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Heat Transfer of a Continuous, Stretching Surface with Suction or Blowing

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An analysis has been carried out to determine the heat transfer occurring in the laminar boundary layer on a linearly stretching, continuous surface subject to suction or blowing. Two cases are considered: the sheet with prescribed wall temperature and heat flux. The results are expressed in terms of Kummer's functions. For specified conditions, the solutions reduced to the published results. Additionally, the results of an impermeable stretching plate with variable wall heat flux are also obtained. Finally, the effects of Prandtl number, suction or blowing parameter, temperature parameter, and heat flux parameter on the temperature distribution are discussed in detail.

INTRODUCTION

Boundary-layer behavior on a moving continuous solid surface is an important type of flow occurring in a number of engineering processes. For example, materials which are manufactured by extrusion processes and heat-treated materials travelling between a feed roll and a wind-up roll or on conveyor belts possess the characteristics of moving continuous surface.

Flow in the boundary layer on a continuous solid surface with constant speed was studied by Sakiadis [1]. Due to entrainment of ambient fluid, this situation represents a different class of boundary-layer problem which has a solution substantially different from that of boundary-layer flow over a semi-infinite flat plate. Erickson, Fan, and Fox [2] extended this problem to the case in which suction or blowing existed at the moving surface. Since polyester is a flexible material, the filament surface may stretch during the course of ejection and therefore the surface velocity deviates from being uniform. Crane [3] considered the moving strip whose velocity is proportional to the distance from the slit. These types of flow usually occur in the drawing of plastic films and artificial fibers. The heat and mass

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transfer on a stretching sheet with suction or blowing was investigated by Gupta and Gupta [4]. They dealt with the isothermal moving plate and obtained the temperature and concentration distributions. Dutta, Roy, and Gupta [5] analyzed the temperature distribution in the flow over a stretching sheet with uniform heat flux. It is shown that the temperature at a point decreases with increase in Prandtl number.

In a more recent study, Grubka and Bobba [6] considered the heat transfer occurring on a linear impermeable stretched surface with a power law surface temperature. In the present investigation the effects of both power law surface temperature and power law surface heat flux variation on the heat transfer characteristics of a continuous, linearly stretching sheet subject to suction or blowing are analyzed. A series solution to the energy equation in terms of Kummer's functions is developed. Several closed-form analytical solutions are also presented for special conditions.

The analysis is carried out in a general form allowing one, in a unified approach, to describe the heat transfer on a stretching sheet with suction or blowing for different versions of thermal boundary conditions on the surface.

Analysis

Consider the problem of a flat plate issuing from a thin slit at $x=0$, $y=0$ and subsequently being stretched, as in a polymer processing application (Fig. 1). It is assumed that the speed of a point on the plate is proportional to its distance from the slit, the boundary-layer approximations are still applicable, and viscous dissipation is neglected in the energy equation.

The velocity boundary-layer equations for the steady, two-dimensional, incompressible Newtonian flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

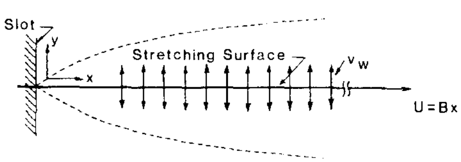

FIG. 1. Boundary layer on a stretching, flat, porous surface.

with the boundary conditions

$$\begin{aligned} u &= Bx, \ v = v_w \qquad && \text{for} \quad y = 0 \\ u &= 0 && \text{as} \quad y \to \infty. \end{aligned} \tag{3}$$

The x-axis is parallel to the surface of the plate in the direction of motion, and the y-axis is perpendicular to the plate; u and v are the velocity components in the direction of x and y, respectively.

For the heat transfer analysis, two different cases are considered: prescribed surface temperature and heat flux. The governing boundary-layer energy equation is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}. \tag{4}$$

The thermal boundary conditions depend on the type of heating process under consideration.

Since the fluid is incompressible, the momentum equation (2) and the energy equation (4) are decoupled and can be solved consecutively. The solution to the momentum equation will be considered first.

Equation (1) implies a stream function ψ given by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

where

$$\psi = (B\nu)^{1/2}\, x f(\eta), \qquad \eta = (B/\nu)^{1/2}\, y, \tag{5}$$

where f is the dimensionless stream function and η is the similarity variable depending on y only. Substituting Eq. (5) into Eq. (2) yields

$$f''' + ff'' - (f')^2 = 0 \tag{6}$$

with boundary conditions

$$f'(0) = 1, \qquad f(0) = -v_w/(B\nu)^{1/2}, \qquad f'(\infty) = 0, \tag{7}$$

where primes denote order of differentiation with respect to η. The solution of Eq. (6) was shown by Gupta and Gupta [4] to be

$$f' = e^{-m\eta}, \qquad f = m - e^{-m\eta}/m. \tag{8}$$

Here m is a positive number and

$$v_w = -(B\nu)^{1/2}\,(m - 1/m). \tag{9}$$

It should be noted that $m > 1$ corresponds to suction ($v_w < 0$), while $m < 1$ corresponds to blowing ($v_w > 0$). In the case when the parameter m is unity, the stretching plate is impermeable.

To solve Eq. (4), we consider two different heating processes:

Case A: *Prescribed Surface Temperature* (*PST*). For this circumstance, the boundary conditions are

$$T_w = T_\infty + Ax^r \qquad \text{for} \quad y = 0$$

and

$$T = T_\infty \qquad \text{as} \quad y \to \infty, \tag{10}$$

where r is the temperature parameter. When $r = 0$, the thermal boundary condition becomes isothermal. The dimensionless temperature is defined as

$$\theta(\eta) = (T - T_\infty)/(T_w - T_\infty). \tag{11}$$

Substitution of Eqs. (5), (8), and (11) in Eq. (4) gives

$$\theta'' + \Pr\left(m - \frac{e^{-m\eta}}{m}\right)\theta' - r \cdot \Pr e^{-m\eta}\theta = 0. \tag{12}$$

The boundary conditions are derived from Eqs. (10) and (11) as

$$\theta(0) = 1, \qquad \theta(\infty) = 0. \tag{13}$$

Introducing a new variable ξ as

$$\xi = -\Pr e^{-m\eta}/m^2 \tag{14}$$

and substituting the solution for f into Eq. (12) yields

$$\xi \frac{d^2\theta}{d\xi^2} + (1 - \Pr - \xi) \cdot \frac{d\theta}{d\xi} + r\theta = 0. \tag{15}$$

The corresponding boundary conditions are

$$\theta(-\Pr/m^2) = 1 \qquad \text{and} \qquad \theta(0) = 0. \tag{16}$$

The solution of Eq. (15) satisfying Eq. (16) in terms of Kummer's functions [7] is

$$\theta(\xi) = \left(\frac{-m^2\xi}{\Pr}\right)^{\Pr} \frac{M(\Pr - r, \Pr + 1, \xi)}{M(\Pr - r, \Pr + 1, -\Pr/m^2)}, \tag{17}$$

where

$$M(a, b, z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{b_n} \frac{Z^n}{n!}$$

$$a_n = a(a+1)(a+2) \cdots (a+n-1)$$

$$b_n = b(b+1)(b+2) \cdots (b+n-1).$$

Equation (17) can be recast in terms of η as

$$\theta(\eta) = e^{-m \Pr \eta} \frac{M(\Pr - r, \Pr + 1, -\Pr \cdot e^{-m\eta}/m^2)}{M(\Pr - r, \Pr + 1, -\Pr/m^2)}. \tag{18}$$

The dimensionless wall temperature gradient derived from Eq. (18) is

$$\theta'(0) = -m \Pr + \frac{\Pr - r}{\Pr + 1} \left(\frac{\Pr}{m}\right) \frac{M(\Pr - r + 1, \Pr + 2, -\Pr/m^2)}{M(\Pr - r, \Pr + 1, -\Pr/m^2)} \tag{19}$$

and the local wall heat flux can be expressed as

$$q_w = -k \left(\frac{\partial T}{\partial y}\right)_w = -kA(B/\nu)^{1/2} x^r \theta'(0).$$

For specific values of r and Pr, closed-form solutions can be readily obtained from Eqs. (18) and (19). Five such cases are reported in Table I.

Case B: *Prescribed Wall Heat Flux* (*WHF*). Now, the boundary conditions are

$$-k \frac{\partial T}{\partial y} = q_w = Dx^s \qquad \text{for} \quad y = 0$$

and

$$T = T_\infty \qquad \text{as} \quad y \to \infty, \tag{20}$$

where s is the heat flux parameter. For $s = 0$, the stretching sheet is subject to uniform heat flux. In order to obtain the temperature distribution, it can be taken in the form of a similar solution as

$$T - T_\infty = \frac{Dx^s}{k} (\nu/B)^{1/2} g(\eta). \tag{21}$$

Putting Eqs. (15), (8), and (21) in Eq. (4), it becomes

$$g'' + \Pr\left(m - \frac{e^{-m\eta}}{m}\right) g' - s \cdot \Pr e^{-m\eta} g = 0 \tag{22}$$

TABLE I

Temperature and Wall Gradient Expressions for Various Pr and r

r	Pr	θ	$\theta'(0)$
Pr	r	$e^{-m\,\mathrm{Pr}\,\eta}$	$-m\,\mathrm{Pr}$
0	Pr	$\gamma(\mathrm{Pr}, (\mathrm{Pr}/m^2)\,e^{-m\eta})/\gamma(\mathrm{Pr}, (\mathrm{Pr}/m^2))$	$-m(\mathrm{Pr}/m^2)^{\mathrm{Pr}}\,e^{-(\mathrm{Pr}/m^2)}/\gamma(\mathrm{Pr}, (\mathrm{Pr}/m^2))$[a]
0	1	$\dfrac{\exp(1/m^2)\cdot(\exp((-1/m^2)\,e^{-m\eta})-1)}{1-\exp(1/m^2)}$	$\dfrac{1}{m\cdot(\exp(1/m^2)-1)}$
-1	Pr	$\exp(\mathrm{Pr}(1-m^3\eta-e^{-m\eta})/m^2)$	$\mathrm{Pr}(1/m-m)$
-2	Pr	$\dfrac{(1+\mathrm{Pr}-(\mathrm{Pr}/m^2)\,e^{-m\eta})\exp(\mathrm{Pr}(1-m^3\eta-e^{-m\eta})/m^2)}{(1+\mathrm{Pr}-\mathrm{Pr}/m^2)}$	$-m\,\mathrm{Pr}+\dfrac{\mathrm{Pr}(\mathrm{Pr}+2-\mathrm{Pr}/m^2)}{m(\mathrm{Pr}+1-\mathrm{Pr}/m^2)}$

[a] Incomplete gamma function,

TABLE II

Dimensionless Temperature Expressions for Various Pr and s

s	Pr	$g(\eta)$
Pr	s	$\frac{\exp(-m\,\mathrm{Pr}\,\eta)}{m\,\mathrm{Pr}}$
0	Pr	$\frac{1}{m}\left(\frac{\mathrm{Pr}}{m}\right)^{-\mathrm{Pr}} \exp(\mathrm{Pr}/m^2)\,\gamma(\mathrm{Pr}, (\mathrm{Pr}\,e^{-m\eta})/m^2)^a$
0	1	$m(1-\exp(-e^{-m\eta})/m^2)$
-1	Pr	$\frac{m}{(m^2-1)\,\mathrm{Pr}} \exp(\mathrm{Pr}(1-m^3\eta-e^{-m\eta})/m^2)$
-2	Pr	$\frac{(1+\mathrm{Pr}-(\mathrm{Pr}/m^2)\,e^{-m\eta})}{m\,\mathrm{Pr}(1+\mathrm{Pr}-2\mathrm{Pr}/m^2-2/m^2+\mathrm{Pr}/m^4)} \exp(\mathrm{Pr}(1-m^3\eta-e^{-m\eta})/m^2)$

[a] Incomplete gamma function.

with the boundary conditions

$$g'(0)=-1 \qquad \text{and} \qquad g(\infty)=0. \tag{23}$$

Using Eq. (14), we find that Eq. (22) becomes

$$\xi \frac{d^2g}{d\xi^2}+(1-\mathrm{Pr}-\xi)\cdot\frac{dg}{d\xi}+sg=0. \tag{24}$$

The corresponding boundary conditions are

$$\frac{dg(-\mathrm{Pr}/m^2)}{d\xi}=-m/\mathrm{Pr} \qquad \text{and} \qquad g(0)=0. \tag{25}$$

The solution of Eq. (24) satisfying Eq. (25) may be expressed in terms of the similarity variable η as

$$g(\eta)=\frac{1}{m\,\mathrm{Pr}}\,e^{-m\,\mathrm{Pr}\,\eta}\,\frac{M(\mathrm{Pr}-s,\ \mathrm{Pr}+1,\ (-\mathrm{Pr}/m^2)\cdot e^{-m\eta})}{M(\mathrm{Pr}-s,\ \mathrm{Pr},\ -\mathrm{Pr}/m^2)}. \tag{26}$$

The wall temperature T_w is obtained from Eq. (21) as

$$T_w-T_\infty=\frac{Dx^s}{k}\,(\nu/B)^{1/2}\,g(0). \tag{27}$$

Again several closed-form solutions are developed from Eq. (26) for specific values of s and Pr. Five such cases are reported in Table II.

RESULTS AND DISCUSSION

For prescribed surface temperature circumstances, Eqs. (18) and (19) were evaluated to determine the temperature distribution and the surface temperature gradient as a function of Pr and r. The results, $\theta(\eta)$, coincide with the previous solutions for an isothermal permeable stretching plate [4] ($r=0$, $m \neq 1$) as well as for an impermeable stretching plate with variable surface temperature [6] ($r \neq 0$, $m=1$).

Figure 2 presents the temperature profiles for selected Pr at $r=1$ and $m=1.5$. It is shown that the temperature at a given point decreases with increase in Pr. The temperature profiles versus similarity variable, η, for selected dimesionless suction or blowing parameter, m, are plotted in Fig. 3. For fixed values of η, Pr, and r, the smaller the m, the larger is the thermal boundary-layer thickness. This implies that the thermal boundary-layer thickness in suction is thinner than that in blowing.

To compare the effects of blowing with those of suction on temperature distribution, temperature profiles were obtained at $\mathrm{Pr}=0.72$, $m=0.8$ (blowing) or $m=2$ (suction) with selected values of r. Plots for the blowing and suction case are presented in Figs. 4 and 5. Figure 4 shows that, for $r>0$, heat flows from the stretching surface to the ambient. The magnitude of the temperature gradient increases as r increases. For $r=-1$ and -2, the wall temperature gradient is positive and heat flows into the stretching surface from the ambient. However, Fig. 5 shows that heat flows from the stretching surface to the ambient because the temperature gradient is negative for r, ranging between -2 and 2.

The wall temperature gradient $\theta'(0)$ as a function of Pr for selected values of m at $r=1$ is shown in Fig. 6. For a given m, the larger the Pr, the larger is the magnitude of the wall temperature gradient. In addition, the magnitude of the wall temperature gradient increases as m increases. Tables III and IV show the variation of $\theta'(0)$ as a function of r in blowing

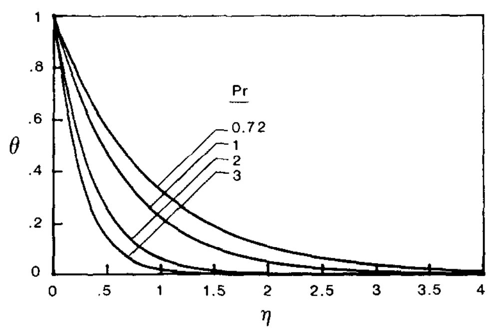

FIG. 2. Dimensionless temperature profiles for various Pr at $r=1$, $m=1.5$ (suction).

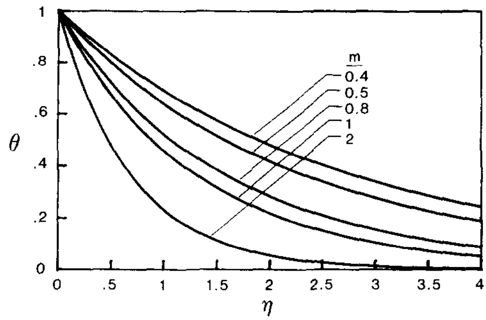

FIG. 3. Dimensionless temperature profiles for various m at $\mathrm{Pr} = 0.72$, $r = 1$.

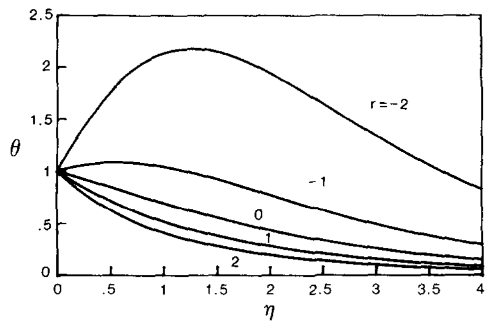

FIG. 4. Dimensionless temperature profiles for various r at $\mathrm{Pr} = 0.72$, $m = 0.8$ (blowing).

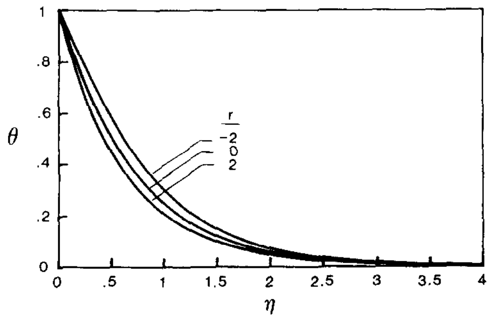

FIG. 5. Dimensionless temperature profiles for various r at $\mathrm{Pr} = 0.72$, $m = 2$ (suction).

TABLE III

Wall Temperature Gradient as a Function of Pr and r at $m=0.7$ (Blowing)

r	Pr 0.01	0.1	0.72	1
−3	0.036460	0.435312	−1.701066	−0.589065
−2.1	0.023182	0.250321	7.413630	−8.565597
−1.2	0.010149	0.102105	0.776315	1.113438
0	−0.006860	−0.058101	−0.195854	−0.213320
1.2	−0.023468	−0.189667	−0.669538	−0.778009
2.1	−0.035671	−0.275158	−0.930378	−1.084757
3	−0.047667	−0.352134	−1.150099	−1.342679

and suction. It is seen in the blowing case that the wall temperature gradient is negative for $r < -2.1$ and certain Pr values. This is physically unrealistic because temperature distribution for these r and Pr values is less than that at the ambient. However, the unrealistic temperature distributions disappear in the suction situation, which is clearly observed in Table IV. It is worth noting that the values of r and Pr for which unrealistic flow is encountered depend on dimensionless suction or blowing parameter m.

For prescribed wall heat flux circumstance, the temperature distribution can be readily obtained from Eqs. (26) and (27). When $s=0$, $m=1$, the solutions, $g(\eta)$, reduce to the published results for an impermeable stretching plate with uniform heat flux [5]. For $s\neq 0$ and $m=1$, the results of an impermeable stretching plate subject to variable heat flux are

$$g(\eta)=\frac{1}{\Pr}e^{-\Pr\eta}\frac{M(\Pr-s,\ \Pr+1,\ -\Pr\cdot e^{-\eta})}{M(\Pr-s,\ \Pr,\ -\Pr)}. \tag{28}$$

TABLE IV

Wall Temperature Gradient as a Function of Pr and r at $m=2$ (Suction)

r	Pr 0.01	0.1	0.72	1
−3	−0.005062	−0.055922	−0.589916	−0.897260
−2.1	−0.009540	−0.098780	−0.821689	−1.184108
−1.2	−0.014008	−0.140780	−1.034853	−1.445049
0	−0.019951	−0.195503	−1.295052	−1.760406
1.2	−0.025875	−0.248838	−1.532560	−2.045739
2.1	−0.030307	−0.287969	−1.698236	−2.243598
3	−0.034729	−0.326386	−1.854735	−2.429752

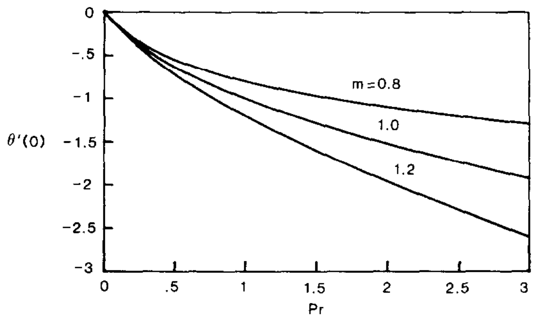

FIG. 6. Dimensionless wall temperature gradient in the PST case for various m at $r = 1$.

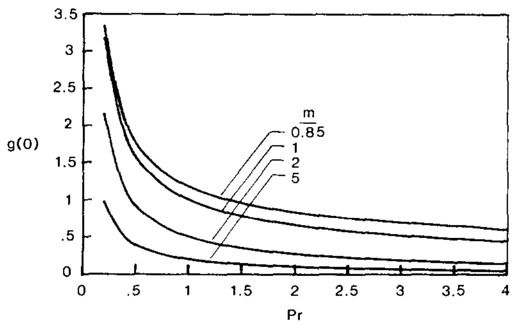

FIG. 7. Dimensionless wall temperature in the WHF case for selected m at $s = 1$.

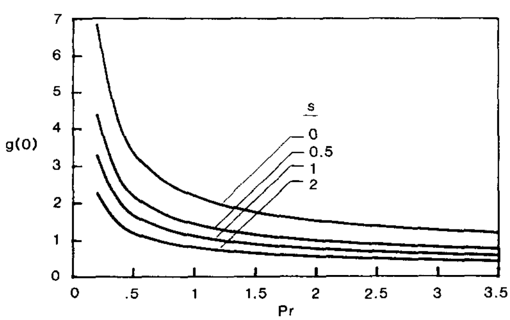

FIG. 8. Dimensionless wall temperature in the WHF case for selected s at $m = 0.9$ (blowing).

Note that the dimensionless temperature distribution $\theta=(T-T_\infty)/(T_w-T_\infty)$ is equal to the ratio of $g(\eta)$ to $g(0)$. Therefore, Fig. 4 with r replaced by s in the prescribed wall heat flux condition shows that the wall temperature gradient is negative for $s=0$, 1, and 2. This implies that heat flows from the stretching surface to the ambient. When $s=-1$ and -2, the sign of the temperature gradient changes but the value of $g(0)$ is negative, and hence the heat flux at the surface flows into the fluid. As can be seen in Fig. 4, zero temperature gradient occurs so the heat flows into the thermal boundary layer from both the ambient and stretching plate.

Figure 7 plots the dimensionless wall temperature $g(0)$ versus Pr for selected m at $s=1$. The wall temperature decreases rapidly as Pr increases from 0 to 1 and then slowly decreases with increase in Pr. Figure 7 also shows that the larger the suction or blowing parameter m, the smaller is the surface temperature. Figure 8 presents the dimensionless wall temperature profiles versus Pr for selected s at $m=0.9$. It can be seen that the wall temperature decreases with increase in Pr. The wall temperature decreases as s increases.

Conclusions

In this study, the laminar boundary-layer heat transfer from a linearly streatching, continuous sheet subject to suction or blowing is studied. Two cases are considered: moving plate with prescribed wall temperature and heat flux. The resulting temperature distribution has been solved in terms of Kummer's functions. Several closed-form solutions for specified conditions are presented.

The heat transfer characteristics for this problem are found to be determined by Prandtl number Pr, suction or blowing parameter m, the temperature parameter r(PST), or the heat flux parameter s(WHF). The thermal boundary-layer thickness decreases with increase in Pr, m, or r. Varying the parameters r and s affects the mechanism of heat transfer. For a specified Pr in the prescribed surface temperature case, the magnitude of the wall temperature gradient increases as m increases. Additionally, the results of an impermeable stretching plate with variable heat flux are also derived.

For the prescribed wall heat flux case, the wall temperature decreases rapidly as Pr increases from 0 to 1. Furthermore, the smaller the wall temperature, the larger is the value of m or s. The unrealistic temperature distributions are encountered on the stretching plate with blowing and on the impermeable plate [6]. Nevertheless, this phenomenon disappears in the suction situation.

It should be noted in conclusion that some of the published results [3–6] are obtained as special cases of the results of the present work.

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