Some further results on $I$-Cauchy sequences and condition (AP)∗

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In this paper we provide answers to two important questions regarding $I$ and $I^*$-Cauchy sequences introduced and studied by Nabiev et al. (2007) [9] which were left unanswered. We then introduce the ideas of $I$ and $I^*$-divergent sequences in a metric space and study their certain properties. Our investigation strengthens and reconfirms importance of condition (AP) in the study of summability through ideals.

1. Introduction and background

The idea of statistical convergence was known to A. Zygmund as early as 1935 and in particular after 1951 when Steinhaus [1] and Fast [2] reintroduced statistical convergence for sequences of real numbers; several generalizations and applications of this notion have been investigated (see [3–11] where many more references can be found). In particular two interesting generalizations of statistical convergence were introduced by Kostyrko et al. [12], using the notion of ideals of the set $\mathbb{N}$ of positive integers who named them as $I$ and $I^*$-convergence, although an equivalent idea named filter convergence was studied earlier by Katetov [13]. Corresponding $I$-Cauchy condition was first introduced and studied by Dems [14] and also by Gurdal [15]. $I^*$-Cauchy sequences have been very recently introduced by Nabiev et al. [16] where they showed that $I^*$-Cauchy sequences are $I$-Cauchy and they are equivalent if the ideal $I$ satisfies the condition (AP) (this was also done to some extent in [3]). However two important questions remained unanswered: firstly construction of an example of an $I$-Cauchy sequence which is not $I^*$-Cauchy and what is the necessary and sufficient condition for their equivalence. In Section 2 of this paper we primarily show that under some general assumption, the condition (AP) is both necessary and sufficient for the equivalence of $I$ and $I^*$-Cauchy sequences and construct an example to prove that in general $I$-Cauchy sequences may not be $I^*$-Cauchy.

In Section 3 of this paper we introduce the notions of $I$-divergent and $I^*$-divergent sequences in a metric space and prove certain properties. We primarily show that, like convergence and the Cauchy condition, condition (AP) is the necessary and sufficient condition for the equivalence of $I$ and $I^*$-divergence under certain conditions. This in a way, strengthens and reconfirms the importance of condition (AP) in the study of summability through ideals.

The following definitions and notions will be needed.

Definition 1. Let $X \neq \emptyset$. A family $I \subset 2^X$ of subsets of $X$ is said to be an ideal in $X$ provided that the following conditions hold:

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(a) $\phi \in I$.
(b) $A, B \in I$ implies $A \cup B \in I$.
(c) $A \in I, B \subseteq A$ implies $B \in I$.

**Definition 2.** Let $X \neq \phi$. A non-empty family $F \subset 2^X$ is said to be a filter on $X$ if the following are satisfied:
(a) $\phi \notin F$.
(b) $A, B \in I$ implies $A \cap B \in I$.
(c) $A \in I, B \subset A$ implies $B \in I$.

**Lemma 1.** Let $I$ be a proper ideal in $X$ (i.e., $X \notin I$), $X \neq \phi$. Then the family of sets $F(I) = \{M \subset X : \text{there exists } A \in I : M = X \setminus A\}$ is a filter in $X$. It is called the filter associated with the ideal.

**Definition 3.** A proper ideal $I$ is said to be admissible if $|x| \in I$ for each $x \in X$.

**Definition 4.** Let $I \subset 2^N$ be a proper ideal in $\mathbb{N}$ and $(X, d)$ be a metric space. The sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ is said to be $I$-convergent to $x \in X$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : d(x_n, x) \geq \epsilon\}$ belongs to $I$.

**Definition 5.** An admissible ideal $I \subset 2^\mathbb{N}$ is said to satisfy the condition $(AP)$ if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \ldots\}$ belonging to $I$ there exists a countable family of sets $\{B_1, B_2, B_3, \ldots\}$ such that $A_j B_j$ is a finite set for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in I$.

Note that $B_j \in I$ for all $j \in \mathbb{N}$.

**Definition 5** is similar to the condition $(APO)$ used in [4].

In [12], the concept of $I^*$-convergence which is closely related to the $I$-convergence had been introduced as follows.

**Definition 6.** The sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ is said to be $I^*$-convergent to $x \in X$ if and only if there exists a set $M \in F(I)$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$ such that $\lim_{n \to \infty} d(x_{m_n}, x) = 0$.

In [12] it was proved that $I$ and $I^*$-convergence are equivalent for admissible ideals with property $(AP)$.

### 2. Equivalence of $I$-Cauchy and $I^*$-Cauchy conditions

In [9] the notions of $I$-Cauchy and $I^*$-Cauchy sequences were introduced as follows.

**Definition 7 (See also [14]).** Let $(X, d)$ be a metric space and $I \subset 2^\mathbb{N}$ be an admissible ideal. Then a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is called an $I$-Cauchy sequence in $X$ if for every $\epsilon > 0$ there exists $K = K(\epsilon)$ such that $\{n \in \mathbb{N} : d(x_n, x_K) \geq \epsilon\} \in I$.

In [14] it was shown that $\{x_n\}_{n \in \mathbb{N}}$ is $I$-Cauchy if and only if for any $\epsilon > 0$, there is a set $B = B(\epsilon) \in F(I)$ such that $m, n \notin B \Rightarrow d(x_m, x_n) < \epsilon$.

**Definition 8.** Let $(X, d)$ be a metric space and $I \subset 2^\mathbb{N}$ be an admissible ideal. Then a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ is called an $I^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subseteq \mathbb{N}$, $M \in F(I)$ such that the subsequence $x_M = \{x_{m_n}\}_{n \in \mathbb{N}}$ is ordinary Cauchy sequence in $X$.

In [9] it was shown that for any admissible ideal $I$, the $I^*$-Cauchy condition of a sequence implies $I$-Cauchy condition (Theorem 3), and the converse is true if the ideal $I$ satisfies the condition $(AP)$ (Theorem 4). However no example was given in [9] to show that in general $I$-Cauchy condition does not imply $I^*$-Cauchy condition. Here we first construct the following example in this direction.

**Example 1.** Let $\mathbb{R}$ be the real number space with usual metric $d$. Let $\mathbb{N} = \bigcup_{i \in \Delta} \Delta_i$ be a decomposition of $\mathbb{N}$ such that each $\Delta_i$ is infinite and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$. Let $I$ be the class of all those subsets $\Delta$ of $\mathbb{N}$ that can intersect only finite number of $\Delta_i$’s. Then $I$ is a non-trivial admissible ideal of $\mathbb{N}$.

Now $\{1, 2, 3, \ldots\}$ is Cauchy in $(\mathbb{R}, d)$. Define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_n = \frac{1}{n}$ if $n \in \Delta_i$. Let $\epsilon > 0$ be given. Then there is a $k \in \mathbb{N}$ such that $d\left(\frac{1}{k+1}, \frac{1}{k}\right) = \frac{1}{k(k+1)} > \frac{\epsilon}{2}$ whenever $n, m \geq k$. Now $B = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k \in I$ and clearly $I \notin B \Rightarrow d(x_m, x_n) < \epsilon$. Hence $\{x_n\}_{n \in \mathbb{N}}$ is $I$-Cauchy.

Next we shall show that $\{x_n\}_{n \in \mathbb{N}}$ is not $I^*$-Cauchy. If possible assume that $\{x_n\}_{n \in \mathbb{N}}$ is $I^*$-Cauchy. Then there is a $A \in F(I)$ such that $\{x_n\}_{n \in A}$ is Cauchy. Since $\mathbb{N} \setminus A \in I$ so there exists a $I \subset \mathbb{N}$ such that $\mathbb{N} \setminus A \subset \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_l$. But then $\Delta_i \not\subseteq A$ for all $i > l$. In particular $\Delta_l \cup \Delta_{l+1} \subset I$. From the construction of $I$’s it clearly follows that given any $m \in \mathbb{N}$ there are $m \in \Delta_{l+1}$ and $n \in \Delta_{l+2}$ such that $m, n \geq k$. Hence there is no $k \in \mathbb{N}$ such that whenever $m, n \in A$ with $m, n \geq k$ then $d(x_m, x_n) < \epsilon_0$ where $\epsilon_0 = \frac{1}{2(3l+2)} > 0$. This contradicts the fact that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy.

In the following theorem we prove the converse of Theorem 4 in [9] showing that under certain general conditions, the condition $(AP)$ is also necessary for the equivalence of the concepts of $I$ and $I^*$-Cauchy conditions.
Theorem 1. Let \((X, d)\) be a metric space containing at least one accumulation point. If for every sequence \(\{x_n\}_{n \in \mathbb{N}}\), I-Cauchy condition implies \(I^*\)-Cauchy condition then \(I\) satisfies the condition (AP).

Proof. Let \(x_0\) be an accumulation point of \(X\). Then there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) of distinct points in \(X\) such that \(\{x_n\}_{n \in \mathbb{N}}\) converges to \(x_0\) and \(x_n \neq x_0\) for all \(n \in \mathbb{N}\). Suppose \(\{A_i : i = 1, 2, 3, \ldots\}\) is a sequence of mutually disjoint non-empty sets from \(I\). Define a sequence \(\{y_n\}_{n \in \mathbb{N}}\) by \(y_n = x_i\) if \(n \in A_i\) and \(y_n = x_0\) if \(n \notin A_i\) for any \(j \in \mathbb{N}\). Let \(\epsilon > 0\) be given. Then there exists \(l \in \mathbb{N}\) such that \(d(x_0, y_n) < \frac{\epsilon}{2}\) for all \(n \geq l\). Then \(A_i(\frac{\epsilon}{2}) = \{n \in \mathbb{N} : d(x_0, y_n) \geq \frac{\epsilon}{2}\} \subseteq A_1 \cup A_2 \cup \cdots \cup A_l\) and \(A_1 \cup A_2 \cup \cdots \cup A_l\) is a Cauchy sequence. Now clearly \(i, j \notin A(\frac{\epsilon}{2})\) implies that \(d(x_0, y_i) < \frac{\epsilon}{2}\) and \(d(x_0, y_j) < \frac{\epsilon}{2}\). So \(d(y_i, y_j) \leq d(y_i, x_0) + d(x_0, y_j) \leq \epsilon\). This shows that \(\{y_n\}_{n \in \mathbb{N}}\) is an \(I\)-Cauchy sequence. By our assumption \(\{y_n\}_{n \in \mathbb{N}}\) is \(I^*\)-Cauchy. Hence there exists \(H \in I\) such that \(B = \mathbb{N} \setminus H \in F(I)\) and \(\{y_n\}_{n \in \mathbb{N}}\) is Cauchy. Now let \(B = A_i \cap H\) for \(j \in \mathbb{N}\). Then each \(B_j \in I\). Further \(\bigcup B_j = H \cap (\bigcup A_i) \in H\). Therefore \(\bigcup B_j \in I\). Now for the sets \(A_i \cap B\), \(i \in \mathbb{N}\) following the cases may arise:

- Case I: Each \(A_i \cap B\) is included in a finite subset of \(\mathbb{N}\).
- Case II: Only one of \(A_i \cap B\)’s namely \(A_i \cap B\) (say) is not included in a finite subset of \(\mathbb{N}\).
- Case III: More than one of \(A_i \cap B\)’s are not included in a finite subset of \(\mathbb{N}\).

If (I) holds, then

\[
A_j \Delta B_j = A_j \setminus B_j = A_j \setminus H = A_j \cap B
\]

is included in a finite subset of \(\mathbb{N}\) and this implies that \(I\) has the (AP) condition.

If (II) holds, then we redefine \(B_k = A_k\) and \(B_j = A_j \cap H\) for \(j \neq k\). Then

\[
\bigcup_{n \in \mathbb{N}} B_j = \left[H \cap \left(\bigcup_{j \neq k} A_j\right)\right] \subseteq H \cup A_k
\]

and so \(\bigcup B_j \in I\). Also since \(A_i \Delta B_i = A_i \cap B\) for \(i \neq k\) and \(A_i \Delta B_k = \emptyset\). So as in Case (I) the criteria for (AP) condition is satisfied.

If (III) holds, then there exists \(k, l \in \mathbb{N}\) with \(k \neq l\) such that \(A_k \cap B\) and \(A_l \cap B\) are not included in any finite subset of \(\mathbb{N}\). Let \(\epsilon_0 = \frac{d(x_0, x_l)}{2} > 0\). As \(\{y_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence, so for the above \(\epsilon_0 > 0\) there exists \(k_0 \in \mathbb{N}\) such that \(d(x_i, x_j) < \epsilon_0\) for all \(i, j \geq k_0\) and \(i, j \in B\). Now since \(A_k \cap B\) and \(A_l \cap B\) are not included in any finite subset of \(\mathbb{N}\), so we can choose \(i \in A_k \cap B\) and \(j \in A_l \cap B\) with \(i, j \geq k_0\). But \(y_i = x_k\) and \(y_j = x_l\) and so \(d(y_i, y_j) = d(x_k, x_l) > \epsilon_0 > 0\) (in fact there are infinite number of indices of \(B\) with that property). This contradicts the fact that \(\{y_n\}_{n \in \mathbb{N}}\) is Cauchy. Therefore Case (III) cannot arise. And in view of Case (I) and Case (II) \(I\)-satisfies (AP) condition. \(\square\)

Note 1. The above result was also simultaneously proved in an asymmetric metric space under certain additional conditional conditions [17].

3. \(I\)-divergence and \(I^*\)-divergence

The concept of divergent sequences of real numbers was generalized to statistically divergent sequences of real numbers by Macaj and Salat in [8]. Though the idea of statistical convergence was later further generalized to \(I\)-convergence and \(I^*\)-convergence (as already mentioned), no such approach has been made as far as divergence is concerned. In this section we precisely intend to do that. However instead of taking only real sequences, we introduce the concept of divergence in a metric space (note that our definition includes the general definition of real divergent sequences as a special case) and extend it with the help of ideals. Our investigation reveals that again the condition (AP) plays the same prominent role as in the case of \(I\)-convergence and \(I\)-Cauchy condition.

We first introduce the following definition.

Definition 9. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a metric space \((X, d)\) is said to be divergent (or properly divergent) if there exists an element \(x \in X\) such that \(d(x, x_n) \to \infty\) as \(n \to \infty\).

Note that a divergent sequence in a metric space cannot have any convergent subsequence.

Definition 10. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a metric space \((X, d)\) is said to be \(I\)-divergent if there exists an element \(x \in X\) such that for any positive real number \(G\), \(A(x, G) = \{n \in \mathbb{N} : d(x, x_n) \leq G\} \in I\).

Definition 11. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a metric space \((X, d)\) is said to be \(I^*\)-divergent if there exists \(M \in F(I)\) i.e., \(\mathbb{N} \setminus M \in I\) such that \(\{x_n\}_{n \in M}\) is divergent i.e., there exists at least one \(x \in X\) such that \(\lim_{n \to \infty} d(x, x_n) = \infty\).

Theorem 2. Let \(I\) be an admissible ideal. If \(\{x_n\}_{n \in \mathbb{N}}\) is \(I^*\)-divergent then \(\{x_n\}_{n \in \mathbb{N}}\) is \(I\)-divergent.

Proof. Since \(\{x_n\}_{n \in \mathbb{N}}\) is \(I^*\)-divergent so there exists \(M \in F(I)\) i.e., \(\mathbb{N} \setminus M \in I\) such that \(\{x_n\}_{n \in M}\) is divergent i.e., there exists at least one \(x \in X\) such that \(\lim_{n \to \infty} d(x, x_n) = \infty\). Then for any positive real number \(G\), there exists \(m \in \mathbb{N}\) such that \(d(x, x_n) > G\) for all \(k \geq m\) and \(k \in M\). Hence we have \(\{n \in \mathbb{N} : d(x, x_n) \leq G\} \subseteq (\mathbb{N} \setminus M) \cup \{1, 2, 3, \ldots, m\} \in I\). This implies that \(\{x_n\}_{n \in \mathbb{N}}\) is \(I\)-divergent. \(\square\)
The following example shows that the converse of the above theorem is not always true.

**Example 2.** Let \( N = \bigcup_{j \in N} \Delta_j \) be a decomposition of \( N \) such that each \( \Delta_j \) is infinite and \( \Delta_j \cap \Delta_i = \emptyset \) for \( i \neq j \). Let \( I \) be the class of all those subsets \( A \) of \( N \) that can intersect only finite number of \( \Delta_j \)'s. Then \( I \) is a non-trivial admissible ideal of \( N \).

Take \( \mathbb{R} \) with the usual metric \( d \). Let \( y_i = n \) if \( i \in \Delta_n \). Now for any positive real number \( G \) there exists a natural number \( m \) such that \( G < m \) which implies that \( \{n \in N : d(0, y_i) \leq G\} \subset \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \cdots \cup \Delta_m \subset I \) and so \( \{y_i\}_{i \in \mathbb{N}} \) is \( I \)-divergent.

Next we shall show that \( \{y_i\}_{i \in \mathbb{N}} \) is not \( I^* \)-divergent. If possible assume that \( \{y_i\}_{i \in \mathbb{N}} \) is \( I^* \)-divergent. Then there is a \( M \in F(I) \) such that \( \{y_i\}_{i \in \mathbb{N}} \) is divergent. Since \( N \setminus M \in I \) so there exists a \( i \in N \) such that \( N \setminus M \subset \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_i \). But then \( \Delta_i \subset M \) for all \( i > 1 \). In particular \( \Delta_{i+1} \subset M \). But this implies that \( \{y_i\}_{i \in \Delta_{i+1}} \) is a constant subsequence of \( \{y_i\}_{i \in \mathbb{N}} \) which is convergent to \( I + 1 \). This contradicts the fact that \( \{y_i\}_{i \in \mathbb{N}} \) is divergent to \( +\infty \).

**Theorem 3.** If \( I \) is an admissible ideal with property \((AP)\) then for any sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \), \( I \)-divergence implies \( I^* \)-divergence.

**Proof.** First suppose that \( I \) satisfies the condition \((AP)\). Since \( \{x_n\}_{n \in \mathbb{N}} \) is \( I \)-divergent so there exists at least one \( x \in X \) such that for any positive real number \( G \), \( A(x, G) = \{n \in N : d(x, x_n) \leq G\} \subset I \). Let \( A_1 = \{n \in N : d(x, x_n) \leq 1\}, A_2 = \{n \in N : 1 < d(x, x_n) \leq 2\}, \ldots, A_k = \{n \in N : k - 1 < d(x, x_n) \leq k\} \) for all \( k \geq 2 \). Thus we get a collection of mutually disjoint sets \( \{A_i\}_{i \in N} \) such that \( \Delta_i \cap A_i = \emptyset \) for all \( i \in N \). By the condition \((AP)\) there exists a family of sets \( \{B_i\}_{i \in N} \) such that \( \Delta_i \setminus B_i = \emptyset \) for any \( i \). Let \( \Delta_i = B_i \cap I \). Then \( M = N \setminus B \). Let \( x \in F(I) \). Let \( G > 0 \) be any real number and choose \( k \in N \) such that \( G < k \). Then \( \{n \in N : d(x, x_n) \leq G\} \subset A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \). Since \( \Delta_i \setminus B_i \) is finite, so there exists \( n_0 \in N \) such that

\[
\left( \bigcup_{i=1}^{k} B_i \right) \cap \{n \in N : n \geq n_0\} = \left( \bigcup_{i=1}^{k} A_i \right) \cap \{n \in N : n \geq n_0\}.
\]

Clearly if \( n \geq n_0 \) and \( m \in M \) then \( n \not\in \bigcup_{i=1}^{k} B_i \Rightarrow n \not\in \bigcup_{i=1}^{k} A_i \). Therefore \( d(x, x_n) > k > G \). Hence \( \{x_n\}_{n \in \mathbb{N}} \) is divergent. \( \square \)

**Theorem 4.** Let \( (X, d) \) be a metric space containing at least one divergent sequence and let \( I \) be an admissible ideal. If for every sequence \( \{x_n\}_{n \in \mathbb{N}} \), \( I \)-divergence implies \( I^* \)-divergence then \( I \) satisfies the condition \((AP)\).

**Proof.** Let \( \{x_n\}_{n \in \mathbb{N}} \) be a divergent sequence in \( X \). Then there exists an element \( x \in X \) such that \( \lim_{n \to \infty} d(x, x_n) = +\infty \). Suppose \( \{A_i : i = 1, 2, 3, \ldots\} \) is a sequence of mutually disjoint non-empty sets from \( I \). Define a sequence \( \{y_n\}_{n \in \mathbb{N}} \) by \( y_n = x_j \) if \( n \in A_j \) and \( y_n = x_n \) if \( n \not\in A_j \) for any \( j \in N \). Let \( G > 0 \) be any real number. Choose \( k \in N \) such that \( d(x, x_n) \geq G \) for all \( n \geq k \). Now \( A(x, G) = \{n \in N : d(x, x_n) \leq G\} \subset A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup \{1, 2, 3, \ldots, k\} \subset I \). So \( \{y_n\}_{n \in \mathbb{N}} \) is \( I \)-divergent. By our assumption \( \{y_n\}_{n \in \mathbb{N}} \) is \( I^* \)-divergent. So there exists \( M \subset N \) such that \( \Delta_M \subset F(I) \) and \( \{y_n\}_{n \in \mathbb{N}} \) is divergent. Let \( B = N \setminus M \). Then \( B \in I \). Put \( B_i = A \cap B \) for all \( j \in N \). Since \( \bigcup_{i} B_i \subset B \), \( \bigcup_{i} B_i \subset I \). Let \( j \in N \). We claim that \( A_i \cap B = \emptyset \) is a finite set. If not then \( M \) must contain an infinite sequence of elements \( y_{m_k} = x_k \) for all \( k \in N \) which forms a convergent subsequence of \( \{y_n\}_{n \in \mathbb{N}} \). But this contradicts the fact that \( \{y_n\}_{n \in \mathbb{N}} \) is divergent. Hence \( \Delta_i \setminus B_i = A_i \setminus B = A_i \setminus B \) is included in a finite subset of \( N \). This proves that \( I \) satisfies the condition \((AP)\). \( \square \)

**References**


