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FORCING WITH TREES AND ORDINAL DEFINABILITY

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Introduction

It is shown that for every ordinal number Θ it is consistent that there is a strictly descending transfinite sequence of models of set theory such that for every $\alpha < \Theta$,

$$\mathfrak{M}_{\alpha+1} = (\mathrm{HOD})^{\mathfrak{M}_\alpha} ,$$

$$\mathfrak{M}_\alpha = \bigcap_{\beta<\alpha} \mathfrak{M}_\beta ,$$

if α is a limit ordinal.

A set is *ordinal definable* if it is definable by a formula with parameters ranging over ordinal numbers, and is *hereditarily ordinal definable* if, in addition, all its elements, elements of its elements, etc., are ordinal definable. The class HOD of all hereditarily ordinal definable sets is a model of ZFC.

Not all sets in HOD are necessarily ordinal definable in the model HOD. Thus the class $\mathrm{HOD}^2 = (\mathrm{HOD})^{\mathrm{HOD}}$, again a model of ZFC, may be strictly included in HOD; similarly HOD^n, for $n = 1, 2, 3, \ldots$. One may or may not be able to define the *sequence* HOD^n, $n = 1, 2, 3, \ldots$. If one can do so, then the intersection $\bigcap_{n<\omega} \mathrm{HOD}^n$ is a model of ZF; call it HOD^ω. In this fashion, we may be able to proceed and define the transfinite sequence

$$\mathrm{HOD}^\alpha, \quad \alpha \text{ an ordinal} .$$

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McAloon in his thesis [7] constructed models of ZFC in which HOD $\neq$ L. His method can easily be adopted to get an example of a finite descending sequence $V \supset \mathrm{HOD} \supset ... \supset \mathrm{HOD}^n = L$. More recently [8], McAloon constructed a descending ω-sequence of models $\mathrm{HOD}^0 \supset \mathrm{HOD}^1 \supset ... \supset \mathrm{HOD}^n \supset ...$, $n < \omega$, using the method of [7], showing that AC might hold or fail in the limit model.

The main result of the present paper is the following.

Theorem 1. *Given an ordinal Θ, there is a model $\mathfrak{M}$ of* ZFC *such that for every $\alpha < \Theta$, $\mathfrak{M} \models \mathrm{HOD}^{\alpha+1} \neq \mathrm{HOD}^{\alpha}$. In $\mathfrak{M}$, the transfinite sequence*

$$\mathrm{HOD}^0 = \mathfrak{M}, \mathrm{HOD}^1 = \mathrm{HOD}, ..., \mathrm{HOD}^{\alpha}, ..., \quad \alpha < \Theta ,$$

is strictly decreasing.

The model $\mathfrak{M}$ in Theorem 1 is a generic extension of the constructible universe. The basic idea is to add generic branches to trees in L. The result is obtained by the construction of trees in L that have suitable automorphism properties.

In addition to the Main Theorem, we use the same method to give another example of nonabsoluteness of the notion of ordinal definability.

Theorem 2. *There are models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with the same cardinals as* L *such that* $L \subset \mathfrak{M}_1 \subset \mathfrak{M}_2$, *and* $(\mathrm{HOD})^{\mathfrak{M}_2} = L$ *whereas* $(\mathrm{HOD})^{\mathfrak{M}_1} = \mathfrak{M}_1$.

Finally, as a further application of the present methods, we construct a model of set theory whose degrees of constructibility have order type $1 + \omega^*$.

Theorem 3. *There is a model $\mathfrak{M} = L[G]$ and a sequence $\{G_n : n < \omega\} \in \mathfrak{M}$ such that $G_0 = G$, $G_{n+1} \in L[G_n]$ and $G_n \notin L[G_{n+1}]$, for all n, and for every set of ordinals $X \in \mathfrak{M}$ either $X \in L$ or there is n such that $X \in L[G_n]$ and $G_n \in L[X]$.*

1. Ordinal definable sets and models HOD$^\alpha$

A set X is *ordinal definable* if it is definable by a formula $\phi(x, p_1, ..., p_n)$ with ordinal parameters, i.e.

$$X = \{x : \Phi(x, \alpha_1, ..., \alpha_n)\}$$

for some ordinals $\alpha_1, ..., \alpha_n$. The notion of ordinal definability was suggested by Gödel in [2], as a natural notion to be used for the construction of models of set theory. Subsequently, several people have given a formally correct (i.e., expressible in the language of set theory) definition of ordinal definability and investigated the model HOD of hereditarily ordinal definable sets. The following definition of ordinal definable sets is due to Vopěnka:

$$\mathrm{OD} = \text{closure } \{V_\alpha : \alpha < \infty\}$$

$$= \bigcup_{\alpha < \infty} \text{closure } \{V_\xi : \xi < \alpha\} ,$$

where "closure" means the closure under Gödel operations, $\infty = \mathrm{On}$ = the class of all ordinals, and V_α = the set of all sets of rank $< \alpha$.

The class of all hereditarily ordinal definable sets is defined as follows:

$$\mathrm{HOD} = \{x \in \mathrm{OD} : x \subseteq \mathrm{HOD}\}$$

$$= \{x : \text{transitive closure of } \{x\} \subseteq \mathrm{OD}\} .$$

It is easily verified that HOD is a model of ZF: it is closed under Gödel operations, transitive, and $V_\alpha \cap \mathrm{HOD}$ is definable from α, thus $\in \mathrm{HOD}$. Moreover, HOD satisfies the axiom of choice, since the class HOD has a definable well ordering.

Let us denote $\mathrm{HOD}^1 = \mathrm{HOD}$ and consider the relativization of the definition of HOD inside the model HOD,

$$\mathrm{HOD}^2 = (\mathrm{HOD})^{\mathrm{HOD}} .$$

HOD^2 is again a model of ZFC, and may be strictly included in HOD^1. This suggests the following "definition by induction":

$$\mathrm{HOD}^{\alpha+1} = (\mathrm{HOD})^{\mathrm{HOD}^\alpha} ,$$

$$\mathrm{HOD}^\lambda = \bigcap_{\alpha<\lambda} \mathrm{HOD}^\alpha , \quad \lambda \text{ a limit ordinal} .$$

If we can express

$$x \in \mathrm{HOD}^\alpha$$

in the language of set theory, then HOD^α, $\alpha < \infty$, is a transfinite sequence of models. Each $\mathrm{HOD}^{\alpha+1}$ is a model of ZFC, and each HOD^λ, for a limit λ, is a model of ZF: it is transitive, closed under Gödel operations, and $V_\alpha \cap \mathrm{HOD}^\lambda$ is definable from α and λ in each HOD^ξ, $\xi < \lambda$; thus $V_\alpha \cap \mathrm{HOD}^\lambda \in \mathrm{HOD}^\lambda$.

In general, even the sequence HOD^n, $n < \omega$, may not be definable in the language of set theory, since the definition of HOD^2 is more complicated than that of HOD, etc. In [3], Grigorieff has shown that $x \in \mathrm{HOD}^\alpha$ is expressible if we assume that $V = \mathrm{L}[X]$, where X is a set. We note that our proof of Theorem 1 involves a model in which $V = \mathrm{L}[X]$; moreover, in our case each HOD^α is a model of ZFC (including limit α).

We prove the possibility of an arbitrarily long strictly descending sequence

$$\mathrm{HOD}^\alpha, \qquad \alpha < \Theta .$$

[In an earlier version of this paper, I have mistakenly claimed a transfinitely long sequence HOD^α, α an ordinal.]

This should not be very surprising since it follows from earlier results of McAloon [7], and several later refinements, that the model HOD is not a very natural model of set theory (without additional assumptions). As another evidence of that we shall give an example of a model $\mathfrak{N} \supset \mathrm{L}$, with the same cardinals as L, whose only ordinally definable sets are constructible, but it has a submodel $\mathfrak{M}$, $\mathrm{L} \subset \mathfrak{M} \subset \mathfrak{N}$, all of whose sets are ordinally definable.

2. Ordinal definable sets in generic extensions of L

The proof of Theorem 1 involves a construction of a generic extension of the constructible universe. We will use a correspondence between ordinal definability in generic extensions and automorphism properties of the corresponding complete Boolean algebra. In the present section we recall the (well known) theorems characterizing ordinal definability in generic extensions and formulate some lemmas that we shall subsequently use.

It is convenient to describe the correspondence in a somewhat more general setting. Let $\mathfrak{M}$ be an inner model of set theory. A set X is $\mathfrak{M}$-definable if it is definable by some formula ϕ with parameters in $\mathfrak{M}$,

$$X = \{x \colon \phi(x, p_1, \ldots, p_n)\}$$

for some $p_1, \ldots, p_n \in \mathfrak{M}$.

The notion of $\mathfrak{M}$-definability is, as in the case of ordinal definability, expressible in the language of set theory. Note that "L-definable" coincides with OD. Also, we can define the class $\mathrm{HD}\mathfrak{M}$ of hereditarily $\mathfrak{M}$-definable sets.

Now let $\mathfrak{M}$ be a model of ZFC, and let B be a complete Boolean algebra in $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic ultrafilter on B. The HD$\mathfrak{M}$ sets of $\mathfrak{M}[G]$ are characterized in the following theorem of Vopěnka (for a proof, see [9] or [3]).

Let (in $\mathfrak{M}$, of course)

$$\mathrm{B}^* = \{b \in \mathrm{B}\colon b \text{ is fixed under all automorphisms of B}\}.$$

Theorem (Vopěnka).

$$\mathrm{HD}\mathfrak{M}^{\mathfrak{M}[G]} = \mathfrak{M}[G \cap \mathrm{B}^*].$$

A complete Boolean algebra B is *rigid* if it has no nontrivial automorphisms. B is *homogeneous* if for all nonzero $b, c \in \mathrm{B}$ there exists an automorphism π of B such that $\pi b \cdot c \neq 0$.

Corollary 1. *If* B *is rigid, then* $\mathrm{HD}\mathfrak{M}^{\mathfrak{M}[G]} = \mathfrak{M}[G]$.

Corollary 2. *If* B *is homogeneous, then* $\mathrm{HD}\mathfrak{M}^{\mathfrak{M}[G]} = \mathfrak{M}$.

In our proof we shall not apply Vopěnka's Theorem directly but rather use the following two lemmas that say more or less the same as the theorem.

If b is a nonzero element of B then B_b denotes the complete Boolean algebra $\{u \in \mathrm{B}\colon u \leqslant b\}$ endowed with the induced Boolean operations.

Lemma 2.1. *Let G_1 and G_2 be two generic ultrafilters on* B *and let $\mathfrak{M}[G_1] = \mathfrak{M}[G_2]$. Then there is an automorphism π of* B *such that $\pi'' G_1 = G_2$.*

Lemma 2.2. *Let $A \in \mathfrak{M}[G]$ be an $\mathfrak{M}$-definable set of ordinals, let $\underset{\sim}{A}$ be its name. Then there is $p \in G$ such that for every automorphism π of B_p, every $q \leqslant p$ and every ordinal α we have*

$$q \Vdash \alpha \in \underset{\sim}{A} \quad \mathit{iff} \quad \pi q \Vdash \alpha \in \underset{\sim}{A}.$$

Lemma 2.1 is due to Vopěnka; for its proof, consult [9] or [3].

Proof of Lemma 2.2. There is $p \in G$ such that for some formula ϕ with parameters $p_1, ..., p_n \in \mathfrak{M}$, $p \Vdash \underset{\sim}{A}$ is the unique set s.t. $\phi(\underset{\sim}{A}, p_1, ..., p_n)$. Then if $\pi p = p$, we have $\pi\underset{\sim}{A} = \underset{\sim}{A}$ and the lemma follows.

3. Forcing with trees and construction of trees in L

The main tool we shall use in the proof of Theorem 1 is the Suslin tree, and its generalization, the κ-Suslin tree. It will be these trees, with certain automorphism properties, that will provide us with complete Boolean algebras that we shall use to construct a generic extension of L.

Jensen [6] has shown that Suslin trees exist in L, and, moreover, that we may require further automorphism properties, e.g. that the tree is rigid. In this section we recall the construction of such trees in L, and describe the complete Boolean algebras associated with such trees.

A *tree* is a partially ordered set such that the predecessors of any element are well ordered. For further terminology and notation we refer the reader to [4]. We shall be dealing with κ^+-trees, where κ is a regular cardinal. We give the construction only in case $\kappa = \omega$, for the general case is a straightforward generalization. We assume that all trees we deal with are *normal,* i.e., (i) every branch of limit length has at most one immediate successor, (ii) every element has κ immediate successors, and (iii) every element has κ^+ successors. Moreover, in case $\kappa > \omega$ we assume that every branch of cofinality $< \kappa$ has a successor.

A normal κ^+-tree is *Suslin* if it has no antichain of cardinality κ^+.

The well-known construction of a Suslin tree in L uses Jensen's principle

($\diamond$) There exists a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ such that for every $X \subseteq \omega_1$, the set $\{\alpha : X \cap \alpha = S_\alpha\}$ is stationary.

[If $\kappa > \omega$, replace ω_1 by κ^+; the set $\{\alpha : \mathrm{cf}\,\alpha = \kappa \text{ and } X \cap \alpha = S_\alpha\}$ is stationary.]

Sketch of the construction: T is constructed by induction, level by level. The only problem is how to construct limit levels T_α. [If $\kappa > \omega$, only limit levels of cofinality κ.] For every x we pick an α-branch b_x through x. In the good case we extend all b_x's. In the bad case, if S_α is a maximal antichain then we extend only those b_x's that go through S_α. Since S_α is maximal, we preserve normality of T. An argument using ($\diamond$) then shows that every antichain in T is at most countable [at most of cardinality κ].

If we want T to be rigid, then we destroy all potential automorphisms in the same fashion. If $\pi = S_\alpha$ is a nontrivial automorphism of $T \restriction \alpha$, then we pick a branch b_x such that $\pi'' b_x \neq b_x$ and extend it. Then we extend only those b_y's that are distinct from $\pi'' b_x$. Then π does not extend to an automorphism of $T \restriction \alpha + 1$ and a ($\diamond$)-argument completes the proof.

We will use a variation of this argument to control definability in models constructed by forcing with trees.

If T is a normal κ^+-tree then we may consider it as a set of forcing conditions, with inverse ordering. That is, x is a stronger condition than y just in case $x \geqslant y$. It is obvious that if $G \subset T$ is generic then G is a κ^+-branch through T. In case that T is a Suslin tree, then if b is a κ^+-branch through T and b is not in the ground model, then b is generic. Hence if T is a Suslin tree, then generic sets and branches coincide. Moreover, T satisfies the κ^+-chain condition.

The complete Boolean algebra associated with T consists of subsets of T. In case of a Suslin tree, the description of B is particularly simple: we can represent each $u \in$ B as $u = \Sigma\{x: x \in U\}$ where $U \subset T$ is a set of elements of the same level of T (for details, see [5]).

Also, if T is Suslin, then there is a nice representation of automorphisms of the Boolean algebra. Obviously, every automorphism of T induces a unique automorphism of B; but the inverse direction is more interesting.

Let C be a closed unbounded subset of κ^+. By T^C we denote the set of all $x \in T$ lying on levels $\alpha \in C$. T^C is a normal Suslin tree and it is easy to see that T^C is dense in T, hence in B. Thus every automorphism of T^C induces a unique automorphism of B. On the other hand, we have

Lemma 3.1. *If π is an automorphism of* B, *then there exists a closed unbounded set C such that $\pi \restriction T^C$ is an automorphism of T^C.*

Proof. Let

$$C = \{\alpha\colon \text{ for every } x \text{ on the } \alpha^{\text{th}} \text{ level of } T,\ \pi x \text{ is an element of the } \alpha^{\text{th}} \text{ level of } T\}.$$

All we have to do is to show that C is closed and unbounded. It is easy to verify that C is closed; we have to show that C is unbounded, or, what is equally difficult (or easy) that C is nonempty. Given β, consider

$$\{\pi x\colon x \in T \restriction \beta\} \cup \{\pi^{-1}x\colon x \in T \restriction \beta\} \subseteq \mathrm{B}\ ;$$

we can represent all these elements of B by subsets of T, and indeed, by subsets of some $T \restriction \alpha_1$, $\alpha_1 > \beta$. In a similar way we find $\alpha_2 > \alpha_1, \alpha_3, \alpha_4, \ldots$ and then let $\alpha = \lim_{n\to\omega} \alpha_n$. The rest is easy: α is a member of C.

This representation of automorphisms of B is very useful; especially so because an obvious modification of the argument above allows us to destroy potential automorphisms of B in the course of construction of T (compare with the construction of a simple complete Boolean algebra in [5]).

In view of Lemma 2.1 we have the following useful lemma about forcing with Suslin trees:

Lemma 3.2. *Let T be a Suslin tree. Let b_1 and b_2 be generic branches through T such that $\mathfrak{M}[b_1] = \mathfrak{M}[b_2]$. Then there exists a closed unbounded set C and an automorphism π of T^C such that $\pi'' b_1 = b_2$. (Strictly speaking: $\pi'' b_1 = b_2 \upharpoonright C$.)*

4. A descending sequence of HOD^n, $n < \omega$

As a first step towards Theorem 1, we shall prove the following:

Theorem $\frac{1}{2}$ (V = L). *There exists a complete Boolean algebra* B *such that the Boolean valued model* L^{B} *satisfies the following: The sequence*

$$\mathrm{HOD}, \mathrm{HOD}^2, \ldots, \mathrm{HOD}^n, \ldots, \qquad n < \omega$$

is strictly descending.

We assume V = L (in the ground model) from now on. In this section, we construct a Suslin ω_1-tree whose corresponding complete Boolean algebra (cBa) satisfies the statement of Theorem $\frac{1}{2}$.

Before giving the construction, we consider the following situation which is crucial in the subsequent considerations. If T is a tree, and $x \in T$, we let T_x denote the tree consisting of all $y \geqslant x$; $o(x)$ denotes the order type of $\{y : y < x\}$ (thus x is on the $o(x)^{\text{th}}$ level).

Let T^0, T^1 be normal ω_1-trees. Consider the following situation.

(I) We have a projection $h : T^0 \xrightarrow{\text{onto}} T^1$ such that

(1) $o(hx) = o(x)$,

(2) $x < y \rightarrow hx < hy$,

(3) if $x \in T^0$ and $y > x$, then there is $z > x$, $z \neq y$ such that $hz = hy$.

(II) T^0 is Suslin (hence T^1 is Suslin).

(III) If $hx = hy$, then T^0_x and T^0_y are isomorphic.

(IV) If $o(x) = o(y)$ and $hx \neq hy$, then there is no closed unbounded set C and no automorphism π of $(T^0)^C$ such that $\pi'' T^0_x \cap T^0_y \neq \emptyset$.

(Note that (IV) states that then there is no automorphism π of the complete Boolean algebra such that $\pi x \cdot y \neq 0$.)

Let B^0, B^1 be the complete Boolean algebras associated with the Suslin trees T^0, T^1. The projection $h : T^0 \to T^1$ induces a complete embedding of B^1 into B^0: if $u \in \mathrm{B}^1$, then $u \subset T^1$ and we let $e(u) = h_{-1}[u]$. It is easily verified that e is a complete one-to-one homomorphism. Thus B^1 can be considered a complete subalgebra of B^0.

We note in passing that not every complete subalgebra of B^0 can be obtained this way. The algebra B^1 has the following property: For $x \in T^0$, denote $\bar{x} = \Pi\{u \in \mathrm{B}^1 : x \leqslant u\}$. If $x \in T^0$ is on a limit level and if $x_1 < x_2 < ... < x_n < ...$ are such that $\lim x_n = x$, then $\bar{x} = \Pi \bar{x}_n$, namely $\bar{x} = \{y : hx = hy\}$. [In general, one can only prove $\bar{x} \leqslant \Pi \bar{x}_n$.]

Lemma 4.1. *Let* T^0, T^1 *satisfy* (I)–(IV). *Let* b_0 *be a generic branch through* T^0, *let* $b_1 = h'' b_0$. *Then*

(a) $\mathrm{L}[b_0] \models \mathrm{HOD} = \mathrm{L}[b_1]$,

(b) $b_0 \notin \mathrm{L}[b_1]$.

Proof. Let b_0 be a branch through T^0. We shall prove first that b_1 is definable in $\mathrm{L}[b_0]$. Namely, b_1 is the unique element of the set

$$\{h'' b : b \text{ is a branch through } T^0 \text{ s.t. } \mathrm{L}[b] = V\} .$$

To verify this, let b be a branch through T^0, and let $\mathrm{L}[b] = \mathrm{L}[b_0]$. By Lemma 3.2, there is a closed unbounded set C and an automorphism π of $(T^0)^C$ such that $\pi'' b_0 = b$. Let us assume that $h'' b \neq b_1 = h'' b_0$. Then there are $x \in b_0$ and $y \in b$ such that $o(x) = o(y)$ and $hx \neq hy$. However, the existence of π contradicts (IV) because $\pi x' = y'$ for some $x' \geqslant x$ and $y' \geqslant y$.

Next we prove that if A is a set of ordinals ordinal definable in $\mathrm{L}[b_0]$, then $A \in \mathrm{L}[b_1]$. That proves (a). Let $\underset{\sim}{A}$ be a name of A. By Lemma 2.2, pick $x \in b_0$ such that every automorphism of B_x preserves the forcing relation $y \Vdash \alpha \in \underset{\sim}{A}$, for $y \geqslant x$. We will construct a new name $\underset{\sim}{A}'$ for A, such that $A' \in \mathrm{L}^{\mathrm{B}^1}$ (where B^1 corresponds to T^1). For every α, let $u_\alpha = [\![\alpha \in \underset{\sim}{A}]\!]$ be a subset of some level $> \alpha(x)$. If $u_\alpha \cap T_x = \emptyset$, we let $v_\alpha = \emptyset$. Otherwise, look at all $y \in u_\alpha \cap T_x$, and let

$$v_\alpha = \{z : hz = hy \text{ for some such } y\} .$$

It is easy to see that each v_α belongs to B^1; thus we define $\underset{\sim}{A}'$ by $[\![\alpha \in \underset{\sim}{A}']\!] = v_\alpha$, for each α. We have only to verify that $\underset{\sim}{A}'$ is a name for A.

If $\alpha \in A$, then there is some $y \geq x$ such that $y \in u_\alpha$, and hence $y \Vdash \alpha \in \underset{\sim}{A}'$. On the other hand, if $\alpha \in A'$, then $b_0 \cap v_\alpha \neq \emptyset$; let $z \in b_0 \cap v_\alpha$. Since $z \in b_0$, we have $z \geq x$; also, $h\,z = h\,y$ for some $y \geq x$ such that $y \Vdash \alpha \in \underset{\sim}{A}$. By (III), T_y^0 and T_z^0 are isomorphic, and so one can get an automorphism π of B_x such that $\pi y = z$. By the choice of x this means that $z \Vdash \alpha \in \underset{\sim}{A}$ also, and hence $\alpha \in A$.

Finally, we show that $\underset{\sim}{b}_0$ is not definable in $\mathrm{L}[b_0]$, and hence $b_0 \notin \mathrm{L}[b_1]$. Here we argue in the same way as above. First find $x \in b_0$ by Lemma 2.2. Then choose arbitrary distinct $y, z \in T_x^0$ such that $y \in b_0$ and $h\,z = h\,y$ (by (I)(3)). Then by (III) get an automorphism π of B_x such that $\pi y = z$ and by Lemma 2.2 argue that also $z \in b_0$; a contradiction.

As the next step we will show that there are T^0, T^1 and h that satisfy (I)–(IV).

Lemma 4.2 (V = L). *There exist ω_1-trees T^0, T^1 and a homomorphism h such that T^0, T^1 and h satisfy* (I)–(IV).

Proof. Actually, ($\Diamond$) is the only property of L we use in the construction. Our construction follows the outline of the construction of a Suslin tree in Section 3.

We construct T^0 and h by induction, level by level (and $T^1 = h''\,T^0$). In the course of the construction, we keep the following conditions satisfied:

(i) For each $x \in T^0$, for each $\alpha > o(x)$, there is a $y > x$ on the α^{th} level.

(ii) Every $x \in T^0$ has infinitely many immediate successors; moreover, there is an infinite set Y of immediate successors of x such that $h\,y_1 \neq h\,y_2$ whenever $y_1, y_2 \in Y$ and $y_1 \neq y_2$.

(iii) If $x \in T^0$ and $y > x$, then there is $z > x$, $z \neq y$ such that $h\,z = h\,y$.

Notice that the conditions (i) and (ii) will guarantee that both T^0 and T^1 are normal trees. The condition (iii) is the same as (I)(3).

Along with h, we construct by induction a collection of isomorphisms π_{xy}: For each pair x, y such that $h\,x = h\,y$, π_{xy} is an isomorphism between T_x^0 and T_y^0. In addition to (i)–(iii), we keep the following condition satisfied:

(iv) For each pair x, y such that $h\,x = h\,y$, and each $u \in T_x$,

$$h(\pi_{xy}\,u) = h\,u\,.$$

Before we start, we fix a mapping $j : \omega \xrightarrow{\text{onto}} \omega$ such that $j^{-1}(n)$ is infinite, for all n. We construct the tree T^0 as a collection of functions $x : \alpha \to \omega$, $\alpha < \omega_1$, ordered by $\subseteq$ and closed under initial segments. The α^{th} level of T^0 then consists of x with domain α. Having constructed the α^{th} level, we construct the $(\alpha+1)^{\text{th}}$ level as follows: For each $x \in T^0$, $x : \alpha \to \omega$, we adjoin all $x^\frown n$, $n < \omega$. We let

$$h(x^\frown n) = (hx)^\frown (jn) \qquad (x : \alpha \to \omega, \quad x \in T^0, \quad n \in \omega),$$

$$\pi_{xy}(z^\frown n) = (\pi_{xy} z)^\frown n \qquad (x, y \in T^0 \restriction \alpha+1, \quad z : \alpha \to \omega)$$

(and $\pi_{xy}(x) = y$ for x, y on the $(\alpha+1)^{\text{th}}$ level, $hx = hy$).

One can verify that (i) and (ii) are satisfied, and that T^0 and T^1 stay being normal.

Now assume that α is a limit ordinal and that we have constructed T^0, h and π_{xy} below level α. We recall that at a limit stage of the construction of a Suslin tree one invokes $\diamond$ to make sure that the resulting tree is Suslin. Here we need a Suslin tree which, in addition, has the property (IV). Thus we distinguish 3 cases: when S_α (of the $\diamond$ sequence) codes a maximal antichain, when S_α codes an isomorphism to be destroyed, and the easy case (the "otherwise" case).

Case I. The easy case. Let G be the set of all finite compositions of the π_{uv}'s, for all $u, v \in T^0 \restriction \alpha$ such that $hu = hv$. For each $x \in T^0 \restriction \alpha$ pick an α-branch b_x. Extend all these b_x's together with all $\rho'' b_x$, $\rho \in G$. Since G is countable, this set of all extended branches is countable; moreover, it is closed under all π_{uv}'s. We define h and the π_{uv}'s in the obvious way, and the conditions (i)–(iv) are satisfied.

Case II. S_α codes a maximal antichain A in $T^0 \restriction \alpha$ and we wish to construct the α^{th} level such that A remains maximal in $T^0 \restriction (\alpha+1)$. We want to extend a branch through every $x \in T^0 \restriction \alpha$, have the α^{th} level closed under all π_{uv}'s and, moreover, we also want to have all the branches on the α^{th} level to go through the antichain. Thus for every $x \in T^0 \restriction \alpha$ we construct a branch b_x in such a way that each $\rho'' b_x$, $\rho \in G$, meets the antichain. This can be done since it involves satisfying only a countable number of conditions.

Case III. The S_α given by ($\diamond$) is an isomorphism π between $(T^0_x)^C$ and $(T^0_y)^C$, where C is a closed unbounded subset of α, and $hx \neq hy$. This π is to be destroyed, so that when the construction of T^0 is completed, the ($\diamond$) is recalled to show that T^0 satisfies (IV). We are going to extend branches b_z, for all $z \in T^0 \restriction \alpha$, but the branches will be chosen such that b_x is extended while $\pi'' b_x$ is not. Hence π will be destroyed as a potential isomorphism.

First pick b_x arbitrary. Notice that since $h(\pi_{uv} b) = h\, b$, we have $h(\rho b_x) = h\, b_x$ for all $\rho \in G$, while $h(\pi b_x) \neq h\, b_x$, because $h\, x \neq h(\pi x)$. Hence $\rho b_x \neq \pi b_x$ for all $\rho \in G$. We extend all ρb_x, $\rho \in G$.

Then for every $z \neq x$ we find, by diagonalization, a branch b_z such that $\rho b_x \neq \pi b_x$ for all $\rho \in G$. And we extend all ρb_z $(z \in T^0 \restriction \alpha, \rho \in G)$.

When the construction is completed, a standard $\Diamond$-argument shows that T^0 is a Suslin tree and that (IV) holds: Case II takes care of the antichains, and Case III takes care of potential isomorphisms. The projection h satisfies (I), and $T^1 = h'' T^0$ is a normal Suslin tree. The π_{uv}'s witness to (III), and so the trees T^0, T^1 satisfy (I)–(IV).

Now we are ready to prove Theorem $\frac{1}{2}$. In view of Lemma 4.1 it is obvious that it is sufficient to construct a sequence of trees with the following properties.

Lemma 4.3 (V = L). *There exists a sequence $T^0, T^1, \ldots, T^n, \ldots, n < \omega$, of normal Suslin ω_1-trees such that for every $n = 0, 1, \ldots$ there exists a projection $h_n : T^n \xrightarrow{\text{onto}} T^{n+1}$ which satisfies* (I)–(IV).

Proof. We construct all T^n at once, by induction on levels. As in Lemma 4.2, we construct also the h_n's, and all the π_{uv}'s (for $u, v \in T^n$ such that $h_n u = h_n v$); also, we keep (i)–(iv) satisfied.

The successor step is exactly as in Lemma 4.2. So let α be limit. We will construct a countable set B of α-branches through the tree $T^0 \restriction \alpha$ (B will serve as the α^{th} level of T^0) such that the following two conditions are satisfied:

(1) For each $x \in T^0 \restriction \alpha$ there is $b \in B$ that goes through x.

(2) If $b \in B$ and $u, v \in T^m$ are such that $h_m u = h_m v$ and if $(b)^m$ (the image of b in T^m, i.e., $(b)^m = h_{m-1} \ldots h_0 b$) goes through u, then there is $b_1 \in B$ such that $(b_1)^m = \pi_{uv}(b)^m$.

If (1) and (2) are satisfied, then, after defining h_n's and π_{uv}'s in the obvious way, the trees $T^n \restriction (\alpha + 1)$ satisfy the conditions (i)–(iv).

Having in mind that T^0 is supposed to be Suslin, and that the trees T^n should satisfy (IV), we construct B such that, in addition to (1) and (2), we take care of the S_α of the $\Diamond$-sequence. Thus the three cases.

Case I. The easy case. We fix an increasing sequence $\{\alpha_n : n < \omega\}$ such that $\alpha = \lim \alpha_n$. We further fix an enumeration $\{x_n : n < \omega\}$ of

$T^0 \restriction \alpha$. Finally, we enumerate all π_{uv}'s, where $u, v \in T^m$, $m < \omega$ and $h_m u = h_m v$: $\{\pi_n: n < \omega\}$.

We construct B in ω steps. After n steps, we will have constructed initial segments of finitely many branches in B. We construct $B = \{b_i: i < \omega\}$ as follows. At step n, we have initial segments of $b_0, ..., b_{k_n}$, and each initial segment has length at least α_n and has a maximal element. To make sure that B satisfies (1), we simply require, at each step n, that

(A) for each $j < n$ there is $i < k_n$ such that b_i goes through x_j.

To satisfy (2), let us enumerate all pairs (j, k). Let $b_l \simeq \pi_j b_k$ denote $(b_l)^m = \pi_j(b_k)^m$, where π_j is an isomorphism in the tree T^m. At step n, assuming that at previous steps we have assigned to each (j, k) among the first n pairs some l, we require that the initial segments of these b_k's and b_l's satisfy

(B) $b_l \simeq \pi_j b_k$.

Furthermore, if (j, k) is the next pair in the enumeration, we add one more (initial segment of) branch, b_l, to B_n, so that $b_l \simeq \pi_j b_k$ (and assign l to (j, k)).

Case II. S_α codes a maximal antichain A in $T^0 \restriction \alpha$. In addition to (1) and (2) we require that each $b \in B$ meets A. We proceed as in Case I, and have to make sure that every time we add a new branch to our collection, it has to go through A.

We are adding a new branch at step n either to satisfy (A), that is, to have a branch through every x_j, or to satisfy (B), to have b_l such that $b_l \simeq \pi_j b_k$. In both cases we make sure, by taking a sufficiently long initial segment, that the new branch meets the antichain.

Case III. S_α codes an isomorphism π between $(T_x^m)^C$ and $(T_y^m)^C$ such that $h_m x = h_m y$, and we wish to destroy π, in addition to satisfying (1) and (2).

Let $x_0 \in T^0$ be such that $x = h_{m-1} \dots h_0 x_0$. We will construct, as in Case I, a countable set of α-branches $B = \{b_i: i < \omega\}$ satisfying (1) and (2), and, moreover, such that b_0 goes through x_0, and for all $b \in B$, $(b)^m \neq \pi(b_0)^m$.

Thus we have to make sure that when adding a new branch b either to satisfy (A) or (B), we have $(b)^m \neq \pi(b_0)^m$ (once the initial segments are unequal they stay unequal). This is easy to arrange if a new branch is added to go through some x_j. If a new branch b_l is added in order to have $b_l \simeq \pi_j b_k$, then π_j is an isomorphism in T^r and we may have $r < m$, $r = m$ or $r > m$. If $r < m$, then $\pi_j(b_k)^m = (b_k)^m \neq \pi(b_0)^m$ and $(b_l)^m \neq \pi(b_0)^m$

is easily arranged. If $r = m$ and $k = 0$ then $\pi_j(b_0)^m \neq \pi(b_0)^m$ because $h_m(\pi_j(b_0)^m) = h_m(b_0)^m$ while $h_m(\pi(b_0)^m) \neq h_m(b_0)^m$. If $r = m$ and $k \neq 0$, then we have the choice of extending b_0 and b_k somewhat to have $\pi_j(b_k)^m \neq \pi(b_0)^m$. In any case, if $r = m$, then b_l can be obtained so that $(b_l)^m \neq \pi(b_0)^m$. The case $r > m$ is handled similarly.

When the construction is completed, a $\diamondsuit$-argument shows that T^0 is a Suslin tree and that (IV) holds for each T^m. This completes the proof of Lemma 4.3 and also the proof of Theorem $\frac{1}{2}$.

5. Proof of the Main Theorem

We will use the technique introduced in Section 4 to construct, for each Θ, a model of ZFC in which the sequence HOD^α, $\alpha < \Theta$, is strictly decreasing.

The construction in Section 4 involves projections of trees. We wish to get a κ-sequence of trees such that each successive tree is a projection of its predecessor and the tree on a limit stage is a limit of these projections. The construction will basically be the same as in Section 4 but we need a nice sequence of projections to work with (in place of the projection j in the proof of Lemma 4.2). Note that projections are the same as equivalence relations.

Lemma 5.1. *Let κ be a cardinal. There exists a sequence $\{\equiv_\alpha : \alpha < \kappa\}$ of equivalence relations on κ such that*

(0) *$\equiv_0$ is the identity,*

(1) *each $\equiv_\alpha$ has κ equivalence classes,*

(2) *each $\equiv_{\alpha+1}$ equivalence class is the union of κ $\equiv_\alpha$ equivalence classes,*

(3) *if λ is a limit ordinal, $\equiv_\lambda$ is the limit of $\equiv_\alpha$, $\alpha < \lambda$; i.e. $s \equiv_\lambda t$ iff $s \equiv_\alpha t$ for some $\alpha < \lambda$,*

(4) *$\lim_{\alpha \to \kappa} \equiv_\alpha$ is trivial, i.e. if $x, y \in \kappa$, then $x \equiv_\alpha y$ for some α.*

Proof. We identify κ with κ-sequences $s : \kappa \to \kappa$ such that $s_\alpha \neq 0$ for only finitely many $\alpha < \kappa$. We define, for each $\alpha < \kappa$,

$$s \equiv_\alpha t \Leftrightarrow s(\beta) = t(\beta) \text{ for all } \beta \geqslant \alpha.$$

It is easy to verify that the sequence $\equiv_\alpha$, $\alpha < \kappa$, satisfies (0)–(4).

The main problem we encounter in the generalization of Theorem $\frac{1}{2}$ is how to handle the limit steps, i.e., how to show that if λ is a limit ordinal, then the λ^{th} model is the intersection of the previous models. We will need the following lemma.

If B is a complete Boolean algebra and κ a cardinal, then B is *κ-distributive* if B satisfies the following distributivity law:

$$\prod_{\alpha<\kappa} \sum_{i\in I} u_{\alpha i} = \sum_{f\in I^\kappa} \prod_{\alpha<\kappa} u_{\alpha f(\alpha)} \, .$$

The significance of this property in forcing is that when forcing via B, one does not add any new κ-sequence of ordinals.

Lemma 5.2. *Let κ be a cardinal and let* B *be a κ-distributive complete Boolean algebra. Let*

$$\mathrm{B}_0, \mathrm{B}_1, \ldots, \mathrm{B}_\alpha, \ldots, \quad \alpha<\kappa \, ,$$

be a κ-sequence of complete Boolean algebras such that $\mathrm{B}_0 = \mathrm{B}$, $\mathrm{B}_{\alpha+1}$ is a complete subalgebra of B_α for each $\alpha<\kappa$, and $\mathrm{B}_\lambda = \bigcap_{\alpha<\lambda} \mathrm{B}_\alpha$ if λ is limit. Let all this be in a ground model $\mathfrak{M}$, let G be an $\mathfrak{M}$-generic ultrafilter on B *and let $G_\alpha = G \cap B_\alpha$ for all $\alpha \leqslant \kappa$. If A is a set of ordinals and $A \in \mathfrak{M}[G_\alpha]$ for all $\alpha<\kappa$, then $A \in \mathfrak{M}[G_\kappa]$.*

This lemma will be used to show that $\mathfrak{M}[G_\kappa] = \bigcap_{\alpha<\kappa} \mathfrak{M}[G_\alpha]$. This is not necessarily true in general; cf. [8].

Proof. Work in $\mathfrak{M}[G]$. Since $A \in \mathfrak{M}[G_\alpha]$ for all $\alpha<\kappa$, it has a B_α-valued name $\underset{\sim}{A}_\alpha$, for each α. Since B is κ-distributive, it follows that the sequence $\{\underset{\sim}{A}_\alpha : \alpha<\kappa\}$ is in $\mathfrak{M}$. Thus work in $\mathfrak{M}$. We are going to define a B_κ-valued name of A. Fix an ordinal ξ. For each $\alpha<\kappa$, let $u_\alpha = [\![\xi \in \underset{\sim}{A}_\alpha]\!] \in \mathrm{B}_\alpha$. By κ-distributivity, there exists a partition $\{p_i : i\in I\}$ of B such that for each $\alpha<\kappa$, each $i\in I$, either $p_i \leqslant u_\alpha$ or $p_i \cdot u_\alpha = 0$. Hence for each α, $u_\alpha = \Sigma\{p_i : i\in J\}$ for some $J \subseteq I$. Now we let

$$w = \sum \{p_i : \exists\, \alpha_0 \forall\, \alpha \geqslant \alpha_0 \; p_i \leqslant u_\alpha\} \, .$$

We will show that

(a) $w \in \mathrm{B}_\kappa$,

(b) $w \in G$ iff $\xi \in A$.

Thus, if we call w_ξ this w obtained for ξ, and let $\underset{\sim}{A}$ be such that

$[\![\xi \in \underset{\sim}{A}]\!] = w_\xi$, then $\underset{\sim}{A}$ is a B_κ-valued name for A. Thus doing this for each ξ, we get a B_κ-valued name for A.

To show that $w \in B_\kappa$, we show that $w \in B_\alpha$ for each $\alpha < \kappa$. Let $\alpha < \kappa$. It suffices to show that if $p_i \leqslant w$, then $\overline{p_i} \leqslant w$, where $\overline{p_i} = \Sigma\{p_j : j \in J\}$, where $J \subseteq I$ is the least J such that $i \in J$ and $\Sigma\{p_j : j \in J\} \in B_\alpha$. Let $p_i \leqslant w$. Then there is $\alpha_0 \geqslant \alpha$ such that $p_i \leqslant u_\beta$ for all $\beta \geqslant \alpha_0$. Let $u = \Pi\{u_\beta : \beta \geqslant \alpha_0\}$. Clearly $u \in B_\alpha$; also $u = \Sigma\{p_j : j \in J\}$ for some $J \subseteq I$. It follows that for each $j \in J$, $p_j \leqslant w$ and hence $u \leqslant w$. However, $\overline{p_i} \leqslant u$ and so $\overline{p_i} \leqslant w$.

To show that $w \in G$ iff $\xi \in A$, let p_i be the unique p_i which is in G. Then $w \in G$ iff $p_i \leqslant w$. If $\xi \in A$, then $u_\alpha \in G$ and $p_i \leqslant u_\alpha$ for each α, and hence $p_i \leqslant w$. On the other hand, if $p_i \leqslant w$, then $p_i \leqslant u_\alpha$ for eventually all α; hence $u_\alpha \in G$ and so $\xi \in A$.

Another problem at limit steps is the following: We want the α^{th} model to be given by a tree T^α. Thus we want , for a limit α, $\bigcap_{\gamma<\alpha} B_\gamma$ to be a cBa associated with a tree T^α, a "limit" of the trees T^γ, $\gamma < \alpha$. The following theorem (or rather its proof) suggests how to handle this problem.

Theorem $\frac{3}{4}$ (V = L). *There exists a sequence $T^0, T^1, ..., T^n, ..., n < \omega$, of normal Suslin ω_1-trees such that for every n there is a projection $h_n : T^n \xrightarrow{\text{onto}} T^{n+1}$ and if b_0 is a generic branch through T^0, $b_1 = h_0 b_0$, $b_2 = h_1 b_1$, etc., then*

(a) $L[b_n] \vDash \mathrm{HOD} = L[b_{n+1}]$,

(b) $b_n \notin L[b_{n+1}]$,

(c) $\bigcap_{n=0}^{\infty} L[b_n] = L$.

We can see that Theorem $\frac{3}{4}$ is just Lemma 4.3 with the clause (c) added. Thus the proof of Theorem $\frac{3}{4}$ is like the proof of Lemma 4.3 but extra care is needed to satisfy condition (c).

Let B_n be the cBa associated with T^n. Since each B_n is $\aleph_0$-distributive (a known property of normal Suslin trees when used to force with), Lemma 5.2 tells us that $\bigcap_{n=0}^{\infty} L[b_n]$ is given by the cBa $\bigcap_{n=0}^{\infty} B_n$. Thus we wish to construct the T^n's so that $\bigcap_{n=0}^{\infty} B_n$ is the trivial algebra.

Proof of Theorem $\frac{3}{4}$. Let us follow the proof of Lemma 4.3. We make a slight change of rotation, namely to call h_m the projection of T^0 onto T^m (what we called then $h_{m-1} \ldots h_0$). We construct T^n, $n < \omega$ to satisfy the conditions (I)–(IV) of Lemma 4.3 (with the notational change), and in addition, the condition

(V) if $x, y \in T^0$ are such that $o(x) = o(y)$, then there is k such that $h_k x = h_k y$.

A typical element of the algebra B_0 is a subset of a level of T^0. A typical element of B_m is a subset u of an α^{th} level of T^0 such that if $o(x) = o(y) = \alpha$, $x \in u$ and $h_m x = h_m y$, then $y \in u$. Hence if $u \in B_m$ for all $m < \omega$, it follows from (V) that $u = 1$ or $u = 0$, and so $\bigcap_{n=0}^{\infty} B_n$ is the trivial algebra. Thus a sequence of trees satisfying (I)–(V) is enough to prove Theorem $\frac{3}{4}$.

We will follow closely the proof of Lemma 4.3. Instead of the projection j (defined in the proof of Lemma 4.2), we use projections j_k, $k < \omega$, given by Lemma 5.1; the property we need is that for every n_1, n_2 there is k such that $j_k(n_1) = j_k(n_2)$.

In Lemma 5.1, let $\kappa = \aleph_0$ and let $\equiv_k$ be the k^{th} equivalence relation on ω ($k < \omega$). We define j_k by

$$j_k(n) = m\,,$$

where n is in the m^{th} equivalence class of $\equiv_k$.

We construct the trees T^k by induction on α. As in Lemma 4.3 (or 4.2), we construct also the h_k's, and the π_{uv}'s. We make sure that the conditions (i)–(iv) (with the notational change) are satisfied, and moreover,

(v) if $o(x) = o(y) = \alpha$, then there is k such that $h_k x = h_k y$.

As before, it is obvious (at limit steps) or easy to define the h_k's and the π_{uv}'s at each step. In particular, we define

$$h_k(z ^\frown n) = (h_k z) ^\frown (j_k n)$$

at successor steps, and

$$h_k(b) = \langle h_k(b(\gamma)) \colon \gamma < \alpha \rangle$$

at a limit step α.

At successor steps, the condition (V) remains satisfied: If $x = z ^\frown n$ and $y = w ^\frown m$, then first there is k_1 such that $h_k(z) = h_k(w)$ for all $k \geqslant k_1$ and second there is k_2 such that $j_k(n) = j_k(m)$ for all $k \geqslant k_2$. Thus for some k,

$$h_k(x) = (h_k z) ^\frown (j_k n) = (h_k w) ^\frown (j_k m) = h_k(y)\,.$$

At a limit step, following Lemma 4.3, it suffices to satisfy

(1) for each $x \in T^0 \restriction \alpha$ there is $b \in \mathrm{B}$ that goes through x,

(2) if $b \in \mathrm{B}$ and $u, v \in T^m$ are such that $h\,u = h\,v$, then there is $b_1 \in \mathrm{B}$ such that $h_m b_1 = \pi_{uv}(h_m b)$.

[Recall that B constitutes the α^{th} level of T^0; h in (2) is the projection of T^m onto T^{m+1}.] And, the additional condition

(3) if b_1 and b_2 are in B, then for some k, $h_k b_1 = h_k b_2$.

In addition to (1), (2), (3), we are also destroying an antichain or an isomorphism, if so required by $\diamond$. We shall only describe the easy case (Case I), since Case II (antichain) and Case III (isomorphism) are handled as in Lemma 4.3 with the additional requirement of (3).

Case I. We proceed as in the proof of Lemma 4.3. In addition, at step n we assume that we have assigned to each (r, s) among the first n pairs, some t, and require that the initial segments of these b_r's and b_s's satisfy

(c) $h_t b_r = b_s$.

Furthermore, if (r, s) is the next pair in the enumeration, we find $t < \omega$ such that $h_t b_r = b_s$ and add this condition to (c) to keep it satisfied at further steps.

Now we are ready to prove the Main Theorem.

Theorem 1 (V = L). *Let κ be a regular cardinal. There is a κ-sequence of normal Suslin κ^+-trees,*

$$T^\gamma, \qquad \gamma < \kappa\,,$$

such that the corresponding cBa's B_γ, $\gamma < \kappa$, *form a descending sequence* $\mathrm{B}_0 \supseteq \mathrm{B}_1 \supseteq \ldots \supseteq \mathrm{B}_\gamma \supseteq \ldots$, $\gamma < \kappa$, *and* $\mathrm{B}_\gamma = \bigcap_{\delta<\gamma} \mathrm{B}_\delta$ *if $\gamma < \kappa$ is a limit ordinal, and if $G \subseteq \mathrm{B}_0$ is generic, then*

$$\mathrm{L}[G] \models \mathrm{HOD}^\gamma = \mathrm{L}[G \cap \mathrm{B}_\gamma]\,,$$

and

$$\bigcap_{\gamma<\kappa} \mathrm{L}[G \cap \mathrm{B}_\gamma] = \mathrm{L}\,.$$

Proof. (Sketch). The proof of Theorem 1 is a more or less straightforward generalization of Theorem $\frac{3}{4}$, but with some caution. We shall first point out the pitfalls.

A Suslin κ^+-tree T is normal if, in addition to obvious generalization of properties of normal Suslin ω_1-trees, it has the property that for every limit α of cofinality $< \kappa$, every α-branch in T is extended. One constructs such trees with the aid of a $\diamond$-sequence $\{S_\alpha : \alpha < \kappa^+ \text{ and } \mathrm{cf}\,\alpha = \kappa\}$, and the induction step at limit α of cofinality $< \kappa$ is trivial: one extends all α-branches.

The corresponding cBa is κ-distributive, and so by Lemma 5.2, $\bigcap_{\delta<\gamma} \mathrm{L}[G \cap \mathrm{B}_\delta] = \mathrm{L}[G \cap \mathrm{B}_\gamma]$, if $\gamma \leqslant \kappa$ is limit. Thus it would be sufficient to construct a κ-sequence of trees T^γ, $\gamma < \kappa$, along with projections $h_\gamma : T^0 \xrightarrow{\text{onto}} T^\gamma$, with properties analogous to (I)–(V) of Theorem $\frac{3}{4}$. However, if $\gamma < \kappa$ is a limit ordinal, then $\mathrm{B}_\gamma = \bigcap_{\delta<\gamma} \mathrm{B}_\delta$ and so we would require that $h_\gamma x = h_\gamma y$ just in case $h_\delta x = h_\delta y$ for some $\delta < \gamma$. But let $\alpha = \mathrm{cf}\, \gamma$. We can easily find two α-branches b_1 and b_2 such that for every $\delta < \gamma$, $h_\delta '' b_1 \neq h_\delta '' b_2$ but $h_\gamma '' b_1 = h_\gamma '' b_2$. Since every α-branch has to be extended, this cannot be avoided.

Here we remark that this is because the algebra $\bigcap_{\delta<\gamma} \mathrm{B}_\delta$, although a complete subalgebra of B_0, cannot be obtained by means of a projection of T^0; see our remark preceding Lemma 4.1.

This difficulty can be avoided by considering the mappings h_γ not to be projections of T^0, but rather defined on a subtree of T^0. Let

$$\Omega = \{\alpha < \kappa^+ \colon \alpha \text{ is a successor or } \mathrm{cf}\, \alpha = \kappa\} .$$

We construct the trees T^γ, and the mappings h_γ, as follows:

I. h_γ is a projection of $(T^0)^\Omega$ onto $(T^\gamma)^\Omega$. For each $\gamma < \kappa$, let $h : (T^\gamma)^\Omega \to (T^{\gamma+1})^\Omega$ be the unique h such that $h_{\gamma+1} = h \circ h_\gamma$.

(1) $o(h_\gamma x) = o(x)$.

(2) $x < y \to h_\gamma x < h_\gamma y$.

(3) If $x \in T^\gamma$ and $y > x$, $o(x) \in \Omega$, then there is $z > x$, $z \neq y$, such that $h z = h y$.

(4) If $\gamma < \kappa$ is a limit ordinal, and $o(x) = o(y) \in \Omega$, then $h_\gamma x = h_\gamma y$ if and only if b for some $\delta < \gamma$, $h_\delta x = h_\delta y$.

II. T^0 is Suslin.

III. If $x, y \in (T^\gamma)^\Omega$ and $h x = h y$, then T^γ_x and T^γ_y are isomorphic.

IV. If $x, y \in (T^\gamma)^\Omega$, $o(x) = o(y)$, and $h x \neq h y$ then there is no closed unbounded set C and no automorphism π of $(T^0)^C$ such that $\pi '' T^\gamma_x \cap T^\gamma_y \neq \emptyset$.

V. If $o(x) = o(y) \in \Omega$, then there is $\gamma < \kappa$ such that $h_\gamma x = h_\gamma y$.

Let $T^\gamma, \gamma < \kappa$, be a sequence of normal κ^+-trees satisfying I – V. Let b_0 be a generic branch through T^0, and let $b_\gamma = h_\gamma(b_0)$. The same argument as in Lemma 4.1 shows that

$$\mathrm{L}[b_\gamma] \models \mathrm{HOD} = \mathrm{L}[b_{\gamma+1}]$$

and that $b_\gamma \notin \mathrm{L}[b_{\gamma+1}]$ for all $\gamma < \kappa$.

In view of V, the intersection $\bigcap_{\gamma<\kappa} \mathrm{B}_\gamma$ is the trivial algebra. Thus to prove Theorem 1, it suffices to show that for each limit ordinal $\lambda \leqslant \kappa$,

$$\bigcap_{\alpha<\lambda} L[b_\alpha] = L[b_\lambda]$$

(where $b_\kappa = \emptyset$).

We do that by induction on $\lambda \leqslant \kappa$. At stage λ, we already know that each $L[b_\alpha]$, $\alpha < \lambda$, is HCD^α in $L[b_0]$; thus the intersection is a model of ZF. To show that it is equal to $L[b_\lambda]$, it suffices to show that it has the same sets of ordinals (by a theorem of Vopěnka and Balcar). Obviously, $L[b_\lambda] \subset L[b_\alpha]$ for each $\alpha < \lambda$. Thus let A be a set of ordinals, $A \in L[b_\alpha]$ for each $\alpha < \lambda$. The algebra B_0 is κ-distributive, so that we may invoke Lemma 5.2, by which $A \in L[b_\lambda]$.

To construct a sequence T^γ, $\gamma < \kappa$, satisfying (I)–(V), we follow the constructions in Lemmas 4.2 and 4.3 and Theorem $\frac{3}{4}$. By induction on $\alpha < \kappa^+$, we construct the α^{th} level of all T^γ's, the mappings h_γ and the isomorphisms π_{uv}'s.

If α is limit and cf $\alpha < \kappa$, then we extend all branches, define π_{uv}'s in the obvious way, and do not care about the h_γ's.

If α is limit and cf $\alpha = \kappa$, then we construct a set of α-branches in $T^0 \upharpoonright \alpha$ in very much the same way as before, paying attention to $\diamondsuit$, and distinguishing 3 cases. The h_γ's and π_{uv}'s are defined in the obvious way. We also verify I (4).

If $\alpha \in \Omega$, then the h_γ's are defined for all $x \in T^0$ with $o(x) = \alpha$. To define the h_γ's on the next level, we use the continuous system of projections j_γ, $\gamma < \kappa$, given by Lemma 5.1. We let

$$h_\gamma(z^\frown i) = (h_\gamma z)^\frown (j_\gamma i)\,,$$

for all z with $o(z) = \alpha$ and $i < \kappa$. Since for every $i_1, i_2 < \kappa$ there is γ such that $j_\gamma i_1 = j_\gamma i_2$, this helps to satisfy V.

If α is limit and cf $\alpha < \kappa$, then we define the h_γ's on the $(\alpha + 1)^{\text{st}}$ level. We define the $h_\gamma(z^\frown i)$, where z is an α-branch and $i < \kappa$, by induction on γ. If γ is limit, then we define h_γ so as to satisfy condition I (4). Otherwise we define h_γ more or less arbitrarily, except for $h_{\gamma+1}$, where γ is limit; then, if $h_\gamma{}'' z_1 = h_\gamma{}'' z_2$, we let $h_{\gamma+1}(z_1{}^\frown i_1) = h_{\gamma+1}(z_2{}^\frown i_2)$, which guarantees that condition V is satisfied for elements of the $(\alpha + 1)^{\text{st}}$ level.

This sketch of the proof of Theorem 1 describes all the departures from a straightforward generalization of the constructions in Section 4 and Theorem $\frac{3}{4}$.

6. Proof of Theorem 2

In view of existing counterexamples, including the results above, it seems that one cannot expect any absoluteness of the notion of ordinal definability. The scope of HOD in one model does not seem to depend on the scope of HOD in another model. In view of the fact that there is a very strong relation between definability and automorphism properties in the models constructed above, the method presented in Section 4 is particularly suitable for getting all sorts of counterexamples. The combinatorial properties of L give us a very good control of automorphisms of trees.

As an example we sketch the proof of the following.

Theorem 2. *There exist models* $\mathfrak{M}_1$ *and* $\mathfrak{M}_2$ *such that*

(a) $\mathrm{L} \subsetneq \mathfrak{M}_2 \subsetneq \mathfrak{M}_1$ *and* L, $\mathfrak{M}_2$, $\mathfrak{M}_1$ *have the same cardinals,*

(b) $\mathfrak{M}_1 \models \mathrm{HOD} = \mathrm{L}$,

(c) $\mathfrak{M}_2 \models \mathrm{HOD} = \mathrm{V}$.

In view of the lemmas in Sections 3 and 4, it suffices to construct Suslin trees T^1 and T^2 such that T^1 is homogeneous and $(T^2)^C$ is rigid for every closed unbounded set C, and T_2 is a homomorphic image of T^1.

Thus we construct T^1 and T^2 along with $h : T^1 \to T^2$, make both trees Suslin and homogenize T^1 while keeping T^2 rigid. The construction is very much like in Lemma 4.2. By induction, we construct T^1, T^2, h, and π_{uv} for *any* $u, v \in T$ on the same level. We keep the following conditions satisfied:

(i) For each $x \in T^1$, each $\alpha > o(x)$, there are infinitely many $y > x$ on the α^{th} level with the same $h\,y$.

(ii) If $h\,u = h\,v$ and $x > u$, then $h(\pi_{uv}x) = hx$.

(iii) If $h\,u \neq h\,v$ and $x > u$ then for each $\alpha > o(x)$ there are $z_1, z_2 > x$ on the α^{th} level such that $h\,z_1 = h\,z_2$ and $h(\pi_{uv}z_1) \neq h(\pi_{uv}z_2)$.

The last condition enables us to annihilate every potential automorphism of $(T^2)^C$ while keeping the π_{uv}'s going (because then $h[\pi_{uv}]$ is not an automorphism if $h\,u \neq h\,v$).

7. Proof of Theorem 3

As another example of forcing with Suslin trees we construct a model whose constructibility degrees of nonconstructible sets have order-type ω^*. The theorem follows from this (V = L):

There is a sequence of Suslin trees T^n, $n < \omega$, and projections $h_n : T^n \xrightarrow{\text{onto}} T^{n+1}$ such that if b_0 is a branch in T^0, $b_1 = h_0 b_0$, $b_2 = h_1 b_1$, etc., then

(1) $b_n \notin \mathrm{L}[b_{n+1}]$.

(2) If A is a set of ordinals in $\mathrm{L}[b_0]$, then either $A \in \mathrm{L}$ or there is n such that $A \in \mathrm{L}[b_n]$ and $b_n \in \mathrm{L}[A]$.

We sketch the construction of a pair of trees T^0, T^1, with the projection $h : T^0 \to T^1$, such that if $A \in \mathrm{L}[b_0]$, then either $b_0 \in \mathrm{L}[A]$ or $A \in \mathrm{L}[b_1]$. This combined with the proof of Theorem $\frac{3}{4}$ (to make $\bigcap_{n<\omega} \mathrm{L}[b_n] = \mathrm{L}$) gives the theorem.

To arrange matters so that $b_0 \notin \mathrm{L}[b_1]$, we construct T^0, T^1 such that

(*) If $x, y \in T^0$ are such that $hx = hy$ and if $u > x$, then there is a $v \neq u$, $v > y$, such that $hu = hv$.

Let B_0 and B_1 be the cBa's associated with T^0 and T^1. We want to construct T^0, T^1 such that if D is a complete subalgebra of B_0, then for some $a \in \mathrm{B}_0$, $D \restriction a = B_0 \restriction a$ and $D \restriction -a = B_1 \restriction -a$.

Let D be a complete subalgebra of B_0. We recall (cf. [5]) that D is represented by partitions of levels of T^0. We say that x, y in the same level of T^0 are *D-separated* if they belong to different parts of the partition of that level. Let $a \in \mathrm{B}_0$ be defined as follows:

$$-a = \sum \{u : D \restriction u \subset B_1 \restriction u\}.$$

For each $u \leq a$, $D \restriction u \not\subseteq \mathrm{B}_1 \restriction u$.

Lemma. *There is a closed unbounded set of α's such that for each $u \leq b$ there are $x, y \in T^0$, $o(x) = o(y) = \alpha$, $hx = hy$, and x, y are D-separated.*

This Lemma is proved using the fact that T^0 satisfies (*).

Proof. We construct T^0, T^1 and h by induction on α, using $\diamond$. We keep the condition (*) satisfied. If ($\diamond$) commands the destruction of an antichain, we do it. To ensure that for every $D \subseteq \mathrm{B}$ there is an a such that $D \restriction a = \mathrm{B}_0 \restriction a$ and $D \restriction -a = \mathrm{B}_1 \restriction -a$, we use the lemma, and destroy a potential counterexample D in a manner similar to the construction of a simple cBa in [5].

If the S_α of $\diamond$ codes $D \restriction (T^0 \restriction \alpha)$ which satisfies the lemma, we construct the α^{th} level of T^0 such that it contains a branch b that is D-separated from all other elements of the α^{th} level. (cf. the argument in [5].)

We also keep the condition (*) satisfied on α^{th} level. This is done roughly as follows: we construct countably many α-branches throught $T^0 \restriction \alpha$; first the "master" branch b, and then the others in such a way that if $x, y \in T^0 \restriction \alpha$, $hx = hy$, and b goes through x, then there is another $b' \neq b$ through y such that $hb = hb'$, and b and b' are D-separated. This is done using (*).

8. Final remarks

Since ordinal definability depends on the model of ZF in question, it is natural to ask:

(i) What can we say about HOD, if we add further axioms?

(ii) What is HOD in "natural" models?

In particular, these questions might be interesting in connection with large cardinals. It is known that virtually every large cardinal axiom is consistent with HOD = V. Moreover, the natural model for a measurable cardinal, L[$\mathfrak{U}$], satisfies HOD = V. And it is expected that the natural models for other large cardinals, when discovered, will satisfy the same.

One problem that might be of some interest is what can we say about HOD if we assume the axiom of determinacy?

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