denote the normalizer of a Sylow-$p$ subgroup of $G$ (in the Lie case, $B$ is a Borel subgroup). If $G$ is of Lie type and Lie rank $n$, there are precisely $n$ subgroups $P_1, \ldots, P_n$ which are minimal with respect to containing $B$. Moreover every subgroup containing $B$ is of the form $P_j = \langle P \mid j \in J \rangle$, $J \subseteq \{1, \ldots, n\}$, where $B = P_{\emptyset}$. The conjugates of these are called the parabolic subgroups of $G$, and they are the simplexes of the building of $G$, using reverse inclusion to say that one is a face of another. If $G$ is sporadic, the situation is different. It may not be possible to generate $G$ using subgroups which are minimal with respect to containing $B$, and even if it there may be other differences such as $P_j \cap P_k \neq P_{J \setminus K}$. This is not surprising, but it does suggest that we should perhaps alter our geometric requirements. We do this by firstly requiring only that the $P_i$ contain a unique maximal subgroup containing $B$ and are $p$-local. The complex obtained by using the $P_i$ may, however, not be a simplicial complex, so we add the requirement that $P_j \cap P_k = P_{J \setminus K}$, and also that the $P_i$ should be a minimal set of subgroups generating $G$ (see Section 1 for more details).

At this point it is appropriate to say a word about chamber systems. In [34], Tits defines a chamber system, and gives an elegant approach to buildings by starting with the set of chambers (i.e. simplexes of maximal dimension), and specifying a set of adjacency relations; two chambers are $i$-adjacent in this sense if they differ in exactly one vertex of type $i$. Group theoretically the chambers are the conjugates of $B$, and $B^g$ is $i$-adjacent to $B^h$ if and only if $gh^{-1} \in P_i$. This idea of starting with the minimal parabolics is what has motivated the work we present. In the case of buildings it makes no difference whether one starts with minimal or maximal parabolics, but the geometries for $p=2$ given in [30] using “maximal parabolics” are for the most part different from the ones we give for $p=2$ here.

Our paper is structured as follows. Section 1 contains the basic definitions and a summary of the results. Sections 2, 3 and 4 then deal with the cases $p=2$, 3, and 5, respectively. In Section 2 some comparisons are drawn with the work of [30] as mentioned above, and in Section 3 we draw some interesting comparisons with the work of Buekenhout [3]. We found in most cases that if there is a geometry satisfying our conditions for a given prime $p$, then it is unique for that prime, although this is not always the case as the McLaughlin group with $p=2$ amply demonstrates.
1. Definitions and Summary of Results

Before giving any definitions we state a simple Proposition valid for any finite group $G$, where $B$ is the normalizer of a Sylow-$p$ subgroup of $G$.

**Proposition.** Let $G = \langle X | B \leq X, O_p(X) \neq 1 \rangle$. Then $G = \langle X | B \leq X, O_p(X) \neq 1 \text{ such that } B \text{ lies in exactly one maximal subgroup of } X \rangle$.

**Proof.** Suppose $B \leq X \leq G$, with $O_p(X) \neq 1$. If $M$ and $N$ are distinct maximal subgroups of $X$ containing $B$, then clearly $O_p(M) \neq 1$, $O_p(N) \neq 1$ and $(M, N) = X$. Thus if $B$ lies in more than one maximal subgroup of $X$, then we replace $\{X\}$ by $\{M, N\}$ and by continuing this procedure as necessary we obtain a set of subgroups containing $B$ and generating $G$ such that $B$ lies in exactly one maximal subgroup of each.

This leads to the following definitions, where $G$ is any finite group, $B$ is the normalizer of a Sylow-$p$ subgroup of $G$, and $P$ is a subgroup containing $B$.

1. We call $P$ a minimal parabolic subgroup if $O_p(P) \neq 1$ and $B$ lies in exactly one maximal subgroup of $P$.

2. Let $P_1, \ldots, P_n$ be minimal parabolic subgroups of $G$. Then we call $\{P_1, \ldots, P_n\}$ a minimal parabolic system of rank $n$ providing that $G$ is generated by $\{P_i\}, i = 1, \ldots, n$ and no proper subset generates $G$.

Given such a minimal parabolic system we immediately obtain a chamber system, as in [34], by taking as chambers the conjugates of $B$, and defining $i$-adjacency, $\sim_i$, by:

$$B^g \sim_i B^h \iff gh^{-1} \in P_i.$$

**Remarks.** If $G$ is a finite group of Lie type in characteristic $p$, with Lie rank at least 2, then there is exactly one such chamber system, namely the building for $G$.

(b) Every sporadic group except $J_1$ and $M_{11}$, admits a minimal parabolic system of rank at least 2 for $p = 2$.

(c) A group can have more than one minimal parabolic system for a fixed $p$ (see for example $M_{23}$, Me for $p = 2$).

Using the above notation, let $J$ be a subset of $I = \{1, \ldots, n\}$, and set

$$P_J = \langle P_j | j \in J \rangle, \quad P_{\emptyset} = B$$

which we call a parabolic subgroup for the system in question.

Then we shall call a minimal parabolic system geometric if for all subsets $J, K \subseteq I$

$$P_J \cap P_K = P_{J \cap K}.$$ 

This is always the case for groups of Lie type in their natural characteristic, and it means that one may define the chamber system as a simplicial complex by starting with the vertices. These are all conjugates of the $P_{I \setminus \{i\}}, i \in I$, and the simplexes are all sets of vertices whose intersection, as groups, contains a conjugate of $B$. Apart from some non-geometric systems given in the appendix to Section 2, all the minimal parabolic systems here are geometric, and we describe them by giving the vertex stabilizers (these may however not be $p$-local—see for example $M_{23}$ for $p = 2$). The simplicial complex is a chamber complex in the sense of Tits [33, p. 3]. We do not consider whether or not it is also a flag complex in the sense of Tits [33, p. 2], although in the rank 2 case this is so by definition.

If $\sigma$ is any simplex of codimension 2 (i.e. corresponding to some $P_{I \setminus \{i,j\}}$), the chambers containing $\sigma$ form a rank 2 chamber system which is a simplicial complex. This is the same thing as a rank 2 flag geometry, and we indicate the type of geometry involved by the appropriate rank 2 piece of the diagram, as in Buekenhout [3].
Our rank 2 subdiagrams have the following meanings:

- Generalized digon (i.e., complete bipartite graph)
- Generalized 3-gon (i.e., projective plane)
- Generalized quadrangle
- Generalized hexagon

\[
\text{generalized m-gon subgeometries, for } i = 1, \ldots, k
\]

**Subgroup Notation**

If the minimal parabolic subgroups for a particular system are \( P_1, \ldots, P_m \), then we give this information by writing, minimal system \( \{ X_1, \ldots, X_n \} \), where \( X_i \) is the permutation group induced on the set of chambers containing the panel corresponding to \( P_i \). Since \( O_p(P_i) \) always acts trivially on this set of chambers, \( X_i \) is always a factor group of \( P_i/O_p(P_i) \). Each node of the diagram corresponds to a vertex stabilizer, which is given above that node. The notation \( B \) refers to an extension of the group \( A \) by a group \( B \) which acts trivially on the residue of the given vertex (notice, however, that \( B \) is not necessarily faithful on the vertex residue—for example \( GL_2(3) \) has a kernel of order 2 when acting on the four edges through a vertex, as in \( M_{12}, p = 3 \)). We use the notation \( n^a_1 \cdots n^a_k \) to denote a group of that order with a subnormal series having sections \( Z_n \).

**Table 1**

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Rank of minimal parabolic system</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( p = 2 )                  ( p = 3 )            ( p = 5 )</td>
</tr>
<tr>
<td>( M_{11} )</td>
<td>1(2^43^25.11 1)</td>
<td>2</td>
</tr>
<tr>
<td>( M_{12} )</td>
<td>2(2^63^25.11 2)</td>
<td>2</td>
</tr>
<tr>
<td>( M_{22} )</td>
<td>12(2^23^57.11 2)</td>
<td>2</td>
</tr>
<tr>
<td>( M_{23} )</td>
<td>1(2^23^57.11 23 1)</td>
<td>3</td>
</tr>
<tr>
<td>( M_{24} )</td>
<td>1(2^103^57.11 23 1)</td>
<td>3</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>1(2^357.11 19 1)</td>
<td>2</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>2(2^33^57 2)</td>
<td>2</td>
</tr>
<tr>
<td>( J_3 )</td>
<td>3(2^33^57.17 19 2)</td>
<td>2</td>
</tr>
<tr>
<td>( J_4 )</td>
<td>1(2^213^57.11 23 29 31 37 43 1)</td>
<td>3</td>
</tr>
<tr>
<td>( \cdot 3 )</td>
<td>1(2^103^57 11 23 1)</td>
<td>3</td>
</tr>
<tr>
<td>( \cdot 2 )</td>
<td>1(2^183^577 11 23 1)</td>
<td>3</td>
</tr>
<tr>
<td>( \cdot 1 )</td>
<td>2(2^213^567 11 13 23 1)</td>
<td>4</td>
</tr>
<tr>
<td>( HS )</td>
<td>2(2^83^577 11 2)</td>
<td>2</td>
</tr>
<tr>
<td>( He )</td>
<td>1(2^103^57 17 2)</td>
<td>3</td>
</tr>
<tr>
<td>( Ly )</td>
<td>1(2^83^577 11 31 37 67 1)</td>
<td>2</td>
</tr>
<tr>
<td>( MC )</td>
<td>3(2^3^5^3^7 11 2)</td>
<td>3, 4</td>
</tr>
<tr>
<td>( ON )</td>
<td>3(2^3^3^5^7 11 19 31 2)</td>
<td>2</td>
</tr>
<tr>
<td>( Ru )</td>
<td>2(2^143^577 13 29 1)</td>
<td>2</td>
</tr>
<tr>
<td>( Suz )</td>
<td>6(2^3^3^5^7 71 11 13 2)</td>
<td>3</td>
</tr>
<tr>
<td>( M(22) )</td>
<td>6(2^173^5^7 11 13 12 2)</td>
<td>3</td>
</tr>
<tr>
<td>( M(23) )</td>
<td>1(2^183^13^5^7 11 13 17 23 1)</td>
<td>4</td>
</tr>
<tr>
<td>( M(24) )</td>
<td>3(2^213^65^2^5^7 11 13 17 23 29 1)</td>
<td>4</td>
</tr>
<tr>
<td>( F_5 )</td>
<td>1(2^43^5^6^7 11 19 2)</td>
<td>2</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>2(2^3^13^5^6^7 11 13 17 19 23 31 47 1)</td>
<td>4</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>2(2^43^13^5^6^7 11 13 17 19 23 31 47 1)</td>
<td>4</td>
</tr>
<tr>
<td>( F_1 )</td>
<td>1(2^63^20^5^8^6^7 11 13 17 19 23 29 31 41 47 59 71 1)</td>
<td>5</td>
</tr>
</tbody>
</table>
(the notation \([m]\) means simply a group of order \(m\)). For example a Sylow-3 normalizer of \(J_3\) has shape \(3^{5+2}8\), meaning that after factoring out the Sylow-3 subgroup, one has \(Z_8\).

**Tabular Summary.** In Table 1 we give the ranks of the various minimal parabolic systems for \(p = 2, 3, 5\) for each sporadic group. If no entry is given, this means that there is no minimal parabolic system of rank at least 2 for the group and prime in question. The only system of rank at least 2 for \(p > 5\) occurs for the Monster, \(F_4\) for \(p = 7\). For each group we also give its order, preceded and followed by the order of its Schur multiplicator and outer automorphism group.

2. 2-Local Geometries

In this section we deal with the case \(p = 2\). The minimal parabolic systems we present are all geometric in the sense of Section 1 (i.e. give rise to simplicial complexes), and they are the only ones possible for each group. In an appendix to this section we give further group theoretical details. In particular we give the partially ordered set of all subgroups containing a Sylow-2 subgroup for most of the more complicated examples.

**A_7—Minimal System \(\{L_2(2), L_2(2), L_2(2)\}\)**

This interesting geometry was first discovered by Neumaier [24]. It is the only known example of a geometry locally isomorphic to a finite building of type \(C_3\) which is not covered by a building (cf. Tits [34]). It may be shown to be simply-connected as a simplicial complex, and hence is its own universal cover. The vertices of the above geometry may be thought of as points, lines and planes, the points being the 7 points on which \(A_7\) acts, the lines being all 3-sets and the planes being one class of \(PG(2,2)\) on the 7 points. Depending on the choice of \(L_3(2)\) (the stabilizer of a plane), the minimal parabolic system is, in notation of the appendix, either \(\{P_1, P_3, P_4\}\) or its image \(\{P'_1, P_3, P_4\}\) under the outer automorphism \(\sigma\) of \(A_7\).

A further minimal system of type \(\{L_2(2), L_2(2), L_2(2)\}\) is \(\{P_1, P_3, P_4\}\) which gives projective 3-space over \(\mathbb{F}_2\), having diagram

The full automorphism group is, however, \(A_8\).

**A_7—Minimal System \(\{L_2(2), L_2(2)\}\)**

Biduads \((ab)(cd)\) and triduads \((ab)(cd)(ef)\) are unordered pairs and triples of disjoint duads (2-sets) of a 7-set. The minimal system is \(\{P_2, P_3\}\) in the notation of the appendix.

**Remark.** If \(A_7\) appears as the top section of a 2-local subgroup, then there is a minimal system of the form \(\{L_2(2), S_3\}\), where the \(L_2(2)\) may arise from any one of \(P_1, P_2, P_3\) or \(P'_7\). This occurs in two cases for \(Mc\).
U₄(2)-2—Minimal System \{L₂(2), S₅\}

\[
\begin{array}{c|c}
S₃ & S₅ \\
2^{1+5} & 2^4 \\
\hline
135 & 27
\end{array}
\]

The building for U₄(2) is a generalized quadrangle with 45 points and 27 lines. In our U₄(2)-2 the parabolic subgroup 2^4A₅ of U₄(2) becomes 2^4S₅ and the 45 points each split into three, with S₅ acting on the set of 15 vertices now incident with a line. The subgroup 2^4S₄ < 2^4S₅, together with 2^{1+5}S₃, generates the subgroup 2^{1+4}S₃ x S₃ which corresponds to generalized digon subgeometries; there are also Sp₄(2) subgeometries with three points per line.

M₁₂—Minimal System \{L₂(2), L₂(2)\}

\[
\begin{array}{c|c}
S₃ & S₅ \\
2^{1+4} & 4^{2}2 \\
\hline
3_5^{5}.11 & 3_5^{5}.11
\end{array}
\]

Points are tetrads (i.e. 4-sets) and lines are trios of three mutually disjoint tetrads in the Steiner system S(12, 5, 6). Both stabilizers are maximal and this geometry appears in Goldschmidt’s work [9] on trivalent graphs (see also Buekenhout [4]). Minimal circuits have eight points and eight lines and lie on flag-graphs of generalized quadrangles with automorphism group Aut(Sp₄(2)). There are 132 such geometries, 8 for each point, which split into two orbits under M₁₂, one corresponding to duads (2-sets) and the other to bihexads (pairs of disjoint hexads) of the Steiner system; they are interchanged by the outer automorphism of M₁₂. A point (tetrad t) is incident with one of these subgeometries (bihexad h₁, h₂, resp. duad d) if t ⊂ h₁ or h₂, resp. t ⋃ d = hexad. If one takes a duad d to be incident with a bihexad h₁ ⋃ h₂ if d ⊂ h₁ or h₂, then using points, duads and bihexads one obtains a geometry belonging to the following diagram.

The lines of our original geometry belong to three bihexads and three duads, one for each pair of points on the line.

J₂—Minimal System \{L₂(4), L₂(2)\}

\[
\begin{array}{c|c}
L₂(4) & S₅ \\
2^{1+4} & 2^{2+4}3 \\
\hline
3_5^{2}.5.7 & 3.5^{2}.7
\end{array}
\]

Both point and line stabilizers are maximal. Minimal circuits have six points and six lines, and lie in generalized hexagon subgeometries with automorphism group G₂(2). The whole geometry embeds in a G₂(4) generalized hexagon, and A. Cohen [8] shows that
it forms a near octagon without quads, in the notation of [31]. The point diameter is 4 and the line diameter is 3.

**J₃—Minimal System** \{L₂(2), L₂(4)\}

\[
\begin{array}{ll}
S₃ & L₂(4) \\
2^{*+*3} & 2^{*+*4} \\
3^3.5.17.19 & 3^4.17.19 \\
points & lines
\end{array}
\]

Both point and line stabilizers are maximal, and are isomorphic to those for J₂ above. It has recently been discovered that J₃ is a subgroup of U₀(2) in such a way that the points and lines of this geometry are points and lines of the U₀(2) polar space; indeed every point of this polar space is a point of our geometry. An elegant combinatorial description of this embedding has been found by Conway [10]. The maximal subgroup \((3 \times A₆)₂\) of J₃ clearly acts on the 45 involutions of A₆ to give a subgeometry with 45 lines and 30 points, being a generalized octagon with two points per line and three lines per point.

**M₂₂—Minimal System** \{L₂(2), S₃\}

\[
\begin{array}{ll}
S₃ & S₃ \\
2^4 & 2^{*+*2} \\
3.7.11 & 3.5.11 \\
points & lines
\end{array}
\]

Points are *duads* \((ab)\) and lines are *triduads* \((abcde)\) forming a hexad of the Steiner system S(22, 3, 6) of which M₂₂ ·2 is the automorphism group. The subgroup \(2^4S₄ < 2^4S₅\) together with the line stabilizer \(2^{*+*2}S₃\) generates \(2^4A₆\) which acts naturally on a generalized quadrangle subgeometry; there are also projective plane subgeometries with automorphism group \(2^3L₃(2)\) (compare [30]). Each point lies on 15 lines, \(\binom{5}{3} = 10\) planes, and 5 quadrangles; each line lies on 3 points, 1 quadrangle, and 2 planes. This M₂₂ geometry appears as a residue in the geometries for M₂₃ ·2, and F₂ below, and a 3-fold covering of it is a residue in our J₄ geometry.

**HS—Minimal System** \{L₂(2), S₃\}

\[
\begin{array}{ll}
S₃ & S₃ \\
4^2 & 4^32^2 \\
3.5^2.7.11 & 3.5^3.7.11 \\
points & lines
\end{array}
\]

The point stabilizer is maximal and its subgroup \(4^2S₃\), together with the line stabilizer \(4^32^2S₃\) generates the maximal subgroup \(4^3L₃(2)\) which acts naturally on projective plane subgeometries. There are also generalized quadrangle subgeometries with automorphism group \(2^3Sp₄(2)\) (compare [30]). Each point lies on 15 lines, 5 planes, and \(\binom{5}{3} = 10\) quadrangles; each line lies on 3 points, 1 planes, and 2 quadrangles (see [30]).

**Ru—Minimal System** \{L₂(2), S₃\}

\[
\begin{array}{ll}
S₃ & S₃ \\
2^{*+*2} & 2^{*+*2} \\
3^2.5^2.7.13.29 & 3^2.5^3.7.13.29 \\
points & lines
\end{array}
\]
The point stabilizer is maximal and its subgroup $2^{1+4+2+4}S_4$, together with the line stabilizer $2^{3+8+2}S_3$, generates the maximal subgroup $2^{3+8}L_3(2)$ which acts naturally on projective plane subgeometries. There are also generalized hexagon subgeometries with automorphism group $2^6G_2(2)$ (compare [30]). Each point lies on 15 lines, 5 planes, and 2 $\binom{5}{3} = 20$ hexagons; each line lies on 3 points, 1 plane, and 4 hexagons. The geometry of the 28-dimensional projective representation of $\text{Ru}$ is studied in detail in [37], and in the 28-dimensional module over $\text{GF}(2)$, our points are certain 1-spaces and our lines are certain 2-spaces (see [22] for more details of this $\text{GF}(2)$-module).

Ly—Minimal System \{L_2(2), S_3\}

There are three points per line and fifteen lines per point. The minimal parabolic system is \{P_1, P_2\} in the notation of the appendix. Using \{P_1, P_3\} one obtains generalized digon subgeometries.

Ly—Minimal System \{L_2(2), S_9\}

There are three points per line and 2835 lines per point. The minimal system is \{P_1, P_6\}; there are generalized digon subgeometries generated by \{P_1, P_2\}, and Mc2 subgeometries generated by \{P_1, P_4\}.

O'N—Minimal System \{L_2(2), L_3(4) \cdot 2\}

There are 3 points per line and 315 lines per point. The minimal parabolic system is \{P_1, P_3\} in the notation of the appendix. Using \{P_1, P_2\}, one obtains projective plane subgeometries with stabilizer $4^3L_3(2)$; there are 105 of these on each point, and one on each line.

F_3—Minimal System \{L_2(2), A_0\}

There are 3 points per line, and 2835 lines per point. The point stabilizer, which is maximal, intersects the non-split $2^3L_4(2)$ subgroup of $F_3$ in $2^{1+8}L_4(2)$, and the line stabilizer is the minimal parabolic subgroup of $2^3L_5(2)$ not contained in this subgroup. As a consequence there are subgeometries with 315 lines per point, in which points and
lines play the role of points and (line, plane, hyperplane) flags of PG(4, 2). Within such
subgeometries there are obviously subplanes, and also generalized digons.

\textbf{F}_5—\textbf{Minimal System} \{L_2(2), A_5[Z_2]\}

\[
\begin{array}{ccc}
A_5[Z_2] & S_3 & \ell \\
2^1+8 & 2^2+3+6+2 & 3
\end{array}
\]

There are three points per line and seventy five lines per point, points corresponding
to 2-central involutions and lines to four-groups \(V\) normal in a Sylow-2 normalizer, \(B_2\).
The subgroup \(P_0 = 2^{1+8}A_4[Z_2]\) maximal in a point stabilizer \(P_1\), together with an incident
line stabilizer \(P_2\), generates the maximal subgroup \(2^{3+2+6}(3 \times L_3(2))\), which clearly stabil­
izes projective plane subgeometries, there being 25 of these on a point, and exactly one
on a line. To obtain \(L_3(2)\) one may work in the centralizer of \(Syl_3(B_2)\), which is \(A_9\).
Clearly \(\langle P_0, P_2 \rangle \cap A_9\) is a rank 2 group fixing a Sylow-2 subgroup of \(A_9\), and it is then
easy to see it must be \(2^1L_3(2) < A_9\); this immediately leads to the structure above.

\textbf{Suz—Minimal System} \{L_2(4), L_2(2), L_2(2)\}

\[
\begin{array}{ccc}
Sp_4(2) & S_3 \times L_2(4) & U_4(2) \\
2^{4+6}3 & 2^{2+8} & 2^1+6
\end{array}
\]

This geometry is described in [30], in which it is explained that its vertices can be
regarded as certain totally isotropic 1, 2 and 4-spaces (reading from left to right of the
diagram) in the \(U_{12}(2)\) geometry. The universal cover is an affine building, by Tits [34].

\textbf{M}_{23—\textbf{Minimal System} \{L_2(2), L_2(2), S_5\}}

\[
\begin{array}{ccc}
(3 \times A_4)2 & L_3(2) & \ell \\
M_{22} & 2^4 & 2^4
\end{array}
\]

Points are the 23 points of the Steiner system \(S(23, 4, 7)\), lines are all \textit{triads} (3-sets)
and planes are one class of projective planes formed from the seven points of a heptad.
The \(A_7\) geometry with diagram

\[
\text{is a subgeometry with automorphism group } 2^4A_7 \text{ and there are also projective spaces}
\text{PG}(3, 2) \text{ as subgeometries with automorphism group } L_4(2). \text{ The minimal parabolics are}
either \{P_1, P_3, P_7\} \text{ or } \{P_4, P_3, P_7\} \text{ in the notation of the Appendix, both systems giving}
geometries as above. Since there is no automorphism of } M_{23} \text{ switching } P_1 \text{ and } P_4, \text{ it}
follows that the two geometries are not isomorphic to one another, although they are
certainly locally isomorphic (i.e. residues are isomorphic). Of course in one case } L_3(2)
\text{ acts on the } 2^4 \text{ as the stabilizer, in } A_7, \text{ of a 1-space, whilst in the other case it is the}
stabilizer of a 3-space.}
**M**\textsubscript{24}—**Minimal System** \{L\textsubscript{2}(2), L\textsubscript{2}(2), L\textsubscript{2}(2)\}

\begin{align*}
3\text{Sp}_4(2) & \quad S_3 \times S_3 & \quad L_3(2) \\
2^6 & \quad 2^{6+2} & \quad 2^{1+6} \\
7, 11, 23 & \quad 3, 5, 7, 11, 23 & \quad 3^2, 5, 11, 23
\end{align*}

In this geometry the points (with stabilizer $2^6\text{Sp}_4(2)$) are sextets (using Conway’s notation [9]) of the Steiner system S(24, 5, 8). Planes are involutions of type $1^82^8$, and lines are sets of three mutually commuting involutions of this type belonging to disjoint octads. In terms of the 12-dimensional module over GF(2) obtained from $2^{24}$ after factoring out the Golay code (see [30]), points, lines and planes are certain 1, 2 and 3-spaces, respectively, all lying in an (11-dimensional) hyperplane. In terms of the maximal 2-local geometry in [30] and [28] with diagram \[ \begin{array}{c}
\text{x} \\
\text{y} \\
\text{z} \\
\text{y'}
\end{array} \], our points, lines and planes are elements of type \( x, (y, \nu) \) and \( (x, \nu) \) where \( \nu \in \text{Res}(x) \) or \( \text{Res}(y) \) as appropriate.

The diagram \[ \sim \] for a point residue indicates a cover of a generalized quadrangle, in this case a 3-fold cover of the Sp\(_4\)(2) quadrangle—its minimal circuits are 5-gons.

**He**—**Minimal System** \{L\textsubscript{2}(2), L\textsubscript{2}(2), L\textsubscript{2}(2)\}

\begin{align*}
3\text{Sp}_4(2) & \quad S_3 \times S_3 & \quad L_3(2) \\
2^6 & \quad 2^{6+2} & \quad 2^{1+6} \\
5, 7^3, 17 & \quad 3, 5^2, 7^3, 17 & \quad 3^2, 5^2, 7^2, 17
\end{align*}

This geometry, locally isomorphic to that for \( M\textsubscript{24} \) above, is easily obtained from the maximal 2-local geometry \( \Delta \) [30] with diagram

\[ \begin{array}{c}
x \\
\text{y} \\
x'
\end{array} \]

where \( \text{y} \) denotes a 3-split cover of the Sp\(_4\)(2) quadrangle, the hollow node referring to split vertices. Our points and planes are vertices \( x \) and \( y \). To obtain the lines look at \( \text{Res}(x') \) in \( \Delta \), which contains 15 sets of triple \( y \)-vertices each incident with the same three \( x \)-vertices; these are our lines. The Held group has a 51-dimensional irreducible module over \( \mathbb{C} \); in recent work [22], Mason and Smith have shown that this module remains irreducible over GF(2) and they have examined the fixed points under \( \text{O}_2(M) \) where \( M \) is a maximal 2-local subgroup, as in [30]. It is clear from their work that the planes of our geometry are certain 3-spaces in this 51-space, and the points and lines are all 1 and 2-spaces in these particular 3-spaces.
\begin{center}
\textbf{3—Minimal System \{L}_2(2), L}_2(2), L}_2(2)\}
\end{center}

Here \(\circ \sim \circ\) represents a 3-fold cover of the complete bipartite graph \(K_{3,3}\); its minimal circuits are 3-gons. The above geometry is easily obtained from the geometry \(\Delta\) formed by the three maximal 2-local subgroups containing a Syl\(_2\)-subgroup, namely \(x = 2\text{Sp}_6(2), z = 2^4\text{L}_4(2)\) and \(y = 2^{2+6}3(S_3 \times S_3)\), having diagram

\begin{center}
\begin{tikzpicture}
    \node (A) at (0,0) {}; \node (B) at (1,0) {}; \node (C) at (0,1) {};
    \draw (A) -- (B) -- (C) -- (A); \draw (A) -- (C) -- (B);
\end{tikzpicture}
\end{center}

This does not appear in \[30\] owing to the fact that \(2\text{Sp}_6(2)\) is not 2-constrained, but as in \[30\], the square node only represents vertices \(v\) when considered in \(\text{Res}(x)\) or \(\text{Res}(z)\). The points, planes, and quads of our geometry are then elements of types \(y, (z, \nu)\), and \((x, \nu)\) where \(\nu \in \text{Res}(x)\) or \(\text{Res}(z)\) as appropriate.

\begin{center}
\textbf{2—Minimal System \{L}_2(2), L}_2(2), S_3\}
\end{center}

This geometry may easily be extracted from the maximal 2-local geometry in \[30\], and it is thereby apparent that it contains subgeometries which are buildings of types \(C_3\) and \(A_3\), stabilized by the subgroups \(2^{1+8}\text{Sp}_6(2)\) and \(2^4 \times 2^{1+6}\text{L}_4(2)\). This contrasts with \(M_{23}\) above where the corresponding subgeometries are for \(A_7\) and \(\text{L}_4(2)\) but with the same diagrams. We now consider the points, lines and planes of this geometry as 1-, 2- and 3-spaces in a 22-space \(V\). Let \(\bar{A}\) denote the Leech lattice \(\text{mod } 2\), so \(\cdot 2\) is the stabilizer of a vector \(\lambda \in \bar{A}\) (we are employing Conway's notation \[9\] as described under \(\cdot 1\) below), and the quadratic form \(Q(v) = v \cdot v/16 \ (\text{mod } 2)\) on \(\bar{A}\) gives us our 22-space \(V = \lambda^+/\langle \lambda \rangle\), on which \(\cdot 2\) acts irreducibly. The special 8-spaces \(\bar{E}\) (as in \(\cdot 1\)) in \(\lambda^+\), containing \(\lambda\), give us 7-spaces \(S = \bar{E}/\langle \lambda \rangle\) of \(V\), with stabilizer \(2^{1+8}\text{O}_7(2)\) in \(\cdot 2\). The totally singular 1-, 2- and 3-spaces (under \(\text{O}_7(2)\)) of such subspaces \(S\) are then the points, lines and planes of our geometry. The seventy seven \(S\) containing a point correspond to the subquadrangles of the \(M_{22}\) geometry above.
Minimal parabolic geometries

J₄—Minimal System \{L₂(2), L₂(2), S₅\}

| 3M₂₂² | S₃×S₅ | L₃(2) |
| 2¹⁺¹² | 2⁴×1²⁺² | 2¹¹⁺¹⁺⁶ |

| points | lines | planes |
| 11², n | 3.7.11³.2₃.n | 3²×5.1₁³.2₃.n |

\( n = 29.3₁.3₇.₄₃ \)

Points, lines, and planes are elements of types \( z, (y, \nu), (x, w, \nu) \) in the maximal 2-local geometry of [30] with diagram

The \( (\bar{5}, 3) \) residue is a 3-fold cover of the M₂₂ geometry above in which the subquadrangles lift to non-trivial covers and the subplanes to trivial covers. Indeed these appear in M₂₄ subgeometries as above, and PG(3, 2) subgeometries, with stabilizers \( 2¹¹M₂₄ \) and \( 2¹⁰+⁴L₄(2) \) respectively, being elements of types \( x \) and \( (w, \nu) \) in the maximal 2-local geometry.

M(22)—Minimal System \{L₂(2), S₅, L₂(2)\}

| U₄(2)² | S₃×S₅ | M₂₂ |
| 2²⁺⁸ | 2⁶⁺6⁺²⁺¹ | 2¹⁰ |

\( 3⁶.5.₇.₁₁ \) \( 3⁷.₅².₇.₁₁.₁₃ \) \( 3⁷.₅.₁₃ \)

The vertices of this geometry are duads, triduads (see M₂₂ above) and 22-ads of Fischer’s 3570 transpositions, reading from left to right. The minimal parabolic corresponding to the middle node has \( S₅ \) on top, and if we use the subgroup with \( S₄ < S₅ \), we generate \( 2⁵⁺₈A₆×S₃ \) which stabilizes a subgeometry which is a building of type \( C₂×A₁ \). There are also \( C₃ \) subgeometries with a stabilizer of shape \( 2⁶Sp₆(2) \) ([23], [40]).

M(23)—Minimal System \{L₂(2), L₂(2), S₅, L₂(2)\}

| (3×U₄(2))² | L₃(2)×S₅ | M₂₃ |
| 2²×2⁺¹⁺₈ | 2⁶⁺6⁺² | 2¹¹ |

\( 3⁴.₁₇.₂₃ \) \( 3⁹.₅.₁₁.₁₃.₁₇.₂₃ \) \( 3¹¹.₅.₁₁.₁₃.₁₇.₂₃ \)

The vertices of this geometry are points, triads, heptad-planes (using one class of \( L₃(2) < A₇ \) as in M₂₃ above), and 23-ads of Fischer’s 31 671 transpositions, reading from left to right. Taking the subgroup \( S₄ < S₅ \) on top of the minimal parabolic corresponding to the second node from the right, one generates, together with the other minimal parabolics, \( 2¹⁰⁺⁴A₇×S₃ \) which stabilizes an appropriate subgeometry with diagram

As in the case of M₂₃, there are two non-isomorphic geometries which have the same local properties discussed above; they arise from the minimal systems \{P₃, P₅, P₆, P₁\} and \{P₄, P₅, P₆, P₁\}, in the notation of the appendix.
Mc—Minimal System \{L_2(2), L_2(2), L_2(2), L_2(2)\}

If we regard the \(U_4(3)\) vertices as points, then the other vertices may be regarded as heptads (two classes), and as quadrics (\(O_7^+(2)\) geometries). A covering of the \(U_4(3)\) residual geometry appears in the geometry for \(M(24)'\). The minimal parabolic system is \(\{P_1, P_3, P_1', P_4\}\) in the notation of the appendix.

Mc—Minimal System \{L_2(2), L_2(2), L_2(2)\}

There are two systems of this type which are interchanged by the outer automorphism of Mc. In the notation of the appendix there are \(\{P_1, P_2, P_1'\}\) and \(\{P_1, P_2', P_1''\}\).

Mc—Minimal System \{L_2(2), L_2(2), S_5\}

Both of the above two geometries involve an \(A_7\) geometry for which one of the minimal parabolics, \(S_5\), is not 2-local in \(A_7\), though it is 2-local in \(2^4A_7\).

\[1—\text{Minimal System} \{L_2(2), L_2(2), L_2(2), L_2(2)\}\]

This geometry arises from the maximal 2-local geometry in [30] in a similar way to that for \(M_{24}\) above. In terms of a GF(2) module let us take the Leech lattice \(\Lambda\), using bars to denote images mod 2. Writing \(\Lambda_n = \{\lambda \in \Lambda | \lambda \cdot \lambda = 16n\}\) as in [9], \(\bar{\Lambda} = \{0\} \cup \bar{\Lambda}_2 \cup \bar{\Lambda}_6\).
Using the construction of $A$ in [9], an octad of the Steiner system $S(24, 5, 8)$ spans an 8-space $E$ of $A$, such that $E \cap A_5 = 0$. For this reason the quadratic form $q(\bar{v}) = v \cdot \bar{v} / 32$ (mod 2) is well-defined on $E$, the totally singular vectors being precisely all vectors of $E \cap A_4$. Under the action of $\cdot 1$ there are 34 . 52 . 7 . 11 . 13 . 23 conjugates of $E$ (they correspond to $E_8$ sublattices of $A$), and they contain two types of totally singular 4-spaces. One type is contained in three different $E$, whilst those of the other type, which we use as the 4-spaces of our geometry, each lies in a unique $E$. Having obtained our 4-spaces, corresponding to the right-most node of the above diagram, our other vertices (points, lines, and planes) are simply all 1-, 2- and 3-spaces in these 4-spaces. The points comprise all vectors of $A_4$.

**F₂—Minimal System** \{L₂(2), L₂(2), L₂(2), S₃\}

<table>
<thead>
<tr>
<th>2</th>
<th>S₃ × M₂2₂</th>
<th>L₃(2) × S₃</th>
<th>L₄(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2₁+22</td>
<td>2²+10+20</td>
<td>2³+32</td>
<td>2⁵+16+6+4</td>
</tr>
</tbody>
</table>

| 3²5³.7.9 | 3¹⁰.5⁶.7.11.23.n | 3¹¹.5⁶.7.11.23.n | 3¹¹.5⁶.7.11.23.n |

Points: $n = 13.17.19.31.47$

This geometry may easily be extracted from the maximal 2-local geometry in [30], and it is thereby apparent that it contains subgeometries which are buildings of types $C₄$ and $A₄$, stabilized by the subgroups $2^{9+16}\text{Sp}_8(2)$ and $2^{5+15+10}\text{L}_₅(2)$.

**M(24)'—Minimal System** \{L₂(2), L₂(2), L₂(2), L₂(2)\}

<table>
<thead>
<tr>
<th>3⁰4(3) : 2</th>
<th>S₃ × Sp₉(2)'</th>
<th>L₃(2) × S₃</th>
<th>M₂₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>2¹+12</td>
<td>2³+12+2</td>
<td>2⁵+6+3</td>
<td>2¹²</td>
</tr>
</tbody>
</table>

| 3⁹.5².7².11.23.n | 3¹³.5³.7³.11.23.n | 3¹⁴.5².7².11.23.n | 3³⁵.5.7².n |

Points: $n = 13.17.29$

This geometry arises from the maximal 2-local geometry in [30] in a similar way to that for $M₂₄$ above, the geometry being a 3-fold cover of the $\text{Sp}_₄(2)$ quadrangle. A block is an $M₂₄$ geometry as given earlier.

**F₁—Minimal System** \{L₂(2), L₂(2), L₂(2), L₂(2), L₂(2)\}

<table>
<thead>
<tr>
<th>1</th>
<th>S₃ × M₂₄</th>
<th>L₄(2) × 3Sp₄(2)</th>
<th>L₄(2) × S₃</th>
<th>L₄(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2¹+24</td>
<td>2²+11+22</td>
<td>2³+36</td>
<td>2⁴+11+2+8+12+4</td>
<td>2⁵+5+16+10</td>
</tr>
</tbody>
</table>

| 3⁹.5.7.13.n | 3⁵.5³.7.11 | 3⁶.5³.7.11 | 3⁷.5³.7.11 |

Points: $n = 3¹¹.5⁶.7⁴.11.1³.17.19.29.31.41.47.59.71$

This arises from the maximal 2-local geometry in [30] in a similar way to that for $M₂₄$ and $\cdot 1$ above.

**Appendix to Section 2**

The partially ordered set (poset) of all subgroups containing a Sylow-2 subgroup is given here for each of the following groups: $M(22)$, $H₄$, $M₂₄$, $\cdot 1$, $J₄$, $F₂$, $A₇$, $M₂₃$, $M(23)$,
Mc, O'N, Ly. For $F_1$ this poset is a straightforward generalization of the ones given for $M_{24}$ and $\cdot 1$, and we omit it. Using each poset we give the possible minimal parabolic systems. As regards notation, the minimal parabolic subgroups are circled and labelled $P_i$. For any subgroup $X$ which is not maximal we show only $X/O_2(X)$; this is no loss since we give the full structure of the maximals. In some cases two of these subgroups are conjugate under an automorphism $\sigma$ (necessarily outer by the Frattini argument), and this is indicated where appropriate.

These posets were obtained using the maximal subgroups which contain a Sylow-2 subgroup. Complete lists of maximal subgroups for $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $J_1$, $J_2$, $J_3$, $\mathbb{M}$, $\cdot 3$, $\cdot 2$, $\cdot 1$, HS, He, Ru, Suz, $F_2$ are given in [9], [12], [14], [15], [13], [38], [39], [21], [7], [37], [36], [25]. For $J_4$, $M(24)$, $F_3$, $F_2$, $F_1$ a complete list of maximal subgroups containing a Sylow-2 subgroup is given by Aschbacher [1]. For $M(22)$ and $M(23)$ the maximal 2-locals were given by Mazurov [23], and a recent paper by R. A. Wilson [40] gives almost complete information on $M(22)$.

For the groups Ly and O'N we know no reference for a list of the relevant maximal subgroups. However we sketch a direct argument to show that the only proper maximal subgroups $H$ containing a Sylow-2 subgroup are: $3\mathbb{M}_2$ and $2A_{11}$ for Ly; and $4^3L_3(2)$ and $4L_3(4) \cdot 2$ for O'N. If $F(H) = 1$, then it is straightforward using group orders and the list of finite simple groups to verify that there is no possibility for $H$ in either case. If $F(H) \neq 1$, then $H$ is the normalizer of some elementary abelian $p$-group. Since we require $2^9 | H |$ for Ly and $2^9 | H |$ for O'N, it is straightforward to check that for $p \geq 3$ the only possibility is $H = 3\mathbb{M}_2$ in the Ly case. This reduces us to the case where $H = N(V)$ with $V$ an elementary abelian 2-group of order $2^i$. In the O'N case $i = 1$ or 3 and $H = 4L_3(4) \cdot 2$ or $4^3L_3(2)$ by [26, sect. 4]. In the Ly case if $i = 1$, then $H = 2A_{11}$ is the involution centralizer. If $i \geq 2$, then since a Sylow-2 subgroup lies in $2A_{11}$, $|V| = 2^2$ and the image of $V$ in $A_{11}$ is of the form $(12)(34)(56)(78)$ (see [20, p. 541]). Therefore $C(V) < 2(A_8 \times 3)2 < 3\mathbb{M}_2$, and since the involutions of $V$ are conjugate in $3\mathbb{M}_2$, we see that $N(V) < 3\mathbb{M}_2$ is not maximal. Therefore $H = 2A_{11}$ or $3\mathbb{M}_2$ for Ly.

The unique minimal parabolic system is $\{P_1, P_2, P_4\}$. 

![Diagram](image-url)
The two minimal parabolic systems, \( \{P_1, P_2, P_4\} \) and \( \{P_1, P_3, P_4\} \) are interchanged by the outer automorphism \( \sigma \).

Clearly \( \{P_1, P_3, P_4\} \) is the unique minimal parabolic system.
It is straightforward to verify that \( \{ P_1, P_2, P_4, P_5 \} \) is the unique minimal parabolic system.
As $P_2$ and $P_5$ are the only minimal parabolics not in $2^{3+12}L_3(2) \times S_3$ and $2^{11}M_{24}$ respectively, they are required in any system. Since $(P_2, P_5) = 2^{1+12}3M_{22}2$, and $P_1$ is the only minimal parabolic not in this subgroup, we obtain the unique system $\{P_1, P_2, P_5\}$. 
The unique minimal parabolic system is \( \{P_1, P_2, P_3, P_3\} \).

If \( X \) is any of the groups on the lower level (i.e. minimal parabolics), then \( X/O_2(X) \cong S_3 \). Both \( L_1 \) and \( L_2 \) have orbits of length 3 and 4 on the 7 points, whilst the others have orbits as indicated. Clearly \( P_3 \) is required in any minimal parabolic system for \( A_7 \), and one obtains the following minimal systems: \( \{P_1, P_3, P_4\} \), \( \{P_1^\sigma, P_3, P_4\} \), \( \{P_1, P_3, P_1^\sigma\} \), and \( \{P_2, P_3\} \): here \( \sigma \) denotes the outer automorphism of \( A_7 \).
Clearly $P_7$ is required in every minimal parabolic system. If, moreover, the system is geometric then it cannot contain $P_6$, and must therefore contain $P_3$. There are two geometric systems, $\{P_1, P_3, P_7\}$ and $\{P_4, P_3, P_7\}$, which are locally isomorphic and differ in the choice of $L_3(2) < A_7$. The non-geometric systems are $\{P_2, P_3, P_7\}$ and $\{P_i, P_6, P_7\}$ where $i = 1, 2, 3$, or 4.
Every minimal parabolic system requires both $P_1$ and $P_8$ as they are the only minimal parabolics outside $2^{11}M_{23}$ and $2^{10+4}S_3 \times A_7$, respectively. A geometric system cannot therefore contain $P_7$, and must consequently contain $P_5$ in order to avoid the $(3 \times U_4(2))2$ parabolic. There are precisely two geometric systems, namely \{P_3, P_5, P_8, P_1\} and \{P_4, P_5, P_8, P_1\}, which are locally isomorphic and differ in the choice of $L_3(2) < A_7$ (cf. $M_{23}$ above).

The non-geometric systems are: \{P_2, P_5, P_8, P_1\} and \{P_7, P_8, P_1\}, and it is worth remarking that only one of the nongeometric systems for $M_{23}$ extends to a system for $M(23)$. 
There are minimal parabolic systems \( \{P_5, P_{51}, P_{52}\} \) where \( P = P_1, P_2, P_3, P_1^* \) or \( P_2^* \), which are clearly not geometric since \( P_5 \cap P_{51}^* = P_4 \). If a system contains \( P_5 \) but not \( P_{51}^* \) then it necessarily contains \( P_1^* \), and if it contains neither \( P_5 \) nor \( P_{51}^* \), then it necessarily contains \( P_1 \) and \( P_1^* \). One obtains the following two additional non-geometric systems: \( \{P_5, P_1^*, P_2\} \) and \( \{P_5, P_1^*, P_3\} \), together with their images under \( \sigma \); and the following two geometric systems involving \( S_3 \): \( \{P_5, P_1^*, P_2^*\} \) and \( \{P_5, P_1^*, P_1\} \), and their images under \( \sigma \). There are just three systems involving only minimal parabolics with \( S_3 \) on top, namely \( \{P_1, P_2^*, P_2^*\} \), \( \{P_1, P_2, P_1^*\} \) and \( \{P_1, P_3, P_1^*, P_4\} \), all of which are geometric.
It is clear that \( \{P_1, P_3\} \) is the unique minimal parabolic system.

Since \( P_1 \) is the only minimal parabolic not contained in 2A\(_{11} \) it is required in every system. Since \( P_5 \) and \( P_6 \) are the only minimal parabolics not in 3Mc2, one therefore obtains the systems \( \{P_1, P_3\} \) and \( \{P_1, P_6\} \) only.

3. 3-LOCAL GEOMETRIES

In each of the groups, \( M_{11}, J_1, J_3, \) and ON, a Sylow-3 normalizer is maximal, of shape \( M_2, S_3 \times D_{10}, 3^{3+2}8, \) and \( 3^42^{1+4}D_{10}, \) respectively. In HS it lies in the unique maximal subgroup \( 2 \times \text{Aut} A_6. \) In \( M_{22}, M_{23}, M_{24}, J_2, Ru, \) and He it lies in more than one maximal
Minimal parabolic geometries

[Page 81]

There is no minimal parabolic system in these cases, nor in the case of $J_4$. Indeed, in $M_{22}$ and $M_{23}$ a Sylow-3 normalizer acts irreducibly on the Sylow-3 subgroup $S$ (of shape $3^2$), and in the other cases, $M_{24}$, $J_2$, $J_4$, Ru, and He, we have the normalizer acting irreducibly on $S/\Phi(S)$ (of shape $3^2$), which leads to a unique 3-local subgroup containing the normalizer. This 3-local subgroup is, respectively: $3S_6$, $3\text{PGL}_2(9)$, $6M_{22}2$, $3(\text{Aut} \ A_6)$, and $3S_7$. For each of the other fourteen sporadic groups, minimal parabolic systems exist, and in the cases of $M_{12}$, $M_{24}$, $J_2$, $J_4$, $F_5$, $Suz$, $M(22)$ they are unique, as may be easily verified from a list of maximal subgroups (see [9, 13, 38, 39, 25, 36, 40]).

A remarkable feature of some of the geometries we give is the existence of subgeometries whose stabilizer $H$ has a Sylow-3 normalizer of index 2 in the Sylow-3 normalizer of the full group $G$, and by adding these subgeometries we obtain a geometry one rank higher which involves a complete graph $\bigcirc \bigcirc$ as a rank 2 residue. Many of these extended geometries appear in the work [3] of Buekenhout, who pioneered the study of geometries involving this $\bigcirc \bigcirc$ residue.

Our first two geometries are for the non-sporadic groups $U_5(2)$ and $D_4(3)S_3$, owing to their appearance in our geometries for $F_1$ and $M(23)$, respectively.

**U$_5$(2)—Minimal System $\{\text{PGL}_2(3), S_3\}$**

\[\begin{array}{ccc}
S_5 & \text{GL}_2(3) \\
3^4 & (4) & 3^2+2 \\
27.5.11 & 2^6.5.11 \\
\text{points} & \text{lines} \\
\end{array}\]

There are 4 points per line and 10 lines per point. These are precisely the points and lines of the dual of the geometry given in [5], with diagram

The 176 quads are $O_5(3)$ quadrangles with stabilizer $3\text{PO}_5(3)$. There are 5 quadrangles on a point, and 2 on a line.

**D$_4$(3)$S_3$—Minimal System $\{\text{PGL}_2(3), L_2(3)^3 2 S_3\}$**

\[\begin{array}{ccc}
(\text{SL}_2(3)^2)S_3 & \text{GL}_2(3) \\
3^{1+4} & (3)(4)(6) & [3^{1+2}] \\
2^4.5.7.13 & 2^6.5.7.13 \\
\text{points} & \text{lines} \\
\end{array}\]

There are four points per line and 64 lines per point. The points are lines (vertices of type 2) of the $D_4(3)$ geometry $Y$, and the lines are all flags of type $(1, 4, 4')$ of $Y$, or in other words

\[\begin{array}{c}
D_4 \\
1 \\
2 \\
\end{array}\]

all $(p, \pi)$ pairs where $p$ is a point on the singular plane $\pi$ of $Y$ ($Y$ has $2^5 \cdot 5 \cdot 7$ points). Taking all points and lines of $D_4(3)S_3$ incident as elements of $Y$ with a fixed flag of type $(1, 4)$, $(1, 4')$, or $(4, 4')$ we obtain a plane. The number of such subplanes is $2^9 \cdot 3 \cdot 5^2 \cdot 7$, there being $2^4 \cdot 3$ on a point, and 3 on a line. Similarly fixing a vertex of type 1, 4 or 4'
in \( Y \) one obtains, in the \( \text{PG}(3, 3) \) residue, \( O_5(3) \) quadrangles formed by flags of type \((4, 4)', (1, 4')\), or \((1, 4)\) and vertices of type 2. Thus these are subquadrangles of the \( \text{D}_4(3)S_3 \) geometry, the number of them being \( 2^5 \cdot 5 \cdot 7 \cdot 3|\text{SL}_4(3): \text{Sp}_4(3)| = 2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \), with \( 2^5 \cdot 3^3 \) on a point, and \( 2 \cdot 3^3 \) on a line. Finally the \( G_2(3) \) generalized hexagons may be obtained from \( Y \) by taking the vertices of type 2 and flags of type \((1, 4, 4')\) fixed under a triality. Consequently we have \(|\text{D}_4(3): G_2(3)| = 2^6 \cdot 3^5 \cdot 5^2 \) generalized hexagon subgeometries, with \( 2^4 \cdot 3^6 \) on a point, and \( 3^6 \) on a line.

**M\(_{12}\)—Minimal System \{\text{PGL}\(_2\)(3), \text{PGL}\(_2\)(3)\}**

\[
\begin{array}{ccc}
\text{GL\(_2\)(3)} & \text{GL\(_2\)(3)} \\
\begin{array}{c}
3^2 \\
220 \\
\end{array} & \begin{array}{c}
(5) \\
220 \\
\end{array} \\
\text{points} & \text{lines} \\
\end{array}
\]

The points are **triads** (3-sets) of the 12 symbols of the Steiner system and the lines are **quartets** of four mutually disjoint triads any two of which form a hexad of the Steiner system. As a bipartite graph it has diameter 6 and girth 10. The outer automorphism of \( M_{12}\) induces a diagram automorphism interchanging points and lines. This geometry was first brought to our attention by Glauberman (personal communication).

**F\(_3\)—Minimal System \{\text{PGL}\(_2\)(3), \text{PGL}\(_2\)(3)\}**

\[
\begin{array}{ccc}
\text{GL\(_2\)(3)} & \text{GL\(_2\)(3)} \\
\begin{array}{c}
3^{1+2}+4+2 \\
211 \cdot 53 \cdot 72 \cdot 13 \cdot 19 \cdot 31 \\
\end{array} & \begin{array}{c}
3^{2+3}+4 \\
211 \cdot 53 \cdot 72 \cdot 13 \cdot 19 \cdot 31 \\
\end{array} \\
\text{points} & \text{lines} \\
\end{array}
\]

There are 4 points per line and 4 lines per point. Let \( G_0 \) and \( G_1 \) denote the normalizer of a \( Z_3 \) and \( Z_3^2 \) subgroup, respectively, both central in a given Sylow-3 subgroup of \( G = \text{F}_3\). Then \( G_i/O_3(G_i) \cong \text{GL}_2(3), \) \( i = 0, 1 \) (see [27]), and we let \( G_0 \) and \( G_1 \) be point and line stabilizers, respectively. Moreover there are subgroups \( X_0 = Z(G_0), X_2, X_4, X_6 \) of \( O_3(G_0) \), and \( X_1, X_3, X_5 \) of \( O_3(G_1) \), where \( X_i/X_{i-2} \) has order \( 3, 3^3, 3^4, 3^2 \) and \( 3^2, 3^3, 3^4 \), respectively, and such that \( X_n \) is the weak closure of \( X_{n-1} (n = 1, \ldots, 6) \) in \( G_0 \) (for \( n \) even) or \( G_1 \) (for \( n \) odd). Let \( d(\ ) \) denote distance in the bipartite graph (i.e. \( d(p, L) = 1 \) if \( p \) is a point incident with a line \( L \)). If we fix a point corresponding to the \( Z_3 \) subgroup \( z \), and a point (resp. line) \( x \) with \( O_3(G_x) = X_6 \) (resp. \( X_3 \)) as above, then \( d(z, x) = 2n \) (resp. \( 2n + 1 \)) if and only if \( z \in X_{2n} \setminus X_{2n-2} \) (resp. \( X_{2n+1} \setminus X_{2n-1} \)).

It follows that the \( Z_3 \) subgroup \( z \) fixes all vertices at distance at most 6 from \( z \). This is a remarkable fact, and the next best known examples are the groups \( ^2\text{F}_4(q) \) where one class of involutions fixes all vertices distant at most 4 from a fixed vertex. Indeed if a graph admits \( L_2(q) \) in its natural permutation representation on the neighbors of any vertex (in particular that is the case here), then Weiss [35] shows there is a vertex \( u \) that \( G_u(u) = 1 \) (\( G_u(u) \) is the subgroup fixing everything distant at most \( n \) from \( u \)). If the graph is bipartite this implies that \( G_0(v) = 1 \) for all vertices \( v \); the \( \text{F}_3 \) example here has \( G_1(v) = 1 \) for all \( v \), but \( G_0(v) \neq 1 \) if \( v \) is a point.

Consider now the 3-local subgroup \( (3 \times G_2(3))2 \); the root groups of \( G_2(3) \) are the 3-central \( Z_3 \) subgroups, and so we obtain generalized 12-gons as subgeometries with automorphism group \( G_2(3)2 \), having two points on a line, the lines corresponding to chambers of the \( G_2(3) \) generalized hexagon. Thus we see that our \( Z_3 \) automorphisms above can be thought of as root groups in \( G_2(3) \). An analogous situation occurs in the generalized octagons for the groups \( ^2\text{F}_4(q) \) which have subgeometries for \( \text{Sp}_4(q)2 \), and
Minimal parabolic geometries

in the $G_2(q)$ generalized hexagons which have subgeometries for $L_3(q)2$. Clearly the
girth of our geometry is 24, though the diameter must be fairly large.

\textbf{Mc—Minimal System \{M_{10}, S_3\}}

\begin{center}
\begin{tabular}{c c c c c}
\hline
2S_5 & M_{10} & M_{10} & 2S_5 & M_{10} \\
\hline
3^5 & 3^5 & 3^5 & (4) & 3^5 \\
2^5.5^2.7.11 & 2^5.5^2.7.11 & & & \\
points & lines & & & \\
\hline
\end{tabular}
\end{center}

The points and lines of this geometry are the points and lines of the dual of Buekenhout’s
gometry [3], [5], with diagram. There are clearly 10 points
per line, and 10 lines per point, and there exist 275 subquadrangles with automorphism
group $U_4(3)$. The number of subquadrangles per point is 5, and there are 2 on each line.
As a bipartite graph viewed from a point vertex it has the following suborbit structure,

orbits corresponding to lines, being circled.

\textbf{3—Minimal System \{M_{11}, S_6\}}

\begin{center}
\begin{tabular}{c c c c c}
\hline
4S_6 & M_{11} \times 2 & M_{11} \times 2 & 4S_6 & M_{11} \times 2 \\
\hline
3^{1+4} & 3^5 & 3^5 & (4) & 3^5 \\
2^4.5^2.7.11 & 2^5.5^2.7.23 & & & \\
points & lines & & & \\
\hline
\end{tabular}
\end{center}

There are 55 points per line, and $\frac{1}{2}(6 \choose 3) = 10$ lines per point. Let us consider this
gometry in relationship to Buekenhout’s geometry [3] with diagram

The points are the points of our geometry, and the 276 $M$-blocks are Mc geometries as
above. There are six $M$-blocks on a point, permuted by $S_6$, and any three of these meet
in an $M$-line $A$ containing points $p$ and $q$. There are three $M$-blocks on $p$ (resp. $q$) not
containing $A$, and these intersect in a further $M$-line $A_1$ (resp. $A_2$). Moreover $A, A_1, A_2$
form a triangle and we obtain subgeometries with 11 $M$-lines and 55 points, stabilized
by the maximal subgroup $3^5(M_{11} \times 2)$; these are our lines. Each $M$-block gives a sub-
geometry with 10 points per line, and any two of these intersect in a generalized quadrangle
with 4 lines per point, stabilized by $U_4(3) \cdot 2^2$. There are 15 quadrangles on a point, and
33 on a line.
Ly—Minimal System \{\text{M}_{11}, S_5\}

\[
\begin{align*}
&\text{8S}_5 \\
&\text{3}_{2+4} \\
&2^2 \cdot 5^5 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67 \\
&\text{points}
\end{align*}
\]

In this geometry there are \(\binom{11}{2} = 55\) points per line, and \(\binom{5}{2} = 10\) lines per point. To obtain it we first construct another geometry whose points are \(Z_3\)-subgroups whose normalizer is \(3\text{Mc}\). In \([20]\) it is shown that the elementary abelian \(3^5\) subgroups with normalizer \(3^5(\text{M}_{11} \times 2)\) contain exactly 11 such \(Z_3\) subgroups, permuted in the natural way by \(\text{M}_{11}\). Any two of these generate the center of a Sylow-3 subgroup, with normalizer \(3^{2+4}8S_5\). Thus we obtain the following rank 3 geometry \(\Delta\),

\[
\begin{align*}
&\text{3Mc} \\
&\text{3}_{2+4} \cdot 8S_5 \\
&\text{3}^5(\text{M}_{11} \times 2)
\end{align*}
\]

where the residue of a point is the \(\text{Mc}\) geometry described above. Clearly our rank 2 minimal parabolic system which gives the geometry comprising vertices of types \(y\) and \(z\) in \(\Delta\) has \(\text{Mc}\) subgeometries (10 points per line and 10 lines per point) with stabilizer \(3\text{Mc}\), there being 2 per point and 11 per line, also within these there are generalized quadrangles with 4 lines per point and stabilizer \(3\text{U}_4(3)\).

As an interesting digression, consider elements of types \((x, \nu), (y, \nu), \) and \(z\) in \(\Delta\), forming a geometry \(\Delta'\). Since the \(\text{M}_{11}\) operating on a \(z\)-residue in \(\Delta\) splits off as a subgroup of the stabilizer of the appropriate vertex, a \(z\)-residue in \(\Delta'\) is a trivial 2-fold cover of \(\text{Mc}\), and consequently, by using the subgroup \(3\text{M}_{11}\), we may split each \(z\)-vertex into two and obtain the following geometry \(\Delta''\).

\[
\begin{align*}
&\text{3U}_4(3) \times 2 \\
&\text{3}_{2+4} \cdot 8S_4 \\
&\text{3}^5\text{M}_{11}
\end{align*}
\]

It is not difficult to see that the subgroup stabilizing such a (sub)-geometry is in fact the entire Lyons group, and so although we have obtained a chamber system with diagram

\[
\begin{align*}
&\text{Mc} \\
&\text{3}_{2+4} \cdot 8S_4 \\
&\text{3}^5\text{M}_{11}
\end{align*}
\]

there is only one \(\nu\)-vertex. This means that if one takes a cell of type \((a, b, c)\), then although it lies in two chambers, these chambers have a common vertex.

\(\cdot 2\)—Minimal System \(\{\text{PGL}_2(9), 2^4S_5\}\)

\[
\begin{align*}
&\text{2}^{1+4}S_3 \\
&\text{3}_{1+4} \\
&2^{10} \cdot 5^2 \cdot 7 \cdot 11 \cdot 23 \\
&\text{points}
\end{align*}
\]

There are 10 points per line, and 40 lines per point. The point stabilizer is maximal, but the line stabilizer lies in the maximal subgroup \(\text{U}_4(3)\) which stabilizes generalized quadrangle subgeometries with ten points per line and four lines per point. There are
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Of these subgeometries on a point and exactly one on each line. They in turn are contained in subgeometries with ten points per line and ten lines per point, stabilized by the maximal subgroup \( M_c \), (see \( M_c \) above); there are sixteen of these on a point, and four on a line.

\[
\text{Fs—Minimal System } \{L_2(3)^2, S_5\} \\
\begin{array}{c}
2S_5 \\
3^{1+4} \\
2^{10}5^57.11.19
\end{array}
\begin{array}{c}
2(A_4 \times A_4)^4 \\
3^4 \\
2^75^67.11.19
\end{array}
\]

There are \( \binom{5}{2} = 10 \) lines per point, and \( 4^2 = 16 \) points per line. The stabilizers are both maximal, and are known [25] to be the only maximal subgroups containing a Sylow-3 normalizer.

\[
\text{Suz—Minimal System } \{\text{PGL}_2(3), M_{11}\} \\
\begin{array}{c}
M_{11} \\
3^5 \\
2^95.7.13
\end{array}
\begin{array}{c}
2(A_4 \times 2^2)^{12} \\
3^{2+4} \\
2^75^67.11.13
\end{array}
\]

There are 4 points per line and 55 lines per point. These are precisely the points and lines of a geometry with diagram

which was determined by Buekenhout (private communication); some of its properties are described in [29]. The 22880 quads correspond to O_6(3) subquadrangles of our geometry, with stabilizer \( 3U_4(3)2 \). There are 11 of these on a point and 2 on a line. The reader may observe that the rank 3 Buekenhout geometry above has the same rank 2 residues as one already obtained for the Lyons group, though in the Ly case none of the stabilizers contains a Sylow-3 normalizer.

For use later in the case of M(24) we remark that the \( \text{Os}(3) \) subquadrangles of the \( O_6(3) \) quadrangles give rise to a subgeometry with diagram

whose automorphism group in Suz is precisely \( U_5(2) \). This geometry has already been discussed above.

\[
\text{M}(22)—\text{Minimal System } \{\text{PGL}_2(3), \text{PGL}_2(3), \text{PGL}_2(3)\} \\
\begin{array}{c}
2^9SL_2(3)^2 \\
3^{1+6} \\
2^95^67.11.13
\end{array}
\begin{array}{c}
2^3L_5(3) \\
\text{planes}
\end{array}
\begin{array}{c}
3^6L_5(3) \\
\text{planes}
\end{array}
\]
We describe this geometry via a geometry with ghost-vertices obtained from three maximal subgroups. From the M(23) transposition graph one obtains a graph $\Gamma$ (a 2-fold cover of $\Gamma$) a suborbit under M(23)) admitting a rank 3 permutation representation (with suborbits $1 + 3159 + 10920 = 14080$ points) of $G = M(22)$ on the maximal subgroup $R \cong \Omega_7(3)$, in which a coclique of four points is stabilized by the subgroup $X = N_G(V_0)$ ($X$ is a 3B-normalizer in Atlas notation), where $V_0 \leq R \cap R'$ for some $g \in G$ (see [16]). Let us take such cocliques as lines and notice there are $10920/3 = 3640$ lines per point. Thus $X \cap R$ is a line stabilizer in the $\Omega_7(3)$ polar space. Now a subgroup $\Omega_7(3)$ not conjugate to $R$ in M(22) (there are two $\Omega_7(3)$ conjugacy classes, fused in M(22)$^\cdot 2$, [16]) gives a subgeometry $S$ comprising 3640 lines (M(22)$^\cdot 2$ fixes a conjugate of $X$) and the points thereon. These lines cannot be mutually disjoint, as there are only 14080 points, and consequently the points of $S$ are the points of an $\Omega_7(3)$ polar space or of its dual. In the latter (dual) case, given two collinear points there are 1045 points collinear with neither, whereas in $\Gamma$ (using non-adjacency) there are at most 324. Therefore the former case holds and we obtain a geometry with diagram

Since $|M(22)|_3 = 3^n$, no 3-local involves $L_4(3)$, and so there are no vertices belonging to the square node. Now taking elements of type $x$ as points and elements of types $(y, \gamma)$ and $(z, \gamma)$ as planes we obtain the geometry given at the start, where $\circ \circ \circ \circ$ denotes a 4-fold cover of the complete bipartite graph $K_{4,4}$ (compare the geometry for $\cdot 3$ with $p = 2$, given earlier).

**M(23)—Minimal System $\{\text{PGL}_2(3), \text{PGL}_2(3), L_2(3)^2\text{S}_3\}$**

From the M(24) transposition graph one obtains a graph $\Gamma$ admitting $G = M(23)$ on the maximal subgroup $D_4(3)S_3$, exhibiting a rank 3 permutation representation with suborbits of size $1 + 3^7.13 + 2^4.3.5^2.7.13 = 2^5.11.17.23$ points. As in the M(22) case one takes certain cocliques of four points in $\Gamma$ [16] as lines, there being $2^4.5^2.7.13$ of these per point, and thus $2^7.5^2.7.11.13.17.23$ in total, with stabilizer $P_{1,3} = N_G(x)$ where $x$ is an appropriate 3-element in $D_4(3)S_3$. The line stabilizer $P_{1,3}$ has two 'minimal parabolics', $P_3 \cong 3^{1+8}2^{1+6}3^2\text{S}_3 \cong 3^{1+8}\text{SL}_2(3)^2\text{S}_3 \cong 3^{1+8}\text{GL}_2(3)$ of index 4, where a 2-element acts as an outer automorphism on the three $\text{SL}_2(3)$, and $P_1 \cong [3^{12}2] \text{GL}_2(3)$, of index 64. If we let $P_{2,3} \cong D_4(3)S_3$ be a point stabilizer sharing a Sylow-3 normalizer with $P_{1,3}$, then $P_{2,3}$ has the minimal parabolics, $P_3$ as above and $P_2 \cong [3^{12}2] \text{GL}_2(3)$. The three subgroups $P_1$, $P_2$, $P_3$ generate our geometry, and by considering $(P_1, P_2) \cap 2M(22) \cong 3^6\text{GL}_3(3)$ we obtain the maximal parabolics above. We omit the group theoretical details and place this argument in a geometric context. Consider two lines on a point. As points of the residual $D_4(3)S_3$ geometry $R$ they lie on a line of $R$ if and only if as lines of a $D_4(3)$ geometry they lie in a plane (on a point) (compare $D_4(3)S_3$ above). But planes of the $D_4(3)$ geometry are planes of $\Omega_7(3)$ subgeometries, and consequently two lines on a point of our $M(23)$ geometry lie in a plane if and only if in the $\Omega_7(3)$ residue of the point in an $M(22)$
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The number of $M(22)$ subgeometries with diagram $\square$ is

$\|M(23) \cdot M(22)\| = 3^4 \cdot 17 \cdot 23$ and so one easily calculates that the number of such subgeometries per point, line, and plane is respectively $2^3 \cdot 3^4 \cdot 5$, $2^2 \cdot 3^4$, and $3^4$. A simple counting argument then shows that every plane meets every $M(22)$ subgeometry in a point, a line, or the plane itself.

$F_1$—Minimal System $\{\text{PGL}_2(3), \text{PGL}_2(\mathbb{F}_2), M_{11}\}$

$2 \cdot \text{Suz} \cdot 2 \cdot \text{GL}_2(3) \cdot \text{M}_{11} \cdot \text{L}_3(3)$

$3^{1+12} \cdot 2^{38} \cdot 5^6 \cdot 7^6 \cdot 11 \cdot 13^2 \cdot n\quad 2^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot n \quad 2^{38} \cdot 5^6 \cdot 7^6 \cdot 11 \cdot 13^2 \cdot n$

The residue of a point is the Suzuki geometry already given, and the quadrics are $\Omega_6(3)$ geometries with stabilizer $3^9 \cdot \Omega_6(3)$ - 2. We note that the points, lines, and planes may be regarded as certain elementary abelian subgroups in $F_1$ of orders $3$, $3^2$, $3^3$, respectively.

$M(24)$—Minimal System $\{\text{PGL}_2(3), \text{PGL}_2(\mathbb{F}_2), S_3\}$

$U_6(2) \cdot 2 \cdot \text{SL}_2(3) \cdot (\text{AG}_2)' \cdot 2 \cdot \text{L}_3(3)$

$3^{1+10} \cdot 2^{10} \cdot 5 \cdot 7^3 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \quad 2^{18} \cdot 5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23 \cdot 29 \quad 2^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 17 \cdot 23 \cdot 29$

The $U_5(2)$ geometry in the residue of a point has already arisen as a subgeometry of our geometry for Suz. Indeed we shall obtain the system above as a subgeometry of that for $F_1$, and by adding subgeometries which are $B_3(3)$ buildings, with stabilizer $3^7 \cdot \Omega_7(3)$, we obtain a geometry belonging to the diagram

$3^7 \cdot \Omega_7(3)$

To construct this consider the $F_1$ geometry above, whose points, lines, and planes can be considered as elementary abelian subgroups of orders $3$, $3^2$, $3^3$ respectively as already mentioned. Now $3M(24)$ is a subgroup of $F_1$, and the centralizer of a 3-central element of $F_1$ in $M(24)$ is known to be $3^{1+10} \cdot U_5(2)$. As explained under Suz, the subgroup $U_5(2)$ in Suz gives a subgeometry $\square$ of the Buekenhout Suz geometry with $O_5(3)$ quadrangles. Since the lines of our $F_1$ geometry correspond to elementary abelian $3^2$ subgroups, they correspond to 1-spaces of the $3^2$ module for 2 Suz, as already
described, and consequently to 1-spaces in the $3^{10}$ submodule of this for $U_5(2)$. That is
to say the lines in the residue of a point for $M(24)$ are $3^2$-subgroups of $M(24)$, and
consequently all points of such a line in the $F_1$ geometry lie in the $M(24)$ submodule
and so the diagram above is verified. We have established the existence of the geometry
using only the knowledge of $3^{1+10}U_5(2)$ -2, but the orders of the other stabilizers may
be immediately calculated and their precise structure may be shown to be as given.

**F₂—MINIMAL SYSTEM** \{PGL₂(3), PGL₂(3), PGL₂(3)\}

<table>
<thead>
<tr>
<th>Points</th>
<th>Lines</th>
<th>Planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{1+6}{\text{PSp}}_4(3) \cdot 2$</td>
<td>$3^{1+8}$</td>
<td>$3^{2+3+6}$</td>
</tr>
<tr>
<td>$2^{27}.5^6.13.n$</td>
<td>$2^{24}.5^6.13.n$</td>
<td>$2^{36}.5^6.n$</td>
</tr>
</tbody>
</table>

$n = 7^2.11.17.19.23.31.47$

To generate this geometry for $G = F₂$ we use the subgroup $M(23)$ and the normalizer
of a $Z_3$-subgroup $\langle \sigma \rangle$ (call it $Q_{2,3}$) which is known to have the structure $3^{1+8}2^{1+6}{\text{PSp}}_4(3) \cdot 2$, where the extension $2^{1+6}{\text{PSp}}_4(3)$ is non-split. Let $Q_2$ and $Q_3$ be the minimal parabolics
generating $Q_{2,3}$, with $Q_3 = N_{Q_{2,3}}(\langle \sigma, \tau \rangle)$, where $\tau \in 3^{1+8}(\sigma)$ is conjugate to $\sigma$. The groups
$Q_2$ and $Q_3$ both contain the extraspecial $3^{1+8}$ of course, and $Q_i/3^{1+8} \cap 3^{1+8}2^{1+6}/3^{1+8} = 2$, for $i = 2, 3$, so the geometry they generate is a 64-fold cover of the ${\text{Sp}}_4(3)$ quadrangle
(denoted above) having 2560 points and 2560 lines. Now let $M$ be an $M(23)$ subgroup of $G$ containing the Sylow-3 normalizer of $Q_{2,3}$. Then $M$ contains a minimal parabolic, $P_2$ in the $M(23)$ notation, normalizing $\langle \sigma, \tau \rangle$, and consequently $(P_2, Q_3)/Q_3(P_2, Q_3)$ is a central product. Moreover $Q_{2,3}$ must met $C$ in a minimal
parabolic, which must therefore be $Q_2 = P_1$. Now using the subgroups $P_2, P_1 = Q_2, Q_3$ we obtain the geometry above.

**·₁—MINIMAL SYSTEM** \{PGL₂(3), PGL₂(3), PGL₂(3)\}

<table>
<thead>
<tr>
<th>Points</th>
<th>Lines</th>
<th>Quads</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2M_{12}$</td>
<td>$\text{GL}_3(3)^2$</td>
<td>$\text{Sp}_4(3) \cdot 2$</td>
</tr>
<tr>
<td>$2^{4}.5^6.13.23$</td>
<td>$2^{14}.5^4.7^2.11.13.23$</td>
<td>$2^{13}.5^5.7^2.11.13.23$</td>
</tr>
</tbody>
</table>

There is a related geometry for ·₁ with diagram

\[
\begin{array}{cccc}
1 & \subset & 2 & \subset \\
\end{array}
\]

discovered by Beukenhout, where the points are the $Z_3$-subgroups of ·₁ with normalizer
3 $\text{Suz} \cdot 2$, the other vertices corresponding to certain duads, triads and dodecads of points.
The dodecads and triads of this latter geometry correspond precisely to the points and
lines of our ·₁ geometry, and consequently there are subgeometries which are $\text{O}_6(3)$
generalized quadrangles with ten lines per point, and $\text{Suz}$ geometries as already described
above with 55 lines per point. In a point stabilizer, $M_{12}$ naturally permutes the twelve
Suz geometries containing the point.

4. $p$-LOCAL GEOMETRIES FOR $p \geq 5$

For each of the following groups there is no minimal parabolic system for $p \geq 5$: $M_{11}$,
$M_{12}, M_{22}, M_{23}, M_{24}, \text{He}, \text{HS}, \text{Ru}, \text{Mc}, \text{ON}, J_1, J_2, J_3, J_4, \text{Suz}, \cdot 3, \cdot 2$, $M(22), M(23)$,
M(24)', F₃. For example a Sylow-7 normalizer of He is maximal of shape $7^{1+2}(S₃ \times 3)$, and as He is a subgroup of M(24)' containing a Sylow-7 subgroup, S, and as $S₃ \times 3$ acts irreducibly on $S/\Phi(S)$, it follows that there is no minimal parabolic system with $p = 7$ for M(24)'. Of course if $|G|_p = p$, it is a trivial observation that no minimal parabolic system for $p$ exists, whilst if $|G|_p > p$ one checks that the Sylow normalizer acts irreducibly on $S/\Phi(S)$, where S is a Sylow-$p$ subgroup. We now deal with each of the remaining groups.

\[ \cdot 1: p = 5—\text{Minimal System } \{\text{PGL}_2(5), \text{PGL}_3(5)\} \]

\[
\begin{array}{c|cc}
(4 \times A_4) & 2 & \text{GL}_2(5) \\
5^3 & & 5^{1+2} \\
2^1.3^8.7^2.11.13.23 & 2^1.3^8.7^2.11.13.23 \\
\hline
\text{points} & \text{lines}
\end{array}
\]

There are 6 points per line, and 6 lines per point. Both stabilizers are maximal in \( \cdot 1 \) (see [39]), and are isomorphic to the point and line stabilizers in $S_{p4}(5)$. It is the only minimal parabolic system for $p = 5$, and there are no minimal parabolic systems for $p > 5$.

\[ F_5: p = 5—\text{Minimal System } \{\text{L}_2(5)2, 2^4 \text{Frob}(20)) \]

\[
\begin{array}{c|cc}
2^{1+4}.5.4 & \text{SL}_2(5)2 \\
5^{1+4} & 5^{2+1+2} \\
2^7.3^6.7.11.19 & 2^0.3^6.7.11.19 \\
\hline
\text{points} & \text{lines}
\end{array}
\]

There are 6 points per line, and 16 lines per point. Both stabilizers are maximal in $F_5$ and this is the only minimal parabolic system for $p = 5$. There are no minimal parabolic systems for $p > 5$, see [25] and [18].

\[ F_2: p = 5—\text{Minimal System } \{\text{PGL}_2(5), 2^4S_3\} \]

\[
\begin{array}{c|cc}
2^{1+4}S_4.2 & \text{GL}_2(5) \\
5^{1+4} & 5^{2+1+2} \\
2^{32}.3^{12}.n & 2^{36}.3^{12}.n \\
\hline
\text{points} & \text{lines}
\end{array}
\]

There are 6 points per line, and 96 lines per point. We shall obtain this geometry as an extension of that for $F_5$ above. It is known that the Monster $F_1$ has two classes of 5-elements, one of which has centralizer $5 \times F_5$ (see [17]). This immediately gives us a 5-element normalizer $5(2 \times F_5)2$ or $5(4 \times F_5)$, since $|\text{Out}(F_5)| = 2$. The involution commuting with $F_5$ must have centralizer $2F_2$ in $F_1$, so $F_5$ is a subgroup of $F_2$, and since it does not lie in an involution centralizer we have $5(2 \times F_5)2$ in $F_1$ and so $\text{Aut}F_5 < F_2$. This immediately gives us the minimal parabolic $5^{2+1+2}\text{GL}_2(5)$. To obtain the other, we notice that $5^{1+2}4S_4 < 2$ involved in $F_2$, together with $5^{1+4}2^{1+4}5.4 < F_2$ generates $5^{1+4}2^{1+4}S_4.2$. Clearly there are $F_5$ subgeometries with stabilizer $F_52$, there being 6 on each point and one on each line.

For $p = 7$, a Sylow-$p$ normalizer has order $2^63^27^2$ and acts irreducibly on its Sylow-7 subgroup (see [32]) and for $p > 7$, $|F_2p| = p$ so there are no minimal parabolic systems for $p > 5$. 
F₁: p = 5—Minimal System {PGL₂(5), J₂2}

There are 6 points per line and 2016 lines per point. The point-stabilizer 5₁+62J₂4 is given in [17], whilst the line stabilizer 5₂⁺²⁺⁴(GL₂(5) × S₃) is the extension of the line stabilizer for 2F₂ or F₅ above. Clearly therefore there are F₂ and F₅ subgeometries as above with stabilizers 2F₂ and 5(F₅ × 2)₂ respectively. A simple counting argument shows there are 3² · 5¹ · 7 (resp. 3 · 5³) F₂ subgeometries and 2 · 3³ · 5² · 7 (resp. 3 · 5³) F₅ subgeometries on a point (resp. line).

F₁: p = 7—Minimal System {PGL₂(7), S₇}

There are 8 points per line and 120 lines per point. The point stabilizer 7₁⁺₄2A₇₁ is given in [17], whilst the line stabilizer may be obtained as follows. Let S denote the extraspecial 7₁⁺₄ which, since it has exponent 7, contains an element s ∈ S[Z(S)] fixed by a 7-element of A₇. Therefore, 7₁⁺₄[S]C₇(S)[S], and since there are only two classes of 7-elements in F₁, the other class having centralizer 7×He, we see that s is conjugate in F₁ to z ∈ Z(S). Our line stabilizer is then NF₁(⟨s, z⟩).

For p > 7, there are no further minimal parabolic systems since in the cases p = 11 and 13, the normalizer of a Sylow-p acts irreducibly on P and P/Φ(P), respectively, and for p > 13, |P| = p.

Ly: p = 5—Minimal System {PGL₂(5), PGL₂(5), PGL₂(5)}

This geometry was first discovered by Buekenhout and Kantor and is described in [19]. Its universal cover is an affine building (of type G₂) by Tits’s theorem [34], though not arising from an algebraic group in view of the action of the line stabilizer on its residue. For p > 5, there are no minimal parabolic systems since |Ly|₀ = p or 1.

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