NOTE

SOME NEW TWO-WEIGHT CODES AND
STRONGLY REGULAR GRAPHS

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It is well known (and due to Delsarte [3]) that the three concepts (i) two-weight projective code, (ii) strongly regular graph defined by a difference set in a vector space, and (iii) subset X of a projective space such that |X ∩ H| takes only two values when H runs over all hyperplanes, are equivalent. Here we construct some new examples (formulated as in (iii)) by taking a quadric defined over a small field and cutting out a quadric defined over a larger field.

Let F be a field with r elements and F_0 a subfield with q elements, so that r = q^e for some e > 1. Let V be a vector space of dimension d over F and write V_0 for the same vector space but now regarded as a vector space of dimension de over F_0. (We shall use the zero subscript to indicate objects or operations in V_0 corresponding to those indicated without this subscript in V.) Let Tr: F → F_0 be the trace map (defined by Tr(x) = x + x^{q^e} + ... + x^{q^{e-1}}). One immediately checks the following observations:

(a) If Q: V → F is a quadratic form on V, then Q_0 = Tr ◦ Q is a quadratic form on V_0.

(b) If B: V × V → F is the bilinear form corresponding to Q (defined by Q(x + y) = Q(x) + Q(y) + B(x, y)), then B_0 = Tr ◦ B is the bilinear form corresponding to Q_0.

(c) B_0 is nondegenerate iff B is nondegenerate.

(d) Q_0 is nondegenerate iff Q is nondegenerate and either q is odd or d is even. [If q is odd, then Q is nondegenerate iff B is; if q is even and d is odd and Q is nondegenerate, then dim rad V = 1 so that dim_0 rad_0 V_0 = e and Q_0 is degenerate.]

(e) If d is even, then Q_0 has maximal (minimal) Witt index iff Q has.

Proof. (For details on orthogonal geometry, see e.g. Artin [1, Chapter III].) Let ε = +1 (−1), then Q(x) = 0 is true for (r^{d/2} − ε)(r^{d/2} − ε + 1) nonzero vectors in V. Since d is even the number of solutions of Q(x) = a does not depend on the a ∈ F \ {0} chosen, so this equation has r^{d-1} − εr^{d/2−1} solutions. Since Tr y = 0 is true for q^{e−1} elements of F among which 0, we see that Tr Q(x) = 0 is true for
nonzero vectors $x$. Thus $Q$ and $Q_0$ have simultaneously maximal or minimal Witt index.

**Remark.** If $U$ is a totally isotropic subspace of $V$ of dimension $\frac{1}{2}d$, then $U_0$ is totally isotropic of dimension $\frac{1}{2}de$ in $V_0$ so that $Q_0$ has maximal index when $Q$ has. But it is not so easy to give a similar proof without counting when $Q$ has minimal index.

If $x^\perp$ is a tangent hyperplane to $Q$ in $PV$, then $x^\perp 0$ is a tangent hyperplane to $Q_0$ in $PV_0$. [Note: the converse does not hold.] [Note: $PV$ is the projective space corresponding to $V$.]

After these preliminaries let us define $X = \{x \in PV_0 | Q_0(x) = 0 \text{ and } Q(x) \neq 0\}$, where $Q_0$ is a nondegenerate quadratic form on $V_0$, and investigate $|X \cap H|$ for hyperplanes $H$ in $PV_0$. Write $H = a^\perp 0$. First assume that $d$ is even. Distinguish three cases.

(i) $a^\perp$ is a tangent hyperplane.

Now $H$ is a tangent hyperplane, and $H \cap Q_0$ is a cone over a nondegenerate quadric in $de - 2$ dimensions and hence contains $1 + q(q^{de/2-1} - \varepsilon)(q^{de/2-2} + \varepsilon)/(q - 1)$ projective points, i.e., $q - 1 + q(q^{de/2-1} - \varepsilon)(q^{de/2-2} - \varepsilon) = q^{de/2 - 2} + \varepsilon q^{de/2-1}(q - 1)$ nonzero vectors.

Similarly $a^\perp \cap Q$ contains $r^{d-2} - 1 + er^{d/2 - 1}(r - 1) = q^{de/2 - 2} - 1 + \varepsilon q^{de/2-1}(q^e - 1)$ nonzero vectors.

Since $Q$ contains $q^{de - e - 1} + \varepsilon q^{de/2 - e}(q^e - 1)$ nonzero vectors and each nonzero value of the inner product $B(a, \cdot)$ occurs equally often on $Q \setminus a^\perp$ we find that each nonzero value of $B(a, \cdot)$ is taken for $q^{de - 2e}$ vectors in $Q \setminus a^\perp$.

Now the number of nonzero vectors $x$ with $Q(x) = 0$ and $B_0(a, x) = 0$ is

\[
q^{de - 2e - 1} + \varepsilon q^{de/2 - e}(q^e - 1) + (q^{e - 1} - 1)q^{de - 2e} = q^{de - e - 1} - 1 + \varepsilon q^{de/2 - e}(q^e - 1).
\]

Finally

\[
|X \cap H| = \frac{1}{q - 1} (q^{e - 1} - 1)(q^{de - e - 1} - \varepsilon q^{de/2 - e}).
\]

(ii) $a^\perp$ is a secant hyperplane but $H$ is tangent.

We find the same value for $|H \cap Q_0|$ as before; this time $a^\perp \cap Q$ is a nondegenerate quadric in $d - 1$ dimensions and hence contains $r^{d-2} - 1$ nonzero vectors.

Each nonzero value of $B(a, \cdot)$ is taken for $q^{de - 2e} + \varepsilon q^{de/2 - e}$ vectors in $Q \setminus a^\perp$ so that $H \cap Q$ contains $q^{de - e - 1} - 1 + \varepsilon q^{de/2 - e}(q^e - 1)$ nonzero vectors. Finally

\[
|X \cap H| = \frac{1}{q - 1} [q^{de - e - 1}(q^e - 1) + \varepsilon q^{de/2 - e}(q^e - 2q^e + 1)].
\]
(iii) Both $a^\perp$ and $H$ are secant.

This time $H \cap Q_0$ contains $q^{de-2}-1$ nonzero vectors, $H \cap Q$ has the same size as under (ii), and

$$|X \cap H| = \frac{1}{q-1} (q^{e-1}-1)(q^{de-e-1} - eq^{de/2-e}),$$

the same value as we found under (i).

**Theorem.** Let $d$ be even. $X$ is a subset of size $(q^{e-1}-1)(q^{de-e}-eq^{de/2-e})/(q-1)$ of $PV_0$ such that $|X \cap H|$ is either $(q^{e-1}-1)(q^{de-e-1} - eq^{de/2-e})/(q-1)$ or

$$[q^{de-e-1}(q^{e-1}-1) + eq^{de-e}(q^{e-2}q^{e-1}+1)]/(q-1)$$

where the latter possibility occurs for precisely $|X|$ hyperplanes $H$.

The corresponding two-weight code over $F_0$ has word length $|X|$ and weights $w_0=0$, $w_1=(q^{e-1}-1)q^{de-e-1}$ and $w_2=(q^{e-1}-1)q^{de-e-1} - eq^{de/2-e}$. The corresponding strongly regular graph has $v=|V_0|=q^{de}$ vertices, valency $k=(q-1)|X|=(q^{e-1}-1)(q^{de-e} - eq^{de/2-e})$ and eigenvalues $k-qw_i$ ($i=0,1,2$).

**Proof.** We already saw the first part. For the connections with two-weight codes and strongly regular graphs see Calderbank & Kantor [2].

**Comparison with known constructions**

For $\varepsilon=+1$ the graphs constructed above have the parameters of Latin square graphs derived from OA$(u, g)$, where

$$u = q^{de/2} \quad \text{and} \quad g = q^{de/2-e}(q^{e-1}-1).$$

Many constructions for graphs with Latin square parameters are known; I do not know whether the graphs constructed above are isomorphic to previously constructed ones.

For $\varepsilon=-1$ these graphs have 'negative Latin square' parameters. When $d=2$ these are known (not surprisingly: $Q$ is empty, so $X=Q_0 \setminus Q_0$) but for $d \geq 4$ they seem to be new. The smallest graph constructed here and not known before has parameters $(q=e=2, d=4)$:

$$v=256, \quad k=68, \quad \lambda=12, \quad \mu=20, \quad r=4, \quad s=-12.$$ 

A cyclotomic description of this same graph can be given by taking $V=GF(256)$, $Q(x)=x^{17}+x^{68}$, $X=\{a^{15i+j} | \ 0 \leq i \leq 16, \ j=1,2,4,8\}$ where $\alpha$ is a primitive element of $GF(256)$.

**Case $d$ odd**

Similar computations when $d$ is odd show that $|X \cap H|$ takes more than two distinct values here, so that this case is not interesting for our purpose.
References