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ON THE ASYMPTOTICS OF MORSE NUMBERS OF FINITE COVERS OF MANIFOLDS

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Let M be a closed connected manifold. We denote by $\mathscr{M}(M)$ the Morse number of M, i.e. the minimal possible number of critical points of a Morse function f on M. M.Gromov posed the following question: Let N_k, $k \in \mathbb{N}$ be a sequence of manifolds, such that each N_k is an a_k-fold cover of M where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathscr{M}(N_k)$ as $k \to \infty$?

In this paper we study the case $\pi_1(M) \approx \mathbb{Z}^m$, $\dim M \geqslant 6$. Let $\xi \in H^1(M, \mathbb{Z})$, $\xi \neq 0$. Let $M(\xi)$ be the infinite cyclic cover corresponding to ξ, with generating covering translation $t: M(\xi) \to M(\xi)$. Let $M(\xi, k)$ be the quotient $M(\xi)/t^k$. We prove that $\lim_{k\to\infty} \mathscr{M}(M(\xi, k))/k$ exists. For ξ outside a subset $\mathfrak{M} \subset H^1(M)$ which is the union of a finite family of hyperplanes, we obtain the asymptotics of $\mathscr{M}(M(\xi, k))$ as $k \to \infty$ in terms of homotopy invariants of M related to the Novikov homology of M. It turns out that the limit above does not depend on ξ (if $\xi \notin \mathfrak{M}$). Similar results hold for the stable Morse numbers. Generalizations for the case of non-cyclic coverings are obtained.

0. INTRODUCTION AND THE STATEMENT OF THE RESULT

Let M be a closed connected smooth manifold. Denote by $\mathscr{M}(M)$ the Morse number of M, i.e. the minimal possible number of critical points of a Morse function on M. In the case $\pi_1(M) = 0$, $\dim M \geqslant 6$, this number is easily computable in terms of homology of M (see [14]). In the case of arbitrary fundamental group (even for $\dim M \geqslant 6$), the number $\mathscr{M}(M)$ is very difficult to compute: it depends on the simple homotopy type of M, the relevant algebraic constructions are rather complicated, and it is not easy to extract the needed numerical invariant (see [15] or [16, Chapter 7]).

M.Gromov posed the following question:

Let N_k, $k \in \mathbb{N}$ be a sequence of manifolds, such that each N_k is an a_k-fold cover of the manifold M where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\mathscr{M}(N_k)$ as $k \to \infty$?

In the present article we study the problem for $\pi_1(M)$ free abelian and $\dim M \geqslant 6$. To formulate our results, we need some terminology from algebra. Denote $\mathbb{Z}[\mathbb{Z}^m]$ by Λ. Let

$$C_* = \{0 \leftarrow C_0 \xleftarrow{\partial_1} C_1 \ \dots \ \xleftarrow{\partial_k} C_k \leftarrow 0\}$$

be a free finitely generated Λ-complex. Denote by $B_i(C_*)$ the rank of the module $H_i(C_*) \otimes_\Lambda \{\Lambda\}$ over the field of fractions $\{\Lambda\}$. Denote by $B(C_*)$ the sum of all $B_i(C_*)$. Consider now the homomorphism $\partial_{i+1}: C_{i+1} \to C_i$, and let $d = \operatorname{rk} C_i$. Recall that the Fitting invariant $\mathscr{F}_t$ of the homomorphism ∂_{i+1} (see, e.g. [4, p. 278]) is the ideal of Λ generated by the $(d-t)\times(d-t)$ subdeterminants of the matrix of ∂_{i+1} (for $t \geqslant d$ one sets $\mathscr{F}_t = \Lambda$ by definition). We shall denote the sequence $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_d$ of the Fitting invariants by $F(i)$.

Define the *reduced Fitting sequence* for ∂_{i+1} to be the sequence

$$FR(i) \qquad \mathscr{F}_s \subset \cdots \subset \mathscr{F}_r$$

where $\mathscr{F}_s$, and $\mathscr{F}_r$, are, respectively, the first and last terms of the Fitting sequence $F(i)$, not equal to 0, respectively to Λ. The sequence $F(i)$ is not a homotopy invariant of C_*, but the sequence $FR(i)$ is (see e.g. [16, Chapter 4, Section 2]). We say that an ideal J of Λ is *numerically prime* if there is no number $l \in \mathbb{Z}$, $l \neq \pm 1$, such that every $R \in J$ is divisible by l, and we denote by $Q_i(C_*)$ the number of ideals in the sequence $FR(i)$ which are *not* numerically prime. Denote by $Q(C_*)$ the sum of all $Q_i(C_*)$.

A subgroup $G \subset \mathbb{Z}^m$ will be called an *integral hyperplane* if it is a direct summand of $\mathbb{Z}^m$ of rank $m-1$.

Now let M be a closed connected manifold, $\pi_1(M) \approx \mathbb{Z}^m$, $m \geqslant 1$. It is convenient to set $m = n+1$, $n \geqslant 0$. For every non-zero $\xi \in H^1(M)$ there is a unique connected infinite cyclic covering $\mathscr{P}_\xi : M(\xi) \to M$ such that $\mathscr{P}_\xi^*(\xi) = 0$. Denote by $M(\xi, k) \to M$ the k-fold cyclic covering of M obtained from $\mathscr{P}_\xi$. Let $C_*(\tilde{M})$ be the cellular chain complex of the universal cover $\tilde{M}$. We shall abbreviate $B(C_*(\tilde{M}))$ to $B(M)$ and $Q(C_*(\tilde{M}))$ to $Q(M)$.

Main Theorem. *Let* $\dim M \geqslant 6$, $\pi_1(M) \approx \mathbb{Z}^{n+1}$, $n \geqslant 0$. *Then*

(1) *For any non-zero* $\xi \in H^1(M)$ *the limit* $\lim_{k\to\infty} \mathscr{M}(M(\xi, k))/k$ *exists.*

(2) *There is a subset* $\mathfrak{M} \subset H^1(M)$ *which is a finite union of integral hyperplanes in* $H^1(M)$, *and for every non-zero* $\xi \notin \mathfrak{M}$ *there is a real number* a *such that for every* $k \in \mathbb{N}$ *we have*

$$k(B(M) + 2Q(M)) - a \leqslant \mathscr{M}(M(\xi, k)) \leqslant k(B(M) + 2Q(M)) + a.$$

Remarks. (1) A similar result holds for the stable Morse numbers of M, see Section 5.

(2) The limit $\lim_{k\to\infty} \mathscr{M}(M(\xi, k))/k$ will be denoted by $\mu(M, \xi)$. The second point of the Main Theorem implies that for a "generic" cohomology class ξ we have $\mu(M, \xi) = B(M) + 2Q(M)$.

(3) Denote by $\mathscr{M}_i(M)$ the minimal number of critical points of index i of a Morse function on M. The methods of the present paper allow also to prove that (under the assumptions of the Main Theorem) the limit $\lim_{k\to\infty} \mathscr{M}_i(M(\xi, k))/k$ exists, and that for all ξ, except those belonging to a finite union of integral hyperplanes, there is a real number a such that for every natural k we have

$$k(B_i(M) + Q_i(M) + Q_{i-1}(M)) - a \leqslant \mathscr{M}_i(M(\xi, k)) \leqslant k(B_i(M) + Q_i(M) + Q_{i-1}(M)) + a.$$

(4) The numbers $B_s(M)$, $Q_s(M)$ are closely related to the Novikov homology of M. Namely, $B_s(M)$ is equal to the Novikov Betti number $b_s(M, \xi)$ [7] for every non-zero class $\xi \in H^1(M)$ (note that $B_s(M)$ is also equal to the L^2-Betti number $b_s^{(2)}(M)$). Further, for every non-zero $\xi \notin \mathfrak{M}$ we have $Q_s(M) \leqslant q_s(M, \xi)$ where $q_s(M, \xi)$ is the Novikov torsion number [7] (that follows from Remark 2.6 and Proposition 3.3 of the present paper).

The proof is outlined as follows. Assume that $\xi \in H^1(M)$ is indivisible. Let $f: M \to S^1$ be a Morse map, representing ξ, and let $V = f^{-1}(\lambda)$ be a regular level surface of f. We can assume that V is connected and that $\pi_1(V) \to \pi_1(M)$ is an isomorphism onto $\operatorname{Ker} \xi$. Cut M along V, and obtain a cobordism W such that the boundary ∂W has two connected components $\partial_0 W$ and $\partial_1 W$, each diffeomorphic to V. The cyclic cover $M(\xi)$ is the union of a countable family of copies of W glued successively to each other. The union W_k of k successive copies is a cobordism. Its boundary ∂W_k has two connected components $\partial_0 W_k$

and $\partial_1 W_k$, each diffeomorphic to V (see Section 4 for details). We show that $\mathscr{M}(M(\xi, k))$ and $\mathscr{M}(W_k, \partial_0 W_k)$ have the same asymptotics as $k \to \infty$ (see Section 4). Further, $\mathscr{M}(W_k, \partial_0 W_k)$ is equal to the Morse number of the $\mathbb{Z}[\mathbb{Z}^n]$-complex $C_*(\tilde{W}_k, \widetilde{\partial_0 W_k})$, see Section 1 for definitions. It turns out that the asymptotic behaviour of this Morse number (as $k \to \infty$) depends only on the chain homotopy type of $C_*(\widetilde{M(\xi)})$ (moreover it depends only on the Novikov completion of this complex). The definition and the properties of the corresponding invariant of chain complexes are the subject of Sections 1–3 of the paper. These sections are purely algebraic. It follows from the author's earlier result [9] that for ξ outside a finite union of integral hyperplanes in $H^1(M)$, the Novikov-completed chain homotopy type of $C_*(\widetilde{M(\xi)})$ is easily computable. (This is the subject of the second half of Sections 2 and of Sections 3.) This leads to the effective computation of the asymptotics presented in the main theorem.

1. MORSE NUMBERS OF CHAIN COMPLEXES

In this section we define the notion of the Morse number for arbitrary chain complexes over $\mathbb{Z}[\mathbb{Z}^n]$ and we develop some basic properties of these numbers. We assume that the reader is familiar with Section 3 of [15] and with Section 1 of [11]. We denote $\mathbb{Z}[\mathbb{Z}^n]$ by R.

Terminological remark

Let A_*, B_* be chain complexes. We shall denote the chain maps from A_* to B_* as follows: $f_*: A_* \to B_*$, so that f_k is a homomorphism $A_k \to B_k$.

Definition 1.1. An *R-complex* is a chain complex $\{0 \leftarrow C_0 \leftarrow C_1 \cdots \leftarrow C_k \leftarrow 0\}$ of finitely generated R-modules. The *length* $l(C_*)$ of an R-complex C_* is the maximal number l such that $C_l \neq 0$. An R-complex C_* is called a *free R-complex* (or simply *f-complex*) if every C_i is a free finitely generated module over R.

Definition 1.2 (Sharko [15]). Let C_* be an f-complex over R. The minimal possible number of free generators of an f-complex D_*, having the same homotopy type as C_*, is called the *Morse number of* C_* and denoted by $\mathscr{M}(C_*)$ (or by $\mathscr{M}_R(C_*)$, if we want to stress the base ring).

One of the consequences of the Quillen–Suslin theorem [13, 17] is that R is an s-ring, i.e. every projective R-module is free (see [4, Chapter 5, Section 4]). R is also an IBN-ring, i.e. the number of free generators of a free module is uniquely determined. Therefore, in the homotopy type of every f-complex over R there exists a minimal chain complex, i.e. a complex D_* such that the number of free generators of D_* in each dimension is minimal over all the free complexes in this homotopy type (see [15, Theorem 3.7]).

Definition 1.3. Let A_* be an R-complex. We call a *free model* of A_* a free R-complex A'_* together with a chain map $\alpha_*: A'_* \to A_*$ which is epimorphic and induces an isomorphism in homology.† If $\alpha_*: A'_* \to A_*$, $\beta_*: B'_* \to B_*$ are free models, and $f_*: A_* \to B_*$ is a chain map, then a chain map $F_*: A'_* \to B'_*$ is called *covering* of f if $\beta_* F_* = f_* \alpha_*$. Similar terminology is accepted for chain homotopies.

†Sometimes we shall say (by abuse of terminology) that the complex A'_* itself is a free model of A_*.

Lemma 1.4. *Let A_* be an R-complex. Then there is a free model of A_* and*

(1) *Every chain map $A_* \to B_*$ admits a covering with respect to any free models of A_* and B_*.*
(2) *Let $h_*: A_* \to B_{*+1}$ be a chain homotopy from f_* to g_*, and F_*, G_* be coverings of f_*, g_*, respectively, with respect to some free models of A_*, B_*. Then there is a chain homotopy H_* from F_* to G_*, covering h_*.*
(3) *Two free models of a complex A_* are homotopy equivalent.*

Proof. To prove the existence of a free model, we proceed by induction in the length of A_*. If $l(A_*) = 0$, then it follows from the fact that every finitely generated module over R has a free finite resolution of finite length. To make the induction step, it suffices to construct a free model for a complex of the type $C_* = \{0 \leftarrow A_0 \overset{\partial_1}{\leftarrow} C_1 \overset{\partial_2}{\leftarrow} C_2 \cdots \overset{\delta_n}{\leftarrow} C_n \leftarrow 0\}$, where C_i are free finitely generated modules and A_0 is a finitely generated module. Let $B_* = \{0 \leftarrow A_0 \overset{\varepsilon}{\leftarrow} E_0 \overset{d_1}{\leftarrow} E_1 \overset{d_2}{\leftarrow} \cdots\}$ be a finite free resolution of A_0. There is a chain map $\phi_*: C_* \to B_*$, such that $\phi_0 = \mathrm{id}$. Define now an R-complex

$$F_* = \{0 \leftarrow E_0 \overset{D_1}{\leftarrow} C_1 \oplus E_1 \overset{D_2}{\leftarrow} C_2 \oplus E_2 \cdots\}$$

setting $D_1(c_1, e_1) = \phi_1(c_1) + d_1(e_1)$ and $D_i(c_i, e_i) = (\partial_i(c_i), d_i(e_i) + (-1)^{i+1}\phi_i(c_i))$ for $i \geqslant 2$.

Define further a map $\gamma_*: F_* \to C_*$ to be the projection $(x, y) \mapsto x$ when $* \geqslant 1$ and set $\gamma_0 = \varepsilon$. It is easy to check that F_* is indeed an f-complex, and that γ_* is a free model. The points (1) and (2) of our lemma are proved by a standard homological algebra argument; (3) follows from (2). □

Definition 1.5. The *Morse number* $\mathscr{M}(C_*)$ of a complex C_* is the Morse number of any of its free models.

Proposition 1.6. *Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of R-complexes. Then* (1) $\mathscr{M}(B_*) \leqslant \mathscr{M}(A_*) + \mathscr{M}(C_*)$, *and* (2) $\mathscr{M}(A_*) \leqslant \mathscr{M}(C_*) + \mathscr{M}(B_*)$.

Proof. (1) The following lemma reduces the assertion to the case of free R-complexes.

Lemma 1.7. *Let $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ be an exact sequence of R-complexes. Then there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longleftarrow & A'_* & \longleftarrow & B'_* & \longleftarrow & C'_* & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow \\
0 & \longleftarrow & A_* & \longleftarrow & B_* & \longleftarrow & C_* & \longleftarrow & 0
\end{array}$$

where $\alpha_*, \beta_*, \gamma_*$ are free models.

Proof of the Lemma. Let $g'_*: C'_* \to B'_*$ be a covering of $C_* \to B_*$ with respect to some free models B'_*, C'_*. We can assume that g'_* is a monomorphism onto a direct summand (the proof repeats almost verbatim the proof of Lemma 1.8 from [11] and will be omitted). Now, setting $A'_* = B'_*/\mathrm{Im}\, C'_*$, we obtain the first line of the commutative diagram above. □

For the case of free complexes the assertion follows from the next one.

LEMMA 1.8. *Let* $0 \leftarrow A_* \leftarrow B_* \leftarrow C_* \leftarrow 0$ *be an exact sequence of free R-complexes. Then there is an exact sequence* $0 \leftarrow A'_* \leftarrow B'_* \leftarrow C'_* \leftarrow 0$ *of free R-complexes such that* $A'_* \sim A_*$, $B'_* \sim B_*$, $C'_* \sim C_*$, *and* A'_*, C'_* *are minimal.*

The proof of this lemma is an exercise in the theory of minimal complexes ([16, Section 4], and will be left to the reader.

To prove (2) observe that there is an exact sequence $\mathscr{X} = \{0 \leftarrow \Sigma C_* \leftarrow D_* \leftarrow B_* \leftarrow 0\}$ where D_* is the mapping cone of j_*, and ΣC_* is the suspension of C_*. Now apply the point (1) to the sequence $\mathscr{X}$. □

In some cases the first inequality of the preceding proposition turns to equality. We shall say that a complex C_* *is concentrated in dimensions* $[k, r]$ if $C_i = 0$ for $i < k$ and for $i > r$. We denote by $F(i, s)_*$ the chain complex $\{0 \leftarrow R^s \leftarrow 0\}$ concentrated in dimensions $[i, i]$.

LEMMA 1.9. (1) *For every f-complex* C_* *we have* $\mathscr{M}(C_* \oplus F(i, s)_*) = \mathscr{M}(C_*) + s$. (2) *Let* C_*, D_* *be f-complexes, concentrated, respectively, in dimensions* $[a, b]$, *and* $[b, c]$. *Then* $\mathscr{M}(C_* \oplus D_*) = \mathscr{M}(C_*) + \mathscr{M}(D_*)$.

The proof of this lemma is easily obtained from V. V. Sharko's criterion of minimality of chain complexes (see [15, Lemma 3.6]). □

2. A NUMERICAL INVARIANT OF FREE CHAIN $R((t))$-COMPLEXES

We denote $\mathbb{Z}[\mathbb{Z}^n]$ by R (as in the previous section). Let us start with a free $R[[t]]$-complex A_*. For $k \in \mathbb{N}$ denote by $A[k]_*$ the free R-complex $A_*/t^k A_*$, and denote its Morse number by $\mu_k(A_*)$, so $\mu_k(A_*) = \mathscr{M}_R(A[k]_*)$. Note that $\mu_k(A_*) + \mu_l(A_*) \geqslant \mu_{k+l}(A_*)$. (Indeed, consider the short exact sequence $0 \leftarrow A[k]_* \leftarrow A[k+l]_* \leftarrow A[l]_* \leftarrow 0$ and apply Proposition 1.6.) Therefore, the sequence $\{\mu_k/k\}_{k \in \mathbb{N}}$ has a limit (see [12, Exercise 98]) which will be denoted by $\sigma(A_*)$. It is clear that $\sigma(A_*)$ is a chain homotopy invariant of A_*.

Now we shall consider free complexes over the ring $R((t)) = \sigma^{-1} R[[t]]$ where σ is the multiplicative set $\{t^l \mid l \in \mathbb{N}\}$. Let C_* be such a complex. We say that a chain subcomplex $D_* \subset C_*$ is a *basic subcomplex* if (1) D_* is a free $R[[t]]$-complex, and (2) $\sigma^{-1} D_* = C_*$. It is clear that each free complex C_* over $R((t))$ has basic subcomplexes.

PROPOSITION 2.1. *Let* C_* *be a free* $R((t))$*-complex. Then the number* $\sigma(D_*)$ *is the same for every basic subcomplex* $D_* \subset C_*$.

Proof. Let D_*, F_* be basic subcomplexes. The Noetherian property of $R[[t]]$ and the condition (2) in the definition of a basic subcomplex imply immediately that there is $s \in \mathbb{N}$ such that $t^s F_* \subset D_*$. Since $\sigma(D_*) = \sigma(t^k D_*)$ we can assume that $t^s F_* \subset D_* \subset F_*$. Now for every $l \in \mathbb{N}$ we obtain two exact sequences of finitely generated chain complexes over R.

$$0 \leftarrow F_*/D_* \leftarrow F_*/t^l D_* \leftarrow D_*/t^l D_* \leftarrow 0 \tag{1}$$

$$0 \leftarrow F_*/t^l D_* \leftarrow F_*/t^{l+s} F_* \leftarrow t^l D_*/t^{l+s} F_* \leftarrow 0. \tag{2}$$

Applying Proposition 1.6 we deduce from (1) and (2) that $\mu_{l+s}(F_*) \leqslant C + \mu_l(D_*)$ where C does not depend on l. This implies easily that $\sigma(F_*) \leqslant \sigma(D_*)$; by symmetry we obtain $\sigma(F_*) = \sigma(D_*)$. □

Now we can define an invariant of $R((t))$-complexes. Namely, if C_* is a free $R((t))$-complex, we set $s(C_*) = \sigma(D_*)$ where D_* is any basic subcomplex of C_*. The number $s(C_*)$ depends only on the homotopy type of the $R((t))$-complex C_*. Indeed, a version of the Cockroft–Swan theorem [11, Proposition 1.7] shows that it is sufficient to check that $\sigma(C_*)$ does not change when we add to C_* a complex of the form $\{0 \leftarrow R((t)) \stackrel{\text{id}}{\leftarrow} R((t)) \leftarrow 0\}$. But this is obvious.

For some free $R((t))$-complexes the asymptotic properties of the Morse numbers are still better. We shall say that a sequence a_k of real numbers is *asymptotically linear* if $\exists\, C, \alpha, \forall\, k$: $\alpha k - C \leqslant a_k \leqslant \alpha k + C$. We shall say that a free $R((t))$-complex C_* is of *asymptotically linear growth* (abbreviation: *aslg*) if for some basic subcomplex $D_* \subset C_*$ the sequence $\mu_k(D_*)$ is asymptotically linear. Similarly to the proof of Proposition 2.1, one can show that in an *aslg*-complex *every* basic subcomplex D'_* has an asymptotically linear sequence $\mu_k(D'_*)$. Note also that the property of being *aslg* is homotopy invariant. We do not know if every $R((t))$-complex is *aslg*, but we shall prove that every complex of a certain class appearing in our geometrical setting is *aslg*, and we shall calculate its s-invariant. We need some definitions. *A monomial* of R is an element of the form ag where $a \in \mathbb{Z}$, and $g \in \mathbb{Z}^n$. Let $Z = z_k t^k + \cdots + z_l t^l \in R[t, t^{-1}]$ where $l, k \in \mathbb{Z}$, $k \leqslant l$, and $z_k, z_l \neq 0$. We shall say that Z is:

- *monic* if $z_k = \pm g$, $g \in \mathbb{Z}^n$ (Our terminology differs here from the standard one.)
- *numerically prime* if it is not divisible by an integer not equal to ± 1.
- *special* if each z_i is a monomial in R.

We denote $R((t))$ by $\mathscr{L}$.

Definition 2.2. Let C_* be a complex over $\mathscr{L}$. We shall say that C_* is *of principal type* if for every i an isomorphism

$$H_i(C_*) \approx \left(\bigoplus_{j=1}^{b_i} \mathscr{L} \right) \oplus \left(\bigoplus_{s=1}^{q_i} \mathscr{L}/a_s^{(i)} \mathscr{L} \right) \tag{3}$$

is fixed, and for every i, s: (1) $a_s^{(i)} \in R[t, t^{-1}]$ and $a_s^{(i)}$ is special, non-zero and not monic (2) $a_s^{(i)} | a_{s+1}^{(i)}$.

For a complex C_* of principal type we denote by $\varkappa_i$ the number of those polynomials $a_s^{(i)}$ which are not numerically prime.

THEOREM 2.3. *Let C_* be a free $\mathscr{L}$-complex of principal type. Then C_* is of asymptotically linear growth, and $s(C_*) = \sum_i b_i + 2\sum_i \varkappa_i$.*

Proof. We can assume that all the elements $a_s^{(i)}$ in (3) are of the form $z_0 + \cdots + z_k t^k$ where $z_0 \in \mathbb{Z}$, $z_0 \neq 0$. Denote by $\mathscr{F}(i)_*$ the free $\mathscr{L}$-complex $\{0 \leftarrow \mathscr{L}^{b_i} \leftarrow 0\}$ concentrated in dimensions $[i, i]$. For $\rho \in \mathscr{L}$ and $i \in \mathbb{N}$, denote by $\tau(\rho, i)_*$ the free complex $\{0 \leftarrow \mathscr{L} \stackrel{\rho}{\leftarrow} \mathscr{L} \leftarrow 0\}$ concentrated in dimensions $[i, i+1]$. Note that if $\rho \in R[[t]]$ then $\tau(\rho, i)_*$ has a standard basic subcomplex $\{0 \leftarrow R[[t]] \stackrel{\rho}{\leftarrow} R[[t]] \leftarrow 0\}$ which will be denoted by $\tau'(\rho, i)_*$.

For a given i denote by π (resp. by ν) the set of all s such that $a_s^{(i)}$ is numerically prime (resp. *not* numerically prime). Set

$$\mathscr{TP}(i)_* = \bigoplus_{s \in \pi} \tau(a_s^{(i)}, i)_*, \qquad \mathscr{TN}(i)_* = \bigoplus_{s \in \nu} \tau(a_s^{(i)}, i)_*, \qquad \mathscr{T}(i)_* = \mathscr{TP}(i)_* \oplus \mathscr{TN}(i)_*.$$

(Morally, $\mathcal{F}(i)_*$ corresponds to the free part of the homology $H_i(C_*)$, and $\mathcal{T}(i)_*$ to the torsion part.) The complexes $\mathcal{T}(i)_*, \mathcal{TP}(i)_*, \mathcal{TN}(i)_*$ have basic subcomplexes $\mathcal{T}'(i)_*$, $\mathcal{TP}'(i)_*, \mathcal{TN}'(i)_*$ which are obtained as direct sums of the corresponding complexes $\tau'(\rho, i)_*$.

Lemma 5.1 of [8] implies that C_* is homotopy equivalent to the direct sum (over all i) of the complexes $\mathcal{F}(i)_* \oplus \mathcal{T}(i)_*$. We call this direct sum *principal model* for C_*. Applying successively Lemma 1.9, it is easy to deduce our theorem from the next lemma.

LEMMA 2.4. *For every i, k we have*: (1) $\mu_k(\mathcal{TN}'(i)_*) = 2k\varkappa_i$, (2) $\mu_k(\mathcal{T}'(i)_*) \geqslant 2k\varkappa_i$. (3) *For every i the sequence* $\{\mu_k(\mathcal{TP}'(i)_*)\}_{k\in\mathbb{N}}$ *is bounded.*

Proof. (1) Fix some i. The condition (2) from the Definition 2.2 implies that there is a prime number p such that every polynomial $a_s^{(i)}$ which is not numerically prime is divisible by p. Abbreviate $\mathcal{TN}'(i)_*$ to L_*; the inequality $\mu_k(L_*) \leqslant 2k\varkappa_i$ is immediate. To prove the inverse inequality consider an $\mathbb{F}_p$-complex $L[k]_* \otimes_R \mathbb{F}_p$ (where $\mathbb{F}_p$ is considered as $R = \mathbb{Z}[\mathbb{Z}^n]$-module via the trivial $\mathbb{F}_p$-representation of $\mathbb{Z}^n$). It is obvious that $\mathcal{M}(L[k]_*)$ is not less than $\dim H_*(L[k]_* \otimes \mathbb{F}_p)$ which equals $2k\varkappa_i$. A similar argument proves the point (2). To prove (3) it suffices to show that if $\rho = a_0 + a_1 t + \cdots + a_r t^r \in R[t]$ is special, numerically prime, and has $a_0 \neq 0$, then $\mu_k(\tau'(\rho, i)_*)$ is bounded. Write $a_j = A_j g_j$ with $g_j \in \mathbb{Z}^n$ and $A_j, a_0 \in \mathbb{Z}$, and note that $a_0, A_1, \ldots, A_r$ are relatively prime. Abbreviate $\tau'(\rho, i)_*$ to S_*. If $k \geqslant 1$ then $S[k+r]_*$ is a free R-complex of the form $\{0 \leftarrow R^{k+r} \overset{\mathcal{A}_k}{\leftarrow} R^{k+r} \leftarrow 0\}$ where $\mathcal{A}_k$ is the following $(k+r)\times(k+r)$-matrix

$$\mathcal{A}_k = \begin{pmatrix} a_0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_r & a_{r-1} & \cdots & 0 & 0 \\ 0 & a_r & \cdots & a_0 & 0 \\ \cdots & \cdots & \cdots & a_1 & a_0 \end{pmatrix}. \tag{4}$$

Denote by $\mathcal{B}_k$ the matrix formed by the first k columns of the matrix $\mathcal{A}_k$, and denote by I_k the ideal of R generated by all $k\times k$ subdeterminants of $\mathcal{B}_k$. The point (2) of the next lemma implies our assertion.

LEMMA 2.5. (1) $I_k = R$; (2) *The submodule* $\mathcal{A}_k(R^{k+r})$ *of* R^{k+r} *contains a direct summand of* R^{k+r}, *which is a free module of rank k.*

Proof. The point (2) follows from the point (1) by a standard argument based on the Quillen–Suslin theorem (we leave the details to the reader). Proceeding to the proof of the point (1), note that $a_0^k \in I_k$. Therefore we can assume that $a_0 \neq \pm 1$. It suffices to show that for every prime number p from the prime decomposition of a_0 there is an element $C \in R$ such that $1 + pC \in I_k$. To show this, recall that the numbers A_j are relatively prime. So there is i such that $p | A_j$ for $j < i$, and $p \nmid A_i$. Consider the $k\times k$ subdeterminant of the matrix $\mathcal{B}_k$ formed by all the columns and by the lines from $i+1$ to $i+k$. The terms of the principal diagonal are all equal to A_i, and the terms above the diagonal are divisible by p. Therefore this subdeterminant equals $Q + pC$ where Q is a monomial of the form $Q = qg$ with $(q, p) = 1$, and Lemma 2.5 follows. This finishes the proof of Theorem 2.3. □

Remark 2.6. Let $L = \mathbb{Z}[t, t^{-1}]$, $\hat{L} = \mathbb{Z}((t))$. The homomorphism $\varepsilon: \mathbb{Z}^n \to \{1\}$ extends to ring homomorphisms $e: R[t, t^{-1}] \to L$ and $\hat{e}: \mathscr{L} \to \hat{L}$. Therefore, for every $\mathscr{L}$-complex C_* we can form an $\hat{L}$-complex $\bar{C}_* = C_* \otimes_{\mathscr{L}} \hat{L}$. Assume that C_* is of principal type. Using the homotopy equivalence $C_* \sim \oplus_i (\mathscr{F}(i)_* \oplus \mathscr{T}(i)_*)$ from the proof of Theorem 2.3, it is easy to see that $\bar{C}_*$ is also of principal type and

$$H_i(\bar{C}_*) \approx \left(\bigoplus_{j=1}^{b_i} \hat{L} \right) \oplus \left(\bigoplus_{s=1}^{q_i} \hat{L}/\alpha_s^{(i)} \hat{L} \right) \tag{5}$$

with $\alpha_s^{(i)} = e(a_s^{(i)})$. Since $a_s^{(i)}$ are special and not monic, b_i and q_i are equal, respectively, to the rank and to the torsion number of $H_i(\bar{C}_*)$ over the principal ring $\hat{L}$. It is easy to see that the above decomposition satisfies Definition 2.2; therefore, $\bar{C}_*$, is *aslg*. Further, $a_s^{(i)}$ is numerically prime if and only if $\alpha_s^{(i)}$ is, and this implies $s(C_*) = s(\overline{C_*})$.

3. A NUMERICAL INVARIANT $S(C_*, \xi)$

In this section $\Lambda = \mathbb{Z}[\mathbb{Z}^{n+1}]$, C_* is a free Λ-complex, and $\xi: \mathbb{Z}^{n+1} \to \mathbb{Z}$ is a non-zero homomorphism. We define a numerical invariant $S(C_*, \xi)$. For the cohomology classes ξ outside a finite union of integer hyperplanes we calculate $S(C_*, \xi)$ in terms of the reduced Fitting sequences of the boundary operators of C_* (the mentioned finite union of integer hyperplanes depends on C_*). An element $z \in \Lambda$ is called *ξ-monic* if $z = \pm g + z_0$ where $g \in \mathbb{Z}^{n+1}$ and $\operatorname{supp} z_0 \subset \{h \in \mathbb{Z}^{n+1} | \xi(h) < \xi(g)\}$. An element z is called *ξ-special* if any two different elements $a, b \in \operatorname{supp} z$ satisfy $\xi(a) \neq \xi(b)$. We denote by S_ξ the multiplicative subset of all ξ-monic polynomials, and we denote by $\Lambda_{(\xi)}$ the localization $S_\xi^{-1}\Lambda$.

Definition 3.1. A subset $X \subset \mathbb{Z}^k$ will be called *small* if it is a finite union of integer hyperplanes.

THEOREM 3.2 [9, Theorem 0.1]). *There is a small subset $\mathfrak{N} \subset \operatorname{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z})$ such that for every $\xi \notin \mathfrak{N}$ and every p we have*

$$S_\xi^{-1} H_p(C_*) \approx \left(\bigoplus_{i=1}^{b_p(C_*, \xi)} \Lambda_{(\xi)} \right) \oplus \left(\bigoplus_{j=1}^{q_p(C_*, \xi)} \Lambda_{(\xi)}/a_j^{(p)} \Lambda_{(\xi)} \right) \tag{6}$$

where $a_j^{(p)} \in \Lambda$ are non-zero and not ξ-monic elements of Λ (depending on ξ), and $a_j^{(p)} \mid a_{j+1}^{(p)}$.

Proof (*Sketch*). We shall recall here the basic idea of the proof of Theorem 3.2 following [8, 9]; see [9] for the full proof. Let $\xi: \mathbb{Z}^{n+1} \to \mathbb{R}$ be a non-zero homomorphism. Similar to the above, we define the notion of ξ-monic polynomial, and we introduce the ring $\Lambda_{(\xi)} = S_\xi^{-1}\Lambda$.† (†We take here the occasion to note that for the first time the localization technique was applied to Novikov rings and Novikov inequalities in paper [2] of Farber. In this paper Farber considers the ring $S_\xi^{-1}\Lambda$, where $\Lambda = \mathbb{Z}[\mathbb{Z}]$, and ξ is the inclusion of $\mathbb{Z}$ to $\mathbb{R}$.) Recall next the definition of the Novikov ring Λ_ξ^- (see, e.g. [11, p. 326]). Denote by $\hat{\hat{\Lambda}}$ the abelian group of all the linear combinations of the form $\lambda = \sum_{g \in \mathbb{Z}^{n+1}} n_g g$ where $n_g \in \mathbb{Z}$ and the sum may be infinite. Let Λ_ξ^- be the subset of $\hat{\hat{\Lambda}}$ consisting of $\lambda \in \hat{\hat{\Lambda}}$ such that for every $c \in \mathbb{R}$ the set $\operatorname{supp} \lambda \cap \xi^{-1}([c, \infty[)$ is finite. This subset is called *Novikov ring* (it is not difficult to see that Λ_ξ^- has a natural ring structure).

Proceeding to Theorem 3.2, recall that Theorem 1.4 of [8] asserts that if ξ is injective then $\Lambda_{(\xi)}$ is euclidean. (The proof is based on a theorem by Sikorav, which asserts that if ξ is injective then Λ_ξ^- is euclidean, see [8, Theorem 1.1]).

Therefore, we obtain the decomposition (6) for any monomorphism ξ. This implies that (6) is true for every homomorphism η belonging to an open conical set containing ξ (see [8, the beginning of Section 7]). Since the monomorphisms are dense in $\mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{R})$, we obtain the decomposition (6) for every ξ belonging to some open and dense conical subset U in $\mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{R})$.

Analysing further the algebraic structure of the rings Λ_{ξ}^{-}, $\Lambda_{(\xi)}$ (it is done in [9]), one can prove that U can be chosen in such a way that the complement $\mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{R})\backslash U$ is a finite union $\mathfrak{R} = \bigcup_i L_i$ of hyperplanes L_i. Moreover, each L_i is of the form $l_i \otimes \mathbb{R}$ where $l_i \in \mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z})$ is an integer hyperplane. That proves Theorem 3.2. (See [9] for more information about the numbers $b_p(C_*, \xi)$, $q_p(C_*, \xi)$.) □

The following proposition relates the above numbers and the elements $a_j^{(p)}$ to the Fitting invariants of the boundary operators of C_*. We need some definitions. Let $A: F_1 \to F_2$ be a homomorphism of free finitely generated Λ-modules. Let $J_0 \subset \cdots \subset J_r$ be the reduced Fitting sequence of A, let $\rho_i \in \Lambda$ be the g.c.d. of the elements of J_i, and denote ρ_i/ρ_{i+1} by $\zeta_i(A)$. Let $\xi: \mathbb{Z}^{n+1} \to \mathbb{Z}$ be a non-zero homomorphism. Denote by $k(A, \xi)$ the number of those ρ_i which are not ξ-monic. Set $R_j(A, \xi) = \zeta_{k-j}(A, \xi)$ where $k = k(A, \xi)$. Now let $C_* = \{0 \leftarrow \cdots \leftarrow C_{i-1} \overset{\partial_i}{\leftarrow} C_i \leftarrow \cdots\}$ be a free Λ-complex.

PROPOSITION 3.3. *Assume that for $\xi \in \mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z})$ and every p the decomposition* (6) *holds. Then* (1) $b_p(C_*, \xi) = B_p(C_*)$. (2) $q_p(C_*, \xi) = k(\partial_{p+1}, \xi)$. (3) *For every p, s the elements $a_s^{(p)}$ and $R_s(\partial_{p+1}, \xi)$ are equal up to multiplication by a ξ-monic element.* (4) $Q_p(C_*)$ *equals to the number of not numerically prime $a_s^{(p)}$.*

Proof. Recall that the reduced Fitting sequences are homotopy invariants of C_*. This implies that $k(\partial_i, \xi)$ and $\varkappa(\partial_i, \xi)$ are homotopy invariants of C_* for fixed ξ. (1) is obvious. Further, let $0 \leqslant p \leqslant n$ and let $J_0 \subset \cdots \subset J_r$ be the reduced Fitting sequence for $\partial_{p+1}: C_{p+1} \to C_p$. Then the reduced Fitting sequence $FR(p)$ of the localized complex is a part of the sequence $S_{\xi}^{-1} J_0 \subset \cdots \subset S_{\xi}^{-1} J_r$, and the g.c.d. of $S_{\xi}^{-1} J_i$ is still ρ_i. Therefore, the sequence $FR(p)$ has $k(\partial_{p+1}, \xi)$ terms. Using the principal model for C_*, it is easy to prove that $FR(p)$ equals to the sequence of principal ideals $(a_1^{(p)} \cdot \ldots \cdot a_N^{(p)}), (a_1^{(p)} \cdot \ldots \cdot a_{N-1}^{(p)}), \ldots, (a_1^{(p)})$ where $N = q_p(C_*, \xi)$. (2)–(4) follow easily. □

Let $\xi = l\bar{\xi}$ where $\bar{\xi}: \mathbb{Z}^{n+1} \to \mathbb{Z}$ is an epimorphism. Choose an isomorphism $\mathrm{Ker}\, \xi \approx \mathbb{Z}^n$, and an element $t \in \mathbb{Z}^{n+1}$ such that $\bar{\xi}(t) = -1$. We obtain a decomposition $\mathbb{Z}^{n+1} = \mathrm{Ker}\, \xi \oplus \mathbb{Z}$ and an isomorphism $I(\xi): \Lambda \approx R[t, t^{-1}]$. Consider the free $\mathscr{L}$-complex $\hat{C}_*(\xi) = C_* \otimes_{\Lambda} \mathscr{L}$, where C_* is an $R[t, t^{-1}]$-module via the isomorphism $I(\xi)^{-1}$. Set $S(C_*, \xi) = s(\hat{C}_*(\xi))$ (it is easy to check that $S(C_*, \xi)$ depends indeed only on ξ and C_*).

THEOREM 3.4. *There is a small subset $\mathfrak{M} \subset \mathrm{Hom}(\mathbb{Z}^{n+1}, \mathbb{Z})$ such that for every $\xi \notin \mathfrak{M}$ the complex $\hat{C}_*(\xi)$ is of asymptotically linear growth and $S(C_*, \xi) = B(C_*) + 2Q(C_*)$.*

Proof. The $I(\xi)$-image of a ξ-monic polynomial is obviously invertible in $\mathscr{L}$; therefore, the homomorphism $\Lambda \to R[t, t^{-1}] \to R((t))$ factors through $\Lambda_{(\xi)}$. Therefore, for every ξ outside a small subset $\mathfrak{N}$ formula (3) holds with $C_* = \hat{C}_*(\xi)$. The complex $\hat{C}_*(\xi)$ is not necessarily of principal type, since the polynomials $a_s^{(i)}$ in the decomposition (3) are not necessarily special. But Proposition 3.3 implies that the elements $a_s^{(i)}$ in the decomposition

(3) can be chosen between the elements of the finite set $\{\zeta_j(\partial_{i+1})\}$. Therefore, adding to $\mathfrak{N}$ some integer hyperplanes if necessary, we can assume that all $a_s^{(i)}$ are special. Now our theorem follows from Theorem 2.3. □

4. PROOF OF THE MAIN THEOREM

Let M be a closed connected manifold and $\xi \in H^1(M, \mathbb{Z})$ be an indivisible cohomology class. Denote by $\mathscr{P}_\xi : M(\xi) \to M$ the infinite cyclic covering such that $\mathscr{P}_\xi^*(\xi) = 0$. Choose a generator $t \in \mathbb{Z} \approx \pi_1(M)/\mathrm{Ker}\,\xi$ of the structure group of $\mathscr{P}_\xi$ such that $\xi(t) = -1$. Let $f : M \to S^1$ be a Morse map representing ξ, and let V be its regular level surface, say, $V = f^{-1}(\lambda)$. Then f lifts to a Morse function $F : M(\xi) \to \mathbb{R}$ and V lifts to $F^{-1}(\lambda') \subset M(\xi)$. Denote by V^- the subset $F^{-1}(]-\infty, \lambda'])$. For $k \geqslant 1$ denote by W_k the cobordism $F^{-1}([\lambda' - k, \lambda'])$, $\partial W_k \approx V \sqcup t^k V$, and denote by $\alpha(k, V)$ its Morse number, i.e. the minimal number of critical points of a Morse function on the cobordism W_k. Note that $\alpha(k + n, V) \leqslant \alpha(k, V) + \alpha(n, V)$. Therefore, the sequence $\alpha(k, V)/k$ has a limit as $k \to \infty$. Denote this limit by $\alpha(V)$. It is easy to see that $\alpha(V)$ depends only on M and ξ, so we denote it by $\alpha(M, \xi)$. An elementary construction, using the gluing of the upper part V of ∂W_k to the lower part $t^k V$, allows to obtain the inequality $\mathscr{M}(M(\xi, k)) \leqslant \alpha(k, V) + 2\mathscr{M}(V)$. In particular, if ξ is represented by a fibration over S^1, the sequence $\mathscr{M}(M(\xi, k))$ is bounded.

In general, only that much we can say about the numbers $\alpha(k, V)$ and their relation to the asymptotics of the Morse numbers of cyclic covers. However, if the fundamental group of M is free abelian and $\dim M \geqslant 6$, one can say much more.

PROPOSITION 4.1. *Let M be a closed connected manifold with a free abelian fundamental group. Assume that $\dim M \geqslant 6$. Let $\xi \in H^1(M)$, $\xi \neq 0$. Then the sequence $\alpha(k, V) - \mathscr{M}(M(\xi, k))$ is bounded.*

Proof. Let $\pi_1(M) \approx \mathbb{Z}^{n+1}$, $n \geqslant 0$. An argument similar to the one used in [3, p. 325] shows that one can choose V above such that the embedding $V \hookrightarrow M$ induces an isomorphism $\pi_1(V) \to \mathrm{Ker}\,\xi$ (such V will be called *admissible ξ-splittings*, see [8, p. 371]). In this case all the embeddings $V \subset W_k \subset M(\xi) \supset t^k V$ induce isomorphisms of π_0 and of π_1. Choose an element $T \in \mathbb{Z}^{n+1}$, such that $\xi(T) = -1$. Let $\overline{M(\xi, k)} = \tilde{M}/T^k$, then there is a $\mathbb{Z}^n$-covering $\overline{M(\xi, k)} \to M(\xi, k)$. Choose a triangulation of M such that V is a subcomplex of M; then we obtain a t-invariant triangulation of $M(\xi)$ and the corresponding triangulations of all the covers. There are two exact sequences of corresponding $\mathbb{Z}[\mathbb{Z}^n]$-complexes:

$$0 \to C_*(\tilde{V}) \to C_*(\overline{M(\xi, k)}) \to C_*(\overline{M(\xi, k)}, \tilde{V}) \to 0 \tag{7}$$

$$0 \to C_*(\tilde{V}) \to C_*(\tilde{W}_k, \widetilde{t^k V}) \to C_*(\overline{M(\xi, k)}, \tilde{V}) \to 0 \tag{8}$$

Proposition 1.6 implies that there is $C = C(V)$ such that for every $k > 0$ we have: $\mathscr{M}(C_*(\overline{M(\xi, k)})) \geqslant \mathscr{M}(C_*(\tilde{W}_k, \widetilde{t^k V})) - C$. Since $\mathscr{M}(C_*(\tilde{W}_k, \widetilde{t^k V})) = \alpha(k, V)$ (see the proof of Corollary 6.3 in [15]), our Proposition is proved. □

Proof of the Main Theorem. The point (1) follows immediately from Proposition 4.1 (with $\alpha(M, \xi) = \lim_{k \to \infty} \mathscr{M}(M(\xi, k))/k$). To prove (2) note that Theorem 3.4 implies that for all $\xi \in H^1(M)$ outside a small subset $\mathfrak{M} \subset H^1(M)$ the complex $(C_*(\tilde{M}))\,\hat{}\,(\xi)$ is *aslg*, and $S(C_*(\tilde{M}), \xi) = B(M) + 2Q(M)$. Note further that for every admissible ξ-splitting V the complex $D_* = C_*(\tilde{V}^-) \otimes_\Lambda R[[t]]$ is a basic subcomplex of $C_*(\tilde{M}) \otimes_\Lambda R((t))$, and that $\mu_k(D_*) = \mathscr{M}(C_*(\tilde{W}_k, \widetilde{t^k V})) = \alpha(k, V)$. Now just apply Proposition 4.1. □

Remark 4.2. A similar argument, together with Remark 2.6, shows that for ξ outside a small subset of $H^1(M)$ the sequence $(B(M) + 2Q(M))k - \mathscr{M}_{\mathbb{Z}}(C_*(M(\xi, k)))$ is bounded where $C_*(M(\xi, k))$ is the chain complex of $M(\xi, k)$, defined over $\mathbb{Z}$ (see Definition 1.2 for the definition of $\mathscr{M}_{\mathbb{Z}}(\cdot)$).

5. FURTHER RESULTS AND CONJECTURES

5.1. Stable Morse numbers

Let M be a closed connected manifold. Recall that a *stable Morse function* on M is a Morse function $f: M \times \mathbb{R}^N \to \mathbb{R}$ such that there is a compact $K \subset M \times \mathbb{R}^N$, and a non-degenerate quadratic form Q of index 0 on $\mathbb{R}^N$ such that $f(x, y) = Q(y)$ outside K. Let $f: M \times \mathbb{R}^N \to \mathbb{R}$ be a stable Morse function. Denote by $\tilde{m}_p(f)$ the number of critical points of f of index $p + N/2$. The Morse–Pitcher inequalities hold: $\tilde{m}_p(f) \geqslant b_p(M) + q_p(M) + q_{p-1}(M)$.

Denote by $\mathscr{MS}(M)$ the minimal possible number of critical points of a stable Morse function on M; we have $\mathscr{MS}(M) \leqslant \mathscr{M}(M)$.

THEOREM 5.1. *Let* $\dim M \geqslant 6$, $\pi_1(M) \approx \mathbb{Z}^{n+1}$, $n \geqslant 0$. *There is a subset* $\mathfrak{M} \in H^1(M)$ *which is a finite union of integral hyperplanes in* $H^1(M)$, *and for every* $\xi \notin \mathfrak{M}$ *there is a real number* a *such that for every* $k \in \mathbb{N}$ *we have*

$$k(B(M) + 2Q(M)) - a \leqslant \mathscr{MS}(M(\xi, k)) \leqslant k(B(M) + 2Q(M)) + a.$$

For the proof just recall that (by Remark 4.2) for every ξ outside a small subset of $H^1(M)$ we have $k(B(M) + 2Q(M)) \leqslant \mathscr{M}_{\mathbb{Z}}(C_*(M(\xi, k))) + C$.

We refer to [1] for a systematic exposition of the theory of stable Morse functions and its applications to Lagrangian intersection theory.

5.2. Non-generic cohomology classes $\xi \in H^1(M)$

Here we construct a manifold M with $\pi_1(M) \approx \mathbb{Z}^2$ and $\dim M \geqslant 6$, and a class $\xi \in H^1(M)$ such that $\mu(M, \xi) \neq B(M) + 2Q(M)$. Let N be a closed connected manifold with $\pi_1(N) \approx \mathbb{Z}$, $\dim N \geqslant 5$ and $B(N) \neq 0$. Set $M = N \times S^1$. Let $\lambda: \pi_1(M) \to \mathbb{Z}$, resp. $\xi: \pi_1(M) \to \mathbb{Z}$, be epimorphisms with $\operatorname{Ker} \lambda = \pi_1(N)$, resp. $\operatorname{Ker} \xi = \pi_1(S^1)$. Then λ is represented by the fibration $N \times S^1 \to S^1$. Therefore, there is an open cone $C \subset H^1(M, \mathbb{R})$ containing λ, such that every integral non-divisible $\lambda' \in C$ can be represented by a fibration, and so $\mu(M, \lambda') = B(M) + 2Q(M) = 0$.

Now we shall show that $\mu(M, \xi) \neq 0$. Note that $M(\xi, k) = N_k \times S^1$ where N_k is the k-fold cyclic cover of N, and therefore $\mathscr{M}(M(\xi, k)) \geqslant \mathscr{M}_{\mathbb{Z}}(C_*(N_k \times S^1))$. We shall obtain a lower estimate for $\mathscr{M}_{\mathbb{Z}}(C_*(N_k \times S^1))$. Let $\xi_0: \pi_1 N \to \mathbb{Z}$ be the restriction $\xi | \pi_1 N$. Let $\bar{N} \to N$ be the infinite cyclic covering and $V \subset N$ be an admissible ξ_0-splitting. Let W_k be the corresponding cobordism in $\bar{N}$. Using exact sequences similar to the exact sequences (7) and (8) from Section 4, it is easy to prove that $\mathscr{M}_{\mathbb{Z}}(C_*(N_k \times S^1)) - \mathscr{M}_{\mathbb{Z}}(C_*(W_k \times S^1, t^k V \times S^1))$ is bounded. Let $X_k = W_k / t^k V$, $Y_k = (W_k / t^k V) \times S^1$. Then $\mathscr{M}_{\mathbb{Z}}(C_*(Y_k)) = \sum_p (b_p(Y_k) + q_p(Y_k) + q_{p-1}(Y_k))$. Since $H_*(X_k \times S^1) = H_*(X_k) \oplus H_{*-1}(X_k)$ we have $q_p(X_k \times S^1) \geqslant q_p(X_k)$, and $\mathscr{M}_{\mathbb{Z}}(C_*(Y_k)) \geqslant$

$\mathscr{M}_{\mathbb{Z}}(C_*(X_k))$. Recall from Proposition 4.1 that the sequence $\mathscr{M}(N_k) - \mathscr{M}_{\mathbb{Z}}(C_*(X_k))$ is bounded. Therefore, $\mathscr{M}_{\mathbb{Z}}(C_*(N_k \times S^1)) \geqslant \mathscr{M}(N_k) + C \geqslant kB(N) + C'$ (where C and C' do not depend on k), and, finally, $\mu(M, \xi) \geqslant B(N)$.

5.3. Non-cyclic finite coverings

The Main Theorem of the present paper allows also to deal with some non-cyclic finite coverings.

PROPOSITION 5.2. *Let M be a closed connected manifold*, $\dim M \geqslant 6$, *and* $\pi_1(M) \approx \mathbb{Z}^{n+1}$, $n \geqslant 0$. *Let $M_k \to M$ be the finite covering corresponding to the subgroup $k\mathbb{Z}^{n+1}$. Then*

$$\lim_{k \to \infty} \frac{\mathscr{M}(M_k)}{k^{n+1}} = B(M) + 2Q(M).$$

Proof. We shall give only the main idea of the proof. Let $\xi : \pi_1(M) \to \mathbb{Z}$ be an epimorphism not belonging to the small set $\mathfrak{M}$ of the Main Theorem. Then $M_k \to M$ factors through $M(\xi, k) \to M$, and therefore $\mathscr{M}(M_k)/k^n \leqslant k(B(M) + 2Q(M)) + C$. To obtain the lower estimate, note that the $\mathbb{Z}^n$-covering $\overline{M(\xi, k)} \to M(\xi, k)$ factors through $\overline{M(\xi, k)} \to M_k$ which corresponds to the subgroup $G_k = k\mathbb{Z}^n \subset \mathbb{Z}^n$. Therefore, $\mathscr{M}(M_k) \geqslant \mathscr{M}_{\mathbb{Z}[G_k]}(C_*(\overline{M(\xi, k)}))$. To obtain the lower estimate for $\mathscr{M}_{\mathbb{Z}[G_k]}(C_*(\overline{M(\xi, k)}))$, use (7) and (8) to reduce the question to finding the corresponding lower estimate for $\mathscr{M}_{\mathbb{Z}[G_k]}(C_*(\tilde{W}_k, \widetilde{t^k V}))$. Then proceed similarly to the proof of the Main Theorem. □

It seems that a similar result must hold for more general systems of non-cyclic finite coverings. To discuss a more general setting we need some definitions. Let G be a group. A sequence of subgroups $G = G_0 \supset G_1 \supset \cdots$ will be called a *tower* if for every i the index of G_i in G is finite. It will be called a *nested tower* if, moreover, $\bigcap_n G_n = \{0\}$.

If M is a closed connected manifold with $\pi_1(M) = G$ and $\mathbb{G} = \{G_n\}$ is a tower of subgroups of $\pi_1(M)$, then there is the corresponding tower of finite coverings $M = M_0 \leftarrow \cdots \leftarrow M_n \cdots$ of M. The sequence $\mathscr{M}(M_k)/|G/G_k|$ is decreasing. Therefore, it has a limit which will be denoted by $\mu(\mathbb{G})$. Recall a theorem of Lück [5], saying that if $\mathbb{G}$ is nested, then the limit of the sequence $b_p(M_k)/|G/G_k|$ exists and is equal to $b_p^{(2)}(M)$.

Problem. Is it true in general (at least for $\dim M \geqslant 6$) that $\mu(\mathbb{G})$ does not depend on the choice of the nested tower $\mathbb{G}$?

We believe that $\mu(\mathbb{G})$ does not depend on $\mathbb{G}$ for the case of free abelian fundamental group. Here is a result in this direction. Let $G = \mathbb{Z}^{n+1}$. For a tower $\mathbb{G} = \{G = G_0 \supset G_1 \supset \cdots\}$ we denote $\max_i m(G/G_i)$ by $r(\mathbb{G})$ (here $m(H)$ stands for the minimal number of generators of H). Denote by $\mathbb{G}^{[i]}$ the tower $G_i \supset G_{i+1} \supset \cdots$ The sequence $r(\mathbb{G}^{[i]})$ is decreasing, denote its limit by $\rho(\mathbb{G})$; then $\rho(\mathbb{G}) \leqslant \operatorname{rk} G$.

PROPOSITION 5.3. *Let M be a closed connected manifold*, $\dim M \geqslant 6$, $\pi_1(M) \approx \mathbb{Z}^{n+1}$, $n \geqslant 0$. *Let $\mathbb{G} = \{G_k\}$ be a tower, and let $M_k \to M$ be the finite covering corresponding to $G_k \subset \pi_1(M)$. Assume that $\rho(\mathbb{G}) = n + 1$. Then*

$$\lim_{k \to \infty} \frac{\mathscr{M}(M_k)}{|G/G_k|} = B(M) + 2Q(M) \tag{9}$$

Proof. It is not difficult to show that if $\rho(\mathbb{G}) = n + 1$ then for each k there are $a_k, b_k \in \mathbb{N}$ such that $a_k G \supset G_k \supset b_k G$ with $a_k, b_k \to \infty$ as $k \to \infty$. Now our proposition follows from Proposition 5.2. □

CONJECTURE. *Equality* (9) *is true for every nested tower in* $\mathbb{Z}^{n+1}$.

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