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Overgroups of classical groups in linear group over Banach algebras ☆

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Abstract

All overgroups of elementary unitary groups in linear group GL_{2n} over Banach algebras with 1 have been described.

Keywords: Overgroup; Unitary group; Linear group; Banach algebra

1. Introduction

Let R be an associative ring with 1 and assume that an anti-automorphism $*: x \mapsto x^*$ is defined on R such that $x^{**} = \varepsilon x \varepsilon^*$ for some unit $\varepsilon = \varepsilon^{*-1}$ of R and every x in R. It also determines an anti-automorphism of the ring $M_n R$ of all n by n matrices (x_{ij}) by $(x_{ij})^* = (x_{ji}^*)$.

Set $R_\varepsilon = \{x - x^*\varepsilon \mid x \in R\}$, $R^\varepsilon = \{x \in R \mid x = -x^*\varepsilon\}$. We fix an additive subgroup Λ of R with the following properties:

(i) $r^*\Lambda r \subset \Lambda$ for all $r \in R$;
(ii) $R_\varepsilon \subset \Lambda \subset R^\varepsilon$.

Let

$$\Lambda_n = \left\{(a_{ij}) \in M_n R \mid a_{ij} = -a_{ji}^*\varepsilon \text{ for } i \neq j \text{ and } a_{ii} \in \Lambda\right\}.$$

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As in [1,11], we define

$$U_{2n}(R,\Lambda)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL_{2n}R\;\middle|\;\alpha^*\delta+\gamma^*\varepsilon\beta=I_n,\alpha^*\gamma,\beta^*\delta\in\Lambda_n\right\}.$$

For fixed $*$ and ε, if $\Lambda\subset\Lambda'$, it is clear that $U_{2n}(R,\Lambda)\subset U_{2n}(R,\Lambda')$. So, in this article we assume that $\Lambda=R^{\varepsilon}$, that is

$$U_{2n}(R,\Lambda)=U_{2n}R=\{\theta\in GL_{2n}R\mid\theta^*\varphi_n\theta=\varphi_n\},$$

where

$$\varphi_n=\begin{pmatrix}0 & I_n\\ \varepsilon I_n & 0\end{pmatrix}.$$

When $*$ is identical on R and $\varepsilon=-1$, $U_{2n}R$ is the symplectic group. When $*$ is identical on R and $\varepsilon=1$, but 2 is torsionfree in R, $U_{2n}R$ is the ordinary orthogonal group. When R is the complex numbers, $*$ the complex conjugation, $\varepsilon=1$ or -1, $U_{2n}R$ is the standard unitary group.

Let

$$GU_{2n}R=\{\theta\in GL_{2n}R\mid\theta^*\varphi_n\theta=\mu\varphi_n,\ \mu\in\operatorname{Cent}R\text{ is invertible}\}.$$

(Note that $\mu^*=\mu$.)

An ideal J of R is said to be dual, if $J^*=J$. For any ideal J of R, let $E_{2n}J$ denote the subgroup of $GL_{2n}R$ generated by all elementary matrices $\xi_{ij}(a)=I_{2n}+aE_{ij}$ with $a\in J$, $i\neq j$, where E_{ij} denotes the matrix with 1 at the position (i,j) and zeros elsewhere. The normal subgroup of $E_{2n}R$ generated by $E_{2n}J$ is denoted by $E_{2n}(R,J)$. With n fixed for any $1\leqslant k\leqslant 2n$, set $\sigma k=k+n$ if $k\leqslant n$ and $\sigma k=k-n$ if $k>n$. For $a\in R$ and $1\leqslant i\neq j\leqslant 2n$ we define the elementary unitary matrices $\rho_{i,\sigma i}(a)$ and $\rho_{ij}(a)$ with $j\neq\sigma i$ as follows: $\rho_{i,\sigma i}(a)=I_{2n}+aE_{i,\sigma i}$ with $a\in\Lambda$ when $n+1\leqslant i$ and $a^*\in\Lambda$ when $i\leqslant n$, $\rho_{ij}(a)=\rho_{\sigma j,\sigma i}(-a')=I_{2n}+aE_{ij}-a'E_{\sigma j,\sigma i}\in U_{2n}R$ with $a'=a^*$ when $i,j\leqslant n$; $a'=\varepsilon^*a^*$ when $i\leqslant n<j$; $a'=a^*\varepsilon$ when $j\leqslant n<i$; and $a'=\varepsilon^*a^*\varepsilon$ when $n+1\leqslant i,j$. The subgroup of $U_{2n}R$ generated by all elementary unitary matrices is denoted by $EU_{2n}R$.

Define

$$EEU_{2n}J=EU_{2n}R\cdot E_{2n}(R,J),\qquad CGU_{2n}J=\{\rho\in GL_{2n}R\mid\rho\ (\operatorname{mod}J)\in GU_{2n}(R/J)\}.$$

(Note that $E_{2n}(R,J)$ is normal in $GL_{2n}R$ when $n\geqslant 2$, $EU_{2n}R\cdot E_{2n}(R,J)$ make sense when $n\geqslant 2$ (see [9]).)

The main result of this paper is stated as follows.

Theorem. *Let R be a Banach algebra with* 1 *and $n\geqslant 4$. Let X be an overgroup of $EU_{2n}R$ in $GL_{2n}R$. Then there is a unique dual ideal J of R such that*

$$EEU_{2n}J\subseteq X\subseteq CGU_{2n}J.$$

King [2,3] and Li [4,5] determined all overgroups of $SU(n,K,f)$ and $\Omega(n,K,Q)$ in GL_nK where K is a division ring, respectively. In the resent years, Vavilov and Petrov [12,13], and the author [14] described the overgroups of symplectic and orthogonal groups (with hyperbolic form) over commutative rings; Petrov [6] also classified under a local stable rank condition with form parameter, the overgroups of unitary groups (with hyperbolic form).

2. Basic lemmas

Lemma 2.1. [11] *The following identities hold for elementary unitary matrices* ($1 \leqslant i \neq j \leqslant 2n$):

(1) $\rho_{ij}(a+b) = \rho_{ij}(a)\rho_{ij}(b)$;
(2) $[\rho_{ij}(a), \rho_{jk}(b)] = \rho_{ik}(ab)$ *when* $i, j, k, \sigma i, \sigma j, \sigma k$ *are all distinct*;
(3) $[\rho_{ij}(a), \rho_{j,\sigma i}(b)] = \rho_{i,\sigma i}(ab - c)$ *when* $j \neq \sigma i$, *where* $c = b^*a^*\varepsilon$ *when* $n+1 \leqslant i$ *and* $c = \varepsilon^*b^*a^*$ *when* $i \leqslant n$;
(4) $[\rho_{ij}(a), \rho_{j,\sigma j}(b)] = \rho_{i,\sigma j}(ab)\rho_{i,\sigma i}(c)$ *when* $j \neq \sigma i$, *where*

$$b^* \in \Lambda \quad \text{and} \quad c = aba^* \quad \text{when } i, j \leqslant n,$$
$$b^* \in \Lambda \quad \text{and} \quad c = aba^*\varepsilon \quad \text{when } j \leqslant n < i,$$
$$b \in \Lambda \quad \text{and} \quad c = -ab^*a^* \quad \text{when } i \leqslant n < j,$$
$$b \in \Lambda \quad \text{and} \quad c = -ab^*a^*\varepsilon \quad \text{when } n+1 \leqslant i, j.$$

Here $[a, b]$ *denotes* $aba^{-1}b^{-1}$.

Lemma 2.2. *The following identities hold* ($1 \leqslant i \neq j \leqslant 2n$):

(1) $\rho_{ij}(a) = \xi_{ij}(a)\xi_{\sigma_j,\sigma_i}(a')$ ($i \neq j$, *the definition of* $\xi_{ij}(a)$ *is indicated in Section* 1), *where*

$$a' = -a^* \quad \text{when } i, j \leqslant n; \qquad a' = -\varepsilon^*a^* \quad \text{when } i \leqslant n < j;$$
$$a' = -a^*\varepsilon \quad \text{when } j \leqslant n < i; \qquad a' = -\varepsilon^*a^*\varepsilon \quad \text{when } n+1 \leqslant i, j.$$

(2) $[\xi_{ij}(a), \rho_{jk}(b)] = \xi_{ik}(ab)$ *when* i, j, k *are distinct and* $j \neq \sigma i$, *where* $b \in \Lambda$ *or* $b^* \in \Lambda$ *if* $k = \sigma j$.
(3) $[\xi_{ij}(a), \rho_{k,\sigma j}(b)] = \xi_{i,\sigma k}(c)$ *when* $i, j, \sigma k$ *are distinct, where*

$$c = -a\varepsilon^*b^* \quad \text{when } j, k \leqslant n; \qquad c = -ab^*\varepsilon \quad \text{when } n+1 \leqslant j, k;$$
$$c = -a\varepsilon^*b^*\varepsilon \quad \text{when } j \leqslant n < k \text{ or } k \leqslant n < j.$$

The following matrices are in $EU_{2n}R$.

$d_A = \begin{pmatrix} A & \\ & A^{*-1} \end{pmatrix}$ where $A \in E_nR$, especially $w_{ij} = \begin{pmatrix} P_{ij} & \\ & (P_{ij}^*)^{-1} \end{pmatrix}$ where $P_{ij} = \xi_{ij}(1)\xi_{ji}(-1)\xi_{ij}(1)$ ($1 \leqslant i, j \leqslant n$).

Set $\tilde{v} = v^*\varphi_n$ for $v \in R^{2n}$. Let $\{e_1, \ldots, e_{2n}\}$ denote the standard basis of R^{2n}, i.e., $(e_1, \ldots, e_{2n}) = I_{2n}$.

Let $\theta \in GL_{2n}R$. By the definition of $U_{2n}R$ and $GU_{2n}R$ we have

Lemma 2.3. $\theta \in U_{2n}R$ *if and only if* $u_i = \varepsilon^*\tilde{v}_{\sigma i}$ *when* $1 \leqslant i \leqslant n$ *and* $u_i = \tilde{v}_{\sigma i}$ *when* $n+1 \leqslant i \leqslant 2n$, *where* u_i *is the* i*th row of* θ^{-1} *and* $v_{\sigma i}$ *is the* σi*th column of* θ.

Proof. We need only to point out that $\theta^{-1} = \varphi_n^{-1}\theta^*\varphi_n$ if and only if $\theta \in U_{2n}R$. $\square$

Similarly, for the unitary *similitudes* $GU_{2n}R$, we have

Lemma 2.4. *$\theta \in GU_{2n}R$ if and only if there is a $\mu \in \operatorname{Cent} R^u$ (R^u, the set of invertible elements in R) such that $u_i = \mu\varepsilon^*\tilde{v}_{\sigma i}$ when $1 \leqslant i \leqslant n$ and $u_i = \mu\tilde{v}_{\sigma i}$ when $n+1 \leqslant i \leqslant 2n$, where u_i is the ith row of θ^{-1} and $v_{\sigma i}$ is the σith column of θ.*

Lemma 2.5. [6] *Let $n \geqslant 2$, $g \in GL_{2n}R$ such that $gEU_{2n}Rg^{-1} \subseteq U_{2n}R$. Then g belongs to $GU_{2n}R$.*

Lemma 2.6. *Let X be an overgroup of $EU_{2n}R$ in $GL_{2n}R$ and $n \geqslant 3$:*

(1) *For Banach algebras R, if X contains an elementary matrix $\xi_{i,\sigma i}(a)$ with $a \bar{\in} R^\varepsilon$ (or $a^* \bar{\in} R^\varepsilon$), then X contains a $\xi_{kl}(c)$ with $l \neq \sigma k$ and $c \in R$ except for the symplectic case, i.e., $*$ is identical on R and $\varepsilon = -1$.*
(2) *If X contains an elementary matrix $\xi_{ij}(a)$ with $j \neq \sigma i$ and $a \in R$, then X contains $E_{2n}J_a$ where J_a is the ideal of R generated by a and a^*, so satisfies that $J_a = J_a^*$.*

Proof. (1) Without loss of generality, suppose that X contains $\xi_{1,n+1}(a)$ with $a^* \bar{\in} R^\varepsilon$. Then X contains $\xi_{1,n+2}(-a)\xi_{2,n+1}(-a)\xi_{2,n+2}(-a) = [\xi_{1,n+1}(a), \rho_{21}(1)]$ and $\xi_{2,n+2}(a) = w_{12}\xi_{1,n+1}(a)w_{12}$. Thus $\xi_{1,n+2}(-a)\xi_{2,n+1}(-a) \in X$ and $\xi_{2,n+1}(-\varepsilon^*a^* - a) = \xi_{2,n+1}(-a)\xi_{1,n+2}(-a)\rho_{1,n+2}(a) \in X$ (note that $a^* \bar{\in} R^\varepsilon$). Since 2 is not torsion in Banach algebra, eventhough $*$ is identical on R and $\varepsilon = 1$, $-\varepsilon^*a^* - a \neq 0$.

(2) By Lemma 2.2, all $\xi_{ij}(aR), \xi_{ij}(Ra)$ for $1 \leqslant i \neq j \leqslant n$ and $n+1 \leqslant i \neq j \leqslant 2n$ lie in X. Further, $\xi_{i,\sigma i}(ab) = [\xi_{ij}(a), \rho_{j,\sigma i}(b)] \in X$ (also, $\xi_{i,\sigma i}(ba) \in X$), where $j \neq \sigma i$, for all $1 \leqslant i \leqslant 2n$; and $\xi_{ij}(ab) = [\xi_{i,\sigma j}(a), \xi_{\sigma j,j}(b)] \in X$(also, $\xi_{ij}(ba) \in X$) for all $1 \leqslant i \leqslant n$, $n+1 \leqslant j \leqslant 2n$ and $n+1 \leqslant i \leqslant 2n$, $1 \leqslant j \leqslant n$ ($j \neq \sigma i$). That means X contains $E_{2n}(aR)$ and $E_{2n}(Ra)$. When X contains $\xi_{ij}(a)$, X also contains $\xi_{\sigma j,\sigma i}(a') = \rho_{ij}(a)\xi_{ij}(-a)$, where $a' = a^*$, or $a' = \varepsilon^*a^*$, or $a' = a^*\varepsilon$, or $a' = \varepsilon^*a^*\varepsilon$. So, by the above argument we have that X contains $E_{2n}J_a$ and $J_a = J_a^*$. $\square$

It is obvious that if an overgroup X of $EU_{2n}R$ contains $E_{2n}J$ where J is an ideal of R, then X contains the subgroup of $GL_{2n}J$, which is denoted by $UE_{2n}(R, J)$, generated by all elements of the form $\rho_{ij}(r)\xi_{kl}(a)\rho_{ij}(-r)$ with $a \in J$ and $r \in R$ for all $k \neq l$, $i \neq j$. Note that $UE_{2n}(R, J)$ is different from $EU_{2n}(R, J)$, the normal subgroup of $EU_{2n}R$ generated by $EU_{2n}J$ (see [11]), and that $EU_{2n}(R, J) \subseteq UE_{2n}(R, J)$.

Lemma 2.7. *For any ideal J of R and $n \geqslant 2$, $UE_{2n}(R, J) = E_{2n}(R, J)$.*

Proof. Since $UE_{2n}(R, J) \subseteq E_{2n}(R, J)$, we only need to show that $E_{2n}(R, J) \subseteq UE_{2n}(R, J)$. By the definition of $E_{2n}(R, J)$ and the commutator formulas of elementary matrices, in fact, $E_{2n}(R, J)$ is generated by $\xi_{ij}(r)\xi_{ji}(a)\xi_{ij}(-r)$ with $a \in J$, $r \in R$ for all $i \neq j$ (see [8]). We distinguish the following two cases on the index (i, j):

(1) $j \neq \sigma i$. In this case, we have $\xi_{ij}(r)\xi_{ji}(a)\xi_{ij}(-r) = \rho_{ij}(r)\xi_{ji}(a)\rho_{ij}(-r) \in UE_{2n}(R, J)$.

(2) $j = \sigma i$. Without loss of generality, we assume that $n = 2$ and $(i, j) = (1, 3)$.

Then

$$\xi_{13}(r)\xi_{31}(a)\xi_{13}(-r) = \begin{pmatrix} 1+ra & -rar \\ a & 1-ra \end{pmatrix} \oplus I_2$$
$$= \xi_{14}\big(-ra\varepsilon^* r^*\big)\rho_{23}(r)\rho_{21}(-1)\xi_{12}(ra)\xi_{34}\big(-a\varepsilon^* r^*\big)\xi_{32}(a)$$
$$\cdot \big(\rho_{23}(r)\rho_{21}(-1)\big)^{-1}\xi_{32}(-a)\xi_{12}(-ra)\xi_{24}\big(r\varepsilon^* ar^*\big) \in UE_{2n}(R, J). \qquad \square$$

Remark 2.8. The above result has been proved in [6, Lemma 12], but, here the proof is direct and simple.

Lemma 2.9. *Suppose that R is a Banach algebra with 1. Let X be an overgroup of $EU_{2n}R$ which is not in $GU_{2n}R$ and let $n \geqslant 3$. Then there exist an element θ in X and an elementary unitary matrix $\rho_{ij}(ra)$ for all real number r with sufficiently small $|r|$ such that $\theta\rho_{ij}(ra)\theta^{-1} \in X$ is not in $GU_{2n}R$.*

Proof. Note that when $n \geqslant 3 EU_{2n}R = [EU_{2n}R, EU_{2n}R]$. If $\xi \in X$ satisfies: $\xi EU_{2n}R\xi^{-1} \subseteq GU_{2n}R$, then $\xi EU_{2n}R\xi^{-1} = [\xi EU_{2n}R\xi^{-1}, \xi EU_{2n}R\xi^{-1}] \subseteq U_{2n}R$. So, by Lemma 2.5 there is $\xi \in X$ such that $\xi EU_{2n}R\xi^{-1} \nsubseteq GU_{2n}R$. Since $EU_{2n}R$ is generated by $\rho_{ij}(a)$ with $a \in R$ when $j \neq \sigma i$ and $a \in R^\varepsilon$ (or $a^* \in R^\varepsilon$) when $j = \sigma i$, there is $\rho_{ij}(a)$ such that $\xi\rho_{ij}(a)\xi^{-1} \bar{\in} GU_{2n}R$. Let $d = \operatorname{diag}(A, A^{*-1})$ where $A = \left(\begin{smallmatrix} r & \\ & r^{-1} \end{smallmatrix}\right) \oplus I_{n-2}$ and r is a real number with sufficiently small $|r|$. Since $A \in E_nR$, $d \in EU_{2n}R$. We have $d\rho_{ij}(a)d^{-1} = \rho_{ij}(r^2a)$ (note that $r^* = r$ for real numbers) and $d\xi d^{-1} = \theta \bar{\in} GU_{2n}R$. It is obvious that $\theta\rho_{ij}(r^2a)\theta^{-1} = d\xi d^{-1}d\rho_{ij}(a)d^{-1}d\xi d^{-1} = d\xi\rho_{ij}(a)\xi^{-1}d^{-1} \bar{\in} GU_{2n}R$. $\square$

Form now, assume that R is a Banach algebra with 1. For Banach algebras R (with 1), if $x \in R$ with $\|x\| < 1$, then $1 + x$ is invertible (see [7]).

So, for any $a \in R$ we have $1 + ra$ invertible for all real number r with sufficiently small $|r|$ (see [9,10]). Moreover, we claim that.

For any finite set $\{a_1, \ldots, a_k\}$ in R, there exists a real number r with sufficiently small $|r|$ such that $1 + rb_1, \ldots, 1 + rb_k$, where b_i is a sum of some a_j and some products of a finite number of ra_i by $(1 + ra_j)^{-1}$ $(1 \leqslant j \leqslant i)$, are all invertible (denote the property *by* (Δ)). In fact, $(1 + rb)^{-1}$ is in a neighborhood of 1 when $|r|$ is sufficiently small.

Lemma 2.10. *Let X be an overgroup of $EU_{2n}R$ in $GL_{2n}R$ which is not in $GU_{2n}R$ where $n \geqslant 4$. Then there is an element θ in X which has the following form and is not in $GU_{2n}R$:*

$$\begin{aligned}
&\theta = (v_1, \ldots, v_n, v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}) \quad \textit{where} \\
&v_1 = e_1 + \alpha e_{n+1} = (1, 0, \ldots, 0, \alpha, 0, \ldots, 0)^t, \\
&v_{n+2} = e_2 + be_{n+1} + \beta e_{n+2} = (0, 1, 0, \ldots, 0, b, \beta, 0, \ldots, 0)^t \quad \textit{and} \\
&v_{n+3} = e_3 + c_1e_{n+1} + c_2e_{n+2} + \gamma e_{n+3} = (0, 0, 1, 0, \ldots, 0, c_1, c_2, \gamma, 0, \ldots, 0)^t.
\end{aligned} \tag{2.1}$$

Proof. By Lemma 2.9 there exist an element ξ in X and $\rho_{ij}(ra)$ with r a real number with sufficiently small $|r|$ such that $\xi\rho_{ij}(ra)\xi^{-1}$ is not in $GU_{2n}R$. When $j \neq \sigma i$, we have $\xi\rho_{ij}(ra)\xi^{-1} = I_{2n} + rv_iau_j - rv_{\sigma j}a'u_{\sigma i}$ with $a' = a^*$ when $i, j \leqslant n$; $a' = \varepsilon^* a^*$ when $i \leqslant n < j$; $a' = a^*\varepsilon$ when $j \leqslant n < i$. When $j = \sigma i$, we have $\xi\rho_{ij}(ra)\xi^{-1} = I_{2n} + rv_iau_{\sigma i}$ with $a \in R^\varepsilon$ or $a^* \in R^\varepsilon$, where

v_i is the ith column of ξ and u_j is the jth row of ξ^{-1}. But in any case, we can write $\xi\rho_{ij}(ra)\xi^{-1}$ as $I_{2n} + r(a_{ij})_{2n\times 2n}$.

Since R is a Banach algebra, we may choose a real number r with sufficiently small $|r|$ such that $1 + ra_{11}, 1 + ra'_{n+2,n+2}, 1 + ra''_{n+3,n+3}$, where $a'_{n+2,n+2}(a''_{n+3,n+3})$ is a sum of $a_{n+2,n+2}(a'_{n+3,n+3})$ and a finite number of products of coefficients in $(a_{ij})_{2n\times 2n}$ by r or by $r(1 + ra_{11})^{-1}(r(1 + ra'_{n+2,n+2})^{-1})$ (for example, $a'_{n+2,n+2} = a_{n+2,n+2} - a_{1,n+2}(ra_{1,n+2} + ra_{n,n+2} + r(ra_{11} - ra_{n1})(1 + ra_{11})^{-1}a_{1,n+2}))$, are all invertible. Now observe $\eta = I_{2n} + r(a_{ij})_{2n\times 2n}$.

Since $1+ra_{11}$ is invertible, multiplying η on the left by $\rho_{1n}(-1)\rho_{n1}((ra_{11}-ra_{n1})(1+ra_{11}^{-1}))$, then by $\prod_{i\neq n+1}\rho_{i1}(-ra'_{i1})$ where ra'_{i1} are the coefficients in the first column of $\rho_{1n}(-1)\rho_{n1}\times((ra_{11} - ra_{n1})(1 + ra_{11})^{-1})\eta$ (denote the product by η_1 and denote $\prod_{i\neq n+1}\rho_{i1}(-ra'_{i1})\eta_1$ by η_2), we have the first column v_1 of η_2 has the form $(1, 0, \ldots, 0, \alpha, 0, \ldots, 0)^t$. Multiplying η_2 on the right by $\rho_{1,n+2}(-ra'_{1,n+2})\rho_{1,n+3}(-ra'_{1,n+3})$ (denote the product by η_3), we get that the first elements in the $(n + 2)$th, $(n + 3)$th columns of η_3 are zero. The $(n + 2, n + 2)-$ element $1 + ra'_{n+2,n+2}$ in η_3 is still invertible, multiplying η_3 on the left by $\rho_{2,2n}(1-ra'_{2,n+2}+\varepsilon^*h^*\varepsilon ra'_{n,n+2})\rho_{2n,n+2}(h)$, where $h = (1-ra'_{2n,n+2})(1+ra'_{n+2,n+2})^{-1}$, then by $\prod_{i\neq 1,n+1,n+2}\rho_{i2}(-ra''_{i,n+2})$ (denote the product by η_4), we get that the $(n+2)$th column v_{n+2} of η_4 has the form $(0, 1, 0, \ldots, 0, b, \beta, 0, \ldots, 0)^t$ and the first column v_1 keeps its form. Multiplying η_4 on the right by $\rho_{n+2,n+3}(-ra''_{2,n+3})$ (denote the product by η_5), we get that the second element of $(n + 3)$th column of η_5 is zero and the forms of v_1, v_{n+2} in η_5 are not changed, since $1 + ra''_{n+3,n+3}$ in η_5 is still invertible, continuing the above procedure, we may get an element θ in X has the required form and is not in $GU_{2n}R$. □

Remark 2.11.

(1) A vector $v \in R^{2n}$ is said to be unitary if $v^*\varphi_n v = 0$. If v_1 in θ is unitary, we may have $v_1 = (1, 0, \ldots, 0, 0, \ldots, 0)^t$; and if v_1, v_{n+2}, v_{n+3} are all unitary, we may have $v_1 = (1, 0, \ldots, 0, 0, \ldots, 0)^t$, $v_2 = (0, 1, \ldots, 0, b, 0, \ldots, 0)^t$, $v_3 = (0, 0, 1, 0, \ldots, 0, c_1, c_2, 0, \ldots, 0)^t$. When v_1 is unitary, no necessary to multiply η_1 on the right by $\rho_{1,n+2}(-a'_{1,n+2})\rho_{1,n+3}(-a'_{1,n+3})$, we may get that $a'_{1,n+2} = a'_{1,n+3} = 0$ by left multiplying η_2 by suitable elementary matrices.
(2) If $\alpha \bar{\in} R^{\varepsilon}$ in v_1, by Lemmas 2.3, 2.4, θ is certainly not in $GU_{2n}R$.

3. Proof of the theorem

Let X be an overgroup of $EU_{2n}R$ in $GL_{2n}R$ which is not in $GU_{2n}R$ where $n \geqslant 4$. By the proof of Lemma 2.10, X contains an element η with the form $I_{2n} + r(a_{ij})_{2n\times 2n}$, where r is a real number with sufficiently small $|r|$, which is not in $GU_{2n}R$.

Lemma 3.1. *Suppose that all columns of $\eta = I_{2n} + r(a_{ij})_{2n\times 2n}$ are unitary. Then X contains an elementary matrix $\xi_{ij}(a)$ with $a \in R$ and $j \neq \sigma i$ ($\xi_{ij}(a)$ is not in $GU_{2n}R$).*

Proof. Since $\eta = I_{2n} + r(a_{ij})_{2n\times 2n}$ is not in $GU_{2n}R$, there exists $\rho_{ij}(a)$, without loss of generality, assume that $\rho_{ij}(a) = \rho_{12}(1)$, such that $\xi = \eta\rho_{12}(1)\eta^{-1} = I_{2n} + v_1u_2 - v_{n+2}u_{n+1}$, where v_i is the ith column of η and u_j is the jth row of η^{-1}, is not in $U_{2n}R$. Note that v_1 and v_{n+2} have the form $(1 + ra_{11}, ra_{21}, \ldots, ra_{n+1,1}, \ldots, ra_{2n,1})^t$ and $(ra_{1,n+2}, \ldots, ra_{n+1,n+2}, 1 + ra_{n+2,n+2}, \ldots, ra_{2n,n+2})^t$, respectively. Refering the proof of Lemma 2.10, we may find $\theta \in EU_{2n}R$ such that $\theta v_1 = (1, 0, \ldots, 0, 0, \ldots, 0)^t$ and $\theta v_{n+2} = (0, 1, 0, \ldots, 0, b, 0, \ldots, 0)^t$. Let $u_2\theta^{-1} = (d_1, \ldots, d_n, d_{n+1}, \ldots, d_{2n})$ and $u_{n+1}\theta^{-1} = (f_1, \ldots, f_n, f_{n+1}, \ldots, f_{2n})^t$, respectively. We have

$$\tau=\theta\xi\theta^{-1}=I_{2n}+\theta v_1u_2\theta^{-1}-\theta v_{n+2}u_{n+1}\theta^{-1}$$

$$=\left(\begin{array}{ccccc|cccc}
1 & d_2 & d_3 & \cdots & d_n & d_{n+1} & d_{n+2} & \cdots & d_{2n}\\
 & 1-f_2 & -f_3 & \cdots & -f_n & -f_{n+1} & -f_{n+2} & \cdots & -f_{2n}\\
 & & 1 & & & & & & \\
 & & & \ddots & & & & & \\
 & & & & 1 & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & -h_1 & -h_2 & \cdots & -h_n & 1-h_{n+1} & -h_{n+2} & \cdots & -h_{2n}\\
 & & & & & & 1 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 1
\end{array}\right)\in X, \tag{3.1}$$

where $h_i=bf_i$ $(1\leqslant i\leqslant 2n)$.

(Note that (i) τ is not in $U_{2n}R$; (ii) $\tau^{-1}=I_{2n}-\theta v_1u_2\theta^{-1}+\theta v_{n+2}u_{n+1}\theta^{-1}$ has the same type (3.1) as τ; (iii) since $u_2v_1=u_{n+1}v_1=0, d_1=f_1=0$.)

(a) If there is $f_i\neq 0$, or $d_i\neq 0$, or $h_i\neq 0$ for $2\leqslant i\neq n+1, n+2\leqslant 2n$, without loss of generality saying $f_3\neq 0$, we have

$$\xi_{4,n+3}(-f_3)=\left[\rho_{42}(1),\left[\left[\tau,\rho_{24}(1)\right],\rho_{4,n+3}(1)\right]\right]\in X \tag{3.2}$$

(if $f_2\neq 0$, then $\xi_{34}(-f_2)=[\rho_{32}(1),[[\tau,\rho_{23}(1)],\rho_{34}(1)]]\in X$).

(b) $d_i=f_i=h_i=0$ for all $2\leqslant i\neq n+1, n+2\leqslant 2n$ in (3.1). In this case, if h_{n+1} or h_{n+2} is not zero, for example, $h_{n+1}\neq 0$, we may have

$$\xi_{24}(-h_{n+1})=\left[\rho_{2,n+1}(1),\left[\left[\tau,\rho_{n+1,3}(1)\right],\rho_{34}(1)\right]\right]\in X. \tag{3.3}$$

So, assume that $h_{n+1}=h_{n+2}=0$. Now if $d_{n+2}\neq\varepsilon^* f_{n+1}^*$, left multiplying τ by $\rho_{2,n+1}(f_{n+1})$, we get that the $(1,n+2)$-coefficient in $\rho_{2,n+1}(f_{n+1})\tau$ is $d_{n+2}-\varepsilon^* f_{n+1}^*\neq 0$ and we can show that X contains an elementary matrix $\xi_{ij}(a)$ with $j\neq\sigma i$. Thus, suppose that $d_{n+2}=\varepsilon^* f_{n+1}^*$. Since $\rho_{2,n+1}(f_{n+1})\tau$ is not in $U_{2n}R$, there is at least one of d_{n+1}^*, f_{n+2}^* not in R^ε (for symplectic case, we must have $f_{n+1}\neq -d_{n+2}$). Modifying the proof of Lemma 2.6 a little, we can get that X contains an elementary matrix $\xi_{ij}(a)$ with $j\neq\sigma i$. □

Lemma 3.2. *Suppose that there is at least one column which is not unitary in $\eta=I_{2n}+r(a_{ij})_{2n\times 2n}$. Then X contains an elementary matrix $\xi_{ij}(a)$ with $a\in R$ and $j\neq\sigma i$ ($\xi_{ij}(a)$ is not in $GU_{2n}R$).*

Proof. By Lemma 2.10. X contains an element (still denote it by η) having the form (2.1). We may assume that v_{n+2} in η is not unitary, i.e., $\beta\neq 0$ in v_{n+2} (see (2.1)), which without loss of generality. Keep the notation of v_1, v_{n+2} and v_{n+3} in (2.1) and let $u_2=(d_1,\dots,d_n,d_{n+1},\dots,d_{2n})$, $u_3=(g_1,\dots,g_n,g_{n+1},\dots,g_{2n})$ and $u_{n+1}=(f_1,\dots,f_n,f_{n+1},\dots,f_{2n})$, where u_j is the jth row of η^{-1}.

Observe

$$\tau=\eta\rho_{12}(1)\eta^{-1}=I_{2n}+v_1u_2-v_{n+2}u_{n+1}$$
$$=\begin{pmatrix}
1+d_1 & d_2 & d_3 & \cdots & d_n & \vdots & d_{n+1} & d_{n+2} & d_{n+3} & \cdots & d_{2n}\\
-f_1 & 1-f_2 & -f_3 & \cdots & -f_n & \vdots & -f_{n+1} & -f_{n+2} & -f_{n+3} & \cdots & -f_{2n}\\
 & & 1 & & & \vdots & & & & & \\
 & & & \ddots & & \vdots & & & & & \\
 & & & & 1 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \\
r_1 & r_2 & r_3 & \cdots & r_n & \vdots & 1+r_{n+1} & r_{n+2} & r_{n+3} & \cdots & r_{2n}\\
-h_1 & -h_2 & -h_3 & \cdots & -h_n & \vdots & -h_{n+1} & 1-h_{n+2} & -h_{n+3} & \cdots & -h_{2n}\\
 & & & & & \vdots & & & 1 & & \\
 & & & & & \vdots & & & & \ddots & \\
 & & & & & \vdots & & & & & 1
\end{pmatrix}\in X, \tag{3.4}$$

where $r_i=\alpha d_i-bf_i$, $1\leqslant i\leqslant 2n$; $h_i=\beta f_i$, $1\leqslant i\leqslant 2n$.

(a) If there is $d_i\neq 0$ or $f_i\neq 0$ for $3\leqslant i\neq n+1, n+2\leqslant 2n$, we can obtain that X contains an elementary matrix $\xi_{ij}(a)$ with $j\neq\sigma i$ by the same method in Lemma 3.1.

(b) Assume that $d_i=f_i=0$ for all $3\leqslant i\neq n+1, n+2\leqslant 2n$ in the 2nd row and $(n+1)$th row of η^{-1}, respectively. Consider

$$\tau'=\eta\rho_{13}(1)\eta^{-1}=I_{2n}+v_1u_3-v_{n+3}u_{n+1}$$
$$=\begin{pmatrix}
1+g_1 & g_2 & g_3 & \cdots & g_n & \vdots & g_{n+1} & g_{n+2} & g_{n+3} & \cdots & g_{2n}\\
0 & 1 & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & \cdots & 0\\
-f_1 & -f_2 & 1 & \cdots & 0 & \vdots & -f_{n+1} & -f_{n+2} & 0 & \cdots & 0\\
 & & & \ddots & & \vdots & & & & & \\
 & & & & 1 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
r'_1 & r'_2 & r'_3 & \cdots & r'_n & \vdots & 1+r'_{n+1} & r'_{n+2} & r'_{n+3} & \cdots & r'_{2n}\\
-h'_1 & -h'_2 & 0 & \cdots & 0 & \vdots & -h'_{n+1} & 1-h'_{n+2} & 0 & \cdots & 0\\
-l'_1 & -l'_2 & 0 & \cdots & 0 & \vdots & -l'_{n+1} & -l'_{n+2} & 1 & & \\
 & & & & & \vdots & & & & \ddots & \\
 & & & & & \vdots & & & & & 1
\end{pmatrix}\in X, \tag{3.5}$$

where $r'_i=\alpha g_i-c_1f_i$, $1\leqslant i\leqslant 2n$; $h'_i=c_2f_i$, $l'_i=\gamma f_i$, $i=1,2,n+1,n+2$.

Same to (a), if there is $f_i \neq 0$ for $i = 2, n+2$, or $g_i \neq 0$ for $4 \leqslant i \neq n+1, n+3 \leqslant 2n$, we can obtain that an elementary matrix $\xi_{ij}(a)$ with $j = \sigma i$ lies in X. For instance, if $f_{n+2} \neq 0$, then

$$\xi_{n+2,n}(-f_{n+2}) = \left[\rho_{n+2,3}(1), \left[\tau', \rho_{n+2,n}(1)\right]\right] \in X. \tag{3.6}$$

(c) Investigating $\tau'' = \eta\rho_{n+2,3}(1)\eta^{-1} = I_{2n} + v_{n+2}u_3 - \varepsilon v_{n+3}u_2$, we can show that if $d_{n+1} \neq 0$, then X contains an elementary matrix $\xi_{ij}(a)$ with $j \neq \sigma i$ by the same method in Lemma 3.1.

Now, assume that $d_i = 0$ for all $3 \leqslant i \neq n+2 \leqslant 2n$ in the 2nd row u_2 and $f_i = 0$ for all $2 \leqslant i \neq n+1 \leqslant 2n$ in the $(n+1)$th row of η^{-1}, respectively. Since $u_2 v_1 = u_{n+1} v_1 = 0$ and $d_{n+1} = 0$, we have $d_1 = 0$ and $f_1 + f_{n+1}\alpha = 0$. Because u_{n+1} is unimodular (a vector $u = (c_1, \ldots, c_n)$ is called unimodular if there are $d_1, \ldots, d_n \in R$ such that $c_1 d_1 + \cdots + c_n d_n = 1$), f_{n+1} should be a unit in R. We have

$$\zeta = \left[\eta, p_{31}(1)\right]$$
$$= \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & \vdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & \vdots & 0 & 0 & f_{n+1} & \cdots & 0 \\
0 & -d_2 & 1 & \cdots & 0 & \vdots & 0 & -d_{n+2} & 0 & \cdots & 0 \\
 & & & \ddots & & \vdots & & & & & \\
 & & & & 1 & \vdots & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & & \vdots & 1 & 0 & bf_{n+1} & & \\
 & & & & & \vdots & & 1 & \beta f_{n+1} & & \\
 & & & & & \vdots & & & 1 & & \\
 & & & & & \vdots & & & & \ddots & \\
 & & & & & \vdots & & & & & 1
\end{pmatrix} \in X. \tag{3.7}$$

Write $\zeta = \rho_{32}(-d_2)\rho_{2,n+3}(f_{n+1})\zeta'$, then $\zeta' \in X$. It is not difficult to show that if $b \neq 0$, or $\beta f_{n+1} \neq d_2^*$, or $d_{n+2} \neq \varepsilon^* f_{n+1}^*$, X contains an elementary matrix $\xi_{ij}(a)$ with $j \neq \sigma i$. Otherwise, i.e., $b = 0$, $\beta f_{n+1} = d_2^*$, and $d_{n+2} = \varepsilon^* f_{n+1}^*$, $\zeta' = \xi_{3,n+3}(d_2 f_{n+1}) \in X$. Since $d_2 = \varepsilon^* f_{n+1}^* \beta^* \varepsilon$ and $\beta^* \bar{\in} R^\varepsilon$, we have $d_2 f_{n+1} \neq -\varepsilon^* f_{n+1}^* d_2^*$ (note that f_{n+1} is invertible), so $\zeta' = \xi_{3,n+3}(d_2 f_{n+1}) \bar{\in} U_{2n}R$. By Lemma 2.6, X contains an elementary matrix $\xi_{ij}(a)$ with $j \neq \sigma i$. □

Lemma 3.3. *Let X be an overgroup of $EU_{2n}R$ in $GL_{2n}R$. Then there is a unique dual ideal J of R such that $EEU_{2n}J = EU_{2n}R \cdot E_{2n}(R, J) \subseteq X$.*

Proof. If $X \subseteq GU_{2n}R$, then $EU_{2n}R \cdot E_{2n}(R, 0) = EU_{2n}R \subseteq X$.

Now suppose that $X \nsubseteq GU_{2n}R$. By Lemma 2.10 there is $\eta = I_{2n} + r(a_{ij})_{2n\times 2n}$ with a real number r with sufficiently small $|r|$ in X which is not in $GU_{2n}R$. By Lemmas 3.1, 3.2 and 2.6, we may get $E_{2n}J_a$ for some ideal J_a of R which is generated by a and a^* lies in X. Let $J = \{x \in R \mid E_{2n}J_x \subseteq X\}$. It is easy to show that J is an ideal of R and satisfies $J^* = J$. Thus $EU_{2n}R \cdot E_{2n}(R, J) \subseteq X$ by Lemma 2.7.

Denote $\bar{R} = R/J$, and $\bar{X} = \lambda_J(x)$, where λ_J denotes the group homomorphism: $GL_{2n}R \to GL_{2n}(R/J)$. We have $EU_{2n}\bar{R} \subseteq \bar{X}$. If $\bar{X} \nsubseteq GU_{2n}\bar{R}$, since $\bar{R}$ still satisfies property (Δ), there exists some $\xi_{ij}(\bar{a})$ with $j \neq \sigma i$ and $\bar{0} \neq \bar{a} \in \bar{R}$ in $\overline{X}$ by Lemmas 3.1, 3.2. Note that $a \bar{\in} J$. Thus there is $\theta \in X$ such that $\lambda_J(\theta) = \lambda_J(\xi_{ij}(a))$. Take $\tau = \xi_{ij}(-a)\theta \in \ker\lambda_J$. Choose $\rho = \rho_{k,\sigma j}(1)$. By [9], $[\rho, \tau] \in E_{2n}(R, J) \subseteq X$. Since $\xi_{ij}(a)[\rho, \tau]\xi_{ij}(-a) \in EU_{2n}(R, J) \subseteq X$, we have

$$\xi_{i,\sigma k}(c) = \big[\xi_{ij}(a), \rho_{k,\sigma j}(1)\big] = \xi_{ij}(a)\big[\rho_{k,\sigma j}(1), \tau\big]\xi_{ij}(-a)\big[\theta, \rho_{k,\sigma j}(1)\big] \in X$$

where $c = -a\varepsilon^*$ when $j, k \leqslant n$; $c = -a\varepsilon$ when $n + 1 \leqslant j, k$; $c = -a$ when $j \leqslant n < k$ or $k \leqslant n < j$. This is contradictory to that $a \bar{\in} J$. Thus $\bar{X}$ must be in $GU_{2n}\bar{R}$. Hence J is maximal such that $EU_{2n}R \cdot E_{2n}(R, J) \subseteq X$, and is uniquely determined. □

Now let us complete the proof of theorem.

By Lemma 3.3, we need only to show that $X \subseteq CGU_{2n}J$. Since J is the maximal ideal of R such that $E_{2n}(R, J) \subseteq X$ and $\lambda_J(EEU_{2n}J) = EU_{2n}(R/J)$, $\lambda_J(X)$ should be in $GU_{2n}(R/J)$ by the proof of Lemmas 3.3 and 2.5, hence $X \subseteq \lambda_J^{-1}(\lambda_J(X)) \subseteq \lambda_J^{-1}(GU_{2n}(R/J))$. Since $X \subseteq GL_{2n}R$, so, $X \subseteq \lambda_J^{-1}(GU_{2n}(R/J)) \cap GL_{2n}R = \{g \in GL_{2n}R \mid \lambda_J(g) \in GU_{2n}(R/J)\} = CGU_{2n}J$.

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