A Comparison Result for a Class of Quasilinear Elliptic Partial Differential Equations*

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A comparison theorem and a uniqueness corollary for positive solutions to the equation

\[ \sum_{i=1}^{n} \left( p_i(x, u)u_{x_i} \right)_{x_i} + q(x, u)u = 0 \]

on the closure of a bounded open set are found. The important hypotheses on the nonlinear coefficients are that each \( p_i \) is positive and monotone increasing in \( u \) while \( q \) is monotone decreasing in \( u \). An application is made to equations arising in the theory of chemical reactors.

INTRODUCTION

A comparison and uniqueness result for ordinary differential equations obtained by the author [4] has been extended to a class of quasilinear partial differential equations. Two solutions, \( u \) and \( v \), to two related differential inequalities are compared. Previous results compared \( u \) and \( v \) on sets which were "rigged" to have a boundary smooth enough to allow application of Green's theorem as is the case in [1, Theorem 2] ([2, 3, 8] contain extensive bibliographies on comparison and oscillation results). In contrast, the present theorem compares \( u \) and \( v \) directly on \( G = \{ x \mid u(x) > v(x) \} \) using only the fact that \( G \) is open and bounded. The result, however, is restricted to equations without mixed derivatives. Finally, an application is made to two classes of equations arising in chemical reactor theory.

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THEOREM. Let $D$ be a bounded open subset of $\mathbb{R}^n$ and let $p_i(x, u), i = 1, \ldots, n$ be $C^1$ and $q$ be $C^0$ on $D \times [0, \infty)$. Further, let $p_i$ be positive and monotone increasing in $u$ for $i = 1, \ldots, n$ and let $q$ be monotone decreasing in $u$. If $u$ and $v$ are $C^1$ on $D$ with $p_i(x, u) u_{x_i}$ and $p_i(x, v) v_{x_i}$ $C^1$ on $D$ for $i = 1, \ldots, n$,

(i) $v \geq u$ on $\partial D$,

(ii) $v > 0$ on $\overline{D}$, $u \geq 0$ on $\overline{D}$,

(iii) $\sum_{i=1}^{n} (p_i(x, v) v_{x_i})_{x_i} + q(x, v) v \leq 0$,

and

(iv) $\sum_{i=1}^{n} (p_i(x, u) u_{x_i})_{x_i} + q(x, u) u \geq 0$, then $v \geq u$ on $\overline{D}$.

Remark. If the monotonicity hypotheses on $p$ and $q$ hold only for $u$ and $v$ in an interval $I$, then the conclusion holds for $u$ and $v$ if $u(x)$ and $v(x)$ are in $I$ when $x$ is in $D$.

Proof. Suppose the theorem is false. Let $G = \{x \mid x$ in $D, u(x) > v(x)\}$. Then $G$ is a bounded open subset of $\mathbb{R}^n$, $G \subset D$ and $u = v$ on $\partial G$.

We shall make use of the following Picone identity.

$$
\sum_{i=1}^{n} \left( \frac{u}{v} \left( p_i(x, u) u_{x_i} v - p_i(x, v) v_{x_i} u \right) \right)_{x_i} - u \left( \sum_{i=1}^{n} (p_i(x, u) u_{x_i})_{x_i} + q(x, u) u \right) - v \left( \frac{v}{u} \right)^2 \left( \sum_{i=1}^{n} (p_i(x, v) v_{x_i})_{x_i} + q(x, v) v \right) + \sum_{i=1}^{n} (p_i(x, u) - p_i(x, v))(u_{x_i})^2 + \sum_{i=1}^{n} p_i(x, v) \left( u_{x_i} - \frac{u}{v} v_{x_i} \right)^2 + (q(x, v) - q(x, u)) u^2.
$$

We assert that the right-hand side of (#) is positive at some point of $G$, and therefore on an open subset of $G$. Consider any $i$. Let $\bar{x}$ be a point of $G$ and let $l_i$ be the line $x_j = \bar{x}_j, j \neq i$. Consider next the restriction of $u$ and $v$ to $l_i \cap G$, which is an open subset of $l_i$ in the $\mathbb{R}^1$ topology. Let $I$ be the maximal interval in $l_i \cap G$ containing $\bar{x}$. At the endpoints of $I$, $u = v$, while $u(\bar{x}) \neq v(\bar{x})$. Now suppose $u_{x_i} - (u/v) v_{x_i} = 0$ for all $x$ in $G$. Then $u/v$ is constant on $I$. But then $u = v$ on $I$, a fact which contradicts $u(\bar{x}) > v(\bar{x})$. 


Thus, at some point of \( G \), \( u_{x_i} - (u/v) v_{x_i} \neq 0 \) and at that point the right-hand side of (\#) is positive.

The integral over \( G \) of the right-hand side of (\#) is positive. The theorem will be proved once we show that the integral over \( G \) of the left-hand side is nonpositive.

Consider again any \( i \). By Fubini's theorem [7, Chap. 7],

\[
\int_{G} \left( \frac{u}{v} \left( p_i(x, u) u_{x_i} v - p_i(x, v) v_{x_i} u \right) \right)_{x_i} \, dV_n = \int_{G_i} \left( \int_{O_i(x)} \frac{u}{v} \left( p_i(x, u) u_{x_i} v - p_i(x, v) v_{x_i} u \right) \right)_{x_i} \, dx_i \, dV_{n-1}
\]

where \( G_i \) is the projection of \( G \) onto the plane \( x_i = 0 \) and for each \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_{i-1}, 0, \bar{x}_{i+1}, \ldots, \bar{x}_n) \) in \( G_i \), \( O_i(\bar{x}) \) is the intersection of the line \( l_i : x_j = \bar{x}_j, j \neq i \), with \( G \). Each \( O_i(\bar{x}) \) is an open subset of \( l_i \) in the \( R^1 \) topology and consequently is an at most countable collection of open intervals. Let \( I \) be such an interval with \( a \) and \( b \) the minimum and maximum values of \( x_i \) in \( I \). Then \( u(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) = v(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \) for \( x_i = a, b \), while \( u(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) > v(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \) when \( a < x_i < b \). In particular, \( u_{x_i}(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \geq v_{x_i}(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \) when \( x_i = a \) and the opposite inequality holds when \( x_i = b \). Thus

\[
\int_{I} \left( \frac{u}{v} \left( p_i(x, u) u_{x_i} v - p_i(x, v) v_{x_i} u \right) \right)_{x_i} \, dx_i = p_i(x, u(x)) \left( u(x)u_{x_i}(x) - v_{x_i}(x) \right) \bigg|_{x=(\bar{x}_1,\ldots,\bar{x}_{i-1},a,\bar{x}_{i+1},\ldots,\bar{x}_n)} - p_i(x, u(x)) \left( u(x)u_{x_i}(x) - v_{x_i}(x) \right) \bigg|_{x=(\bar{x}_1,\ldots,\bar{x}_{i-1},b,\bar{x}_{i+1},\ldots,\bar{x}_n)} \leq 0.
\]

It now follows easily that the integral of the left-hand side of (\#) is nonpositive.

**Corollary.** Let \( D, p_i, i = 1, \ldots, n \) and \( q \) be as in the theorem. Then the boundary value problem

\[
\sum_{i=1}^{n} (p_i(x, u) u_{x_i})x_i + q(x, u)u = 0 \quad \text{in } D, \quad (*)
\]

\[
u = \varphi > 0 \quad \text{on } \partial D \quad (**) 
\]
has at most one solution $u$ that is positive, $C^1$ on $D$ and such that $p_t(x,u)u_x$ is $C^1$ on $D$. Further, the solution $u(\cdot, \varphi)$ to the boundary value problem (*)-(**), when it exists, is monotone increasing in $\varphi$, i.e., $\varphi_1(x) \leq \varphi_2(x)$ for $x$ in $\partial D$ implies that $u(x; \varphi_1) \leq u(x; \varphi_2)$ for $x$ in $D$.

### Applications

Consider the boundary value problem

$$\sum_{i=1}^n (p_i(x,u)u_x)_x + f(x,u) = 0 \quad \text{in } D, \quad (*)$$

$$u = \varphi > 0 \quad \text{on } \partial D, \quad (**)$$

where $D$ and $p_t$, $i = 1, \ldots, n$ are as in the theorem.

First let $f$ be as in [6]. That is, $f$ is $C^2$ w.r.t. $u$ and has the properties (i) $f(x,u) > 0$ for $x$ in $D$ and $u > 0$, and (ii) there is a $c > 0$ for which $f_{uu}(x,u) \leq 0$ for $x$ in $D$ and $c \leq u$. An $f$ which arises in applications is $f(x,u) = f_0(x,u) = \lambda \exp(-1/u)$. Reference [6] contains further references to applications in chemical reactor theory. Property (ii) implies that there is a $\tau_0 > 0$ with $\tau_0 < c$ such that $f(x,u + \tau)/u$ is decreasing in $u$ for each $\tau > \tau_0$ and $x$ in $D$ (cf. [6, Corollary 5.3.11]). $\tau_0 = \frac{1}{4}$ when $f = f_0$.

The maximum principle shows that a positive solution to (*) achieves its minimum on $\partial D$. Let $\tau \geq \tau_0$. Applying the theorem after replacing $u$ by $u + \tau$ shows (*)-(**) has at most one positive solution when $\varphi > \tau$.

Next, consider the case where $f$ is $C^1$ w.r.t. $u$ and there is a constant $c > 0$ such that (i) $f(x,u) < 0$ for $c < u$ and $x$ in $D$ and (ii) $(f(x,u)/u)_u \leq 0$ for $0 < u < c$ and $x$ in $D$. Conditions (i) and (ii) are weaker than H-3 and H-4 of [5]. An application of the maximum principle shows that any solution of (*) satisfies $u \leq c$ on $D$ if $\varphi \leq c$ on $D$. The theorem can then be applied to show that (*)-(**) has at most one positive solution in the case that $0 < \varphi \leq c$ on $\partial D$.

### References

