Journal of Mathematical Analysis and Applications **245,** 189–203 (2000)
doi:10.1006/jmaa.2000.6751, available online at http://www.idealibrary.com on IDEAL®

Regularity of Solutions of Retarded Equations and Application to Sensitivity of Linear Quadratic Controllers to Small Delays

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Received November 3, 1997

We analyze the behaviour of the solution of a linear differential equation of retarded type when the delays vary, for L^p initial conditions, given some $p \in [1, \infty]$. First, we state continuity and differentiability results for the solution viewed as function of the delays. Those regularity results are applied to analyze the small delay sensitivity for quadratic performance index associated to stabilization with a finite dimensional a priori feedback. © 2000 Academic Press

1. INTRODUCTION

In this paper, we study the delay dependence of the solution of the retarded equation

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{I} A_i x(t - k_i) + f(t), \quad \text{a.e. } t \in [0, T], \qquad T > 0,$$

$$x(0) = x_{00} \in \mathbb{R}^n, \qquad x = x_0 \quad \text{a.e. } t \in [-K, 0],$$

$$x_0 \in L^p(-K, 0; \mathbb{R}^n), \tag{1}$$

$$f \in L^p(0, T; \mathbb{R}^n), \quad \text{given } p \geq 1$$

where $A_i \in \mathscr{L}(\mathbb{R}^n)$, $i \in \{0, \dots, I\}$, $k = \min_{i=1,\dots,I} k_i \geq 0$, and $K = \max_{i=1,\dots,I} k_i < +\infty$. We denote the delay-vector $\mathbf{k} = (k_1, k_2, \dots, k_I)$ with

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0022-247X/00 $35.00

the convention $\mathbf{k} \geq 0$, and when necessary, $x(t; \mathbf{k})$ the solution of Eq. (1), and use the shorthand $A = \sum_{i=0}^{I} A_i$.

Controlled linear systems of retarded type are well known to be of theoretical and practical interest as shown, for instance, in Bellman & Cooke [2], Bensoussan et al. [3], Curtain & Zwart [6], Halanay [8], Hale [9] and Kolmanovski & Nosov [10]. In some situations connected to computer control it is assumed some regularity with respect to the delays, e.g. Chan & Özguner [4] and Krtolica et al. [11]. This asks for clarifications mainly in two directions: regularity of the solution and influence of the delays on performance of the controlled system. For the former, it seems that little attention has been paid to the study of continuity and especially differentiability with respect to the delays. For a single delay case, uniform continuity is well known for continuous initial condition since Bellman & Cooke [1] and Sugiyama [12]. A derivative formula at zero was given by Dontchev [7, Theorem 2] for an absolutely continuous initial condition, and rediscovered by Clarke & Wolenski [5, Proposition 8.1] for Lipschitz continuous initial data. But for the general case with non-smooth data, the question of both continuity and differentiability remains open, especially when $\mathbf{k} \neq \mathbf{0}$. For the latter, differentiability results are also given in the finite horizon case in the papers of Dontchev [7] and Clarke & Wolenski [5].

The objective of this article is twofold: (1) to provide some insight into the continuity and differentiability of the function $\mathbf{k} \mapsto x(\cdot; \mathbf{k}) \in W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^n)$; (2) to consider a control application of the differentiability result. For that application, we give a sensitivity result for some quadratic cost associated to a retarded system with small delays.

The results of regularity are stated in Section 2, and the control application in Section 3. The proofs are in Section 4 and followed by some concluding remarks in Section 5.

2. CONTINUITY AND DIFFERENTIABILITY WITH RESPECT TO THE DELAY

First of all, we focus on the status of Eq. (1). When there is no delay ($K = 0$), the solution is given by the usual variation of constant formula; its restriction to $\mathbb{R}_+$ belongs to $W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^n)$ and depends continuously on the initial data x_{00} and f. Otherwise ($k > 0$), we notice that starting from the initial data x_0 the solution can be recursively computed on intervals of the form $[-K, \ell k]$, where $\ell \geq 0$ is an integer (step method). More precisely, denoting x_ℓ the solution on $[-K, \ell k]$, the sequence x_ℓ

and the unique solution x of equation (1) are given by

$$x_1(t) = e^{A_0 t}x_{00} + \int_0^t e^{A_0(t-s)}\left\{\sum_{i=1}^{I} A_i x_0(s-k_i) + f(s)\right\} ds \text{ for } t \in [0,k]$$

and, for $\ell \geq 1$ and $t \in [\ell k, (\ell+1)k]$:

$$x_{\ell+1}(t) = e^{A_0(t-\ell k)}x_\ell(\ell k) + \int_{\ell k}^t e^{A_0(t-s)}\left\{\sum_{i=1}^{I} A_i x_\ell(s-k_i) + f(s)\right\} ds,$$

$$\begin{aligned} x_\ell &= x_{\ell-1} \text{ on } [-K, \ell k]; \\ x &= x_\ell \text{ on } [-K, \ell k]. \end{aligned} \tag{2}$$

So it is clear that the solution belongs to $W^{1,p}_{\text{loc}}(0,\infty;\mathbb{R}^n)$. Moreover, it is known, e.g. [6], that this solution depends continuously on the initial data on compact intervals; i.e., for any $T > 0$ and $K \geq 0$, there exists a constant $C > 0$ such that

$$\|x\|_{W^{1,p}(0,T)} \leq C\big(|x_{00}|_{\mathbb{R}^n} + \|x_0\|_{L^p(-K,0)} + \|f\|_{L^p(0,T)}\big).^2 \tag{3}$$

Remark 2.1. The constant C in Eq. (3) depends continuously[3] on T and K. Therefore, since $K = \max_{i=1,\ldots,I} k_i < +\infty$, there exists some $\overline{C}$ independent of $\mathbf{k}$ such that $C < \overline{C} < +\infty$.

PROPOSITION 2.1. Continuity of the solution.

(i) *When* $x_0 \in L^p(-K,0;\mathbb{R}^n)$ *and* $f \in L^p_{\text{loc}}(0,\infty;\mathbb{R}^n)$ *with* $p \in [1,\infty[$, *the solution of Eq.* (1) *satisfies* $\lim_{\mathbf{h}\to\mathbf{k}} \|x(\cdot;\mathbf{k}) - x(\cdot;\mathbf{h})\|_{W^{1,p}_{\text{loc}}(0,\infty)} = 0$ *for any delay-vectors* $\mathbf{k} \geq 0$, $\mathbf{h} \geq 0$.

(ii) *If, moreover,* $\lim_{s\to 0} x_0(s) = x_{00}$, *then* i) *extends to* $p = +\infty$.

One can observe that in Proposition 2.1, the compatibility condition, $\lim_{s\to 0} x_0(s) = x_{00}$, is required only for the case $p = +\infty$. So this proposition is a generalization to any set of L^p initial conditions of the classical results of Bellman & Cooke [1] and Sugiyama [12] who investigated the continuous initial conditions case.

Proposition 2.1 allows to prove the following.

THEOREM 2.1. Differentiability of the solution. *Assume* $x_0 \in C^1(-K,0;\mathbb{R}^n)$ *and* $x_0(0) = x_{00}$. *Then for any fixed* $T > 0$ *with* f *continuous on* $[0,T]$, *the function* $\mathbf{k} \mapsto x(\cdot;\mathbf{k})$ *from* $\mathbb{R}^I_+$ *to* $C^1(0,T;\mathbb{R}^n)$ *is differentiable*

[2] $|\cdot|$ denotes a vector norm and $\|\cdot\|$ the norm of a matrix or a function. Given a function g with values in a space $\mathscr{E}$, we also use $\|g\|_{L^p(a,b)}$ for $\|g\|_{L^p(a,b;\mathscr{E})}$ and $\|g\|_{W^{1,p}(a,b)}$ for $\|g\|_{W^{1,p}(a,b,\mathscr{E})}$.

[3] see e.g. [6, Lemma 2.4.3] for the case $p = 2$.

at the point $\mathbf{k} \geq 0$ *as soon as*

$$\dot{x}_0(0) = A_0 x_0(0) + \sum_{i=1}^{I} A_i x_0(-k_i) + f(0). \tag{4}$$

Its partial derivatives are solutions of

$$\begin{aligned} \dot{\overline{\frac{\partial x}{\partial k_i}}}(t) &= A_0 \frac{\partial x}{\partial k_i}(t) \\ &\quad + \sum_{j=1}^{I} A_j \frac{\partial x}{\partial k_i}(t - k_j) - A_i \dot{x}(t - k_i), \quad \text{a.e. } t \in [0, T], \end{aligned} \tag{5}$$

$$\frac{\partial x}{\partial k_i} = 0 \quad \text{on } [-K, 0],$$

for each $i \in \{1, \dots, I\}$. ∎

Remark 2.2. If $\mathbf{k} = 0$, the existence result for derivatives may be derived under weaker regularity assumptions on x_0 that need only to be smooth in zero. More precisely:

(i) If for some p in $[1, \infty[$, $x_0 \in L^p(-K, 0; \mathbb{R}^n)$ satisfies $\overline{\lim}_{t \to 0} |(x_0(t) - x_{00})/t|_{\mathbb{R}^n} < +\infty$, then for any fixed $T > 0$ with f continuous on $[0, T]$, the function $\mathbf{k} \mapsto x(\cdot; \mathbf{k})$ from $\mathbb{R}_+^I$ to $W^{1,p}(0, T; \mathbb{R}^n)$ has partial derivatives at the point $\mathbf{k} = 0$. Those are given by

$$\left.\frac{\partial x(t; \mathbf{k})}{\partial k_i}\right|_{\mathbf{k}=0} = -\int_0^t e^{A(t-s)} A_i \dot{x}(s, 0)\, ds. \tag{6}$$

(ii) If in addition $\dot{x}_0(0)$ exists and satisfies $\dot{x}_0(0) = \dot{x}(0, 0)$, the previous result extends to $p = +\infty$.

In particular when $f = 0$ and $\mathbf{k} = 0$, we have $x(s, 0) = e^{As} x_{00}$, in such a way that

$$\left.\frac{\partial x(t; \mathbf{k})}{\partial k_i}\right|_{\mathbf{k}=0} = -\int_0^t e^{A(t-s)} A_i A e^{As} x_{00}\, ds. \tag{7}$$

If, moreover, the A_i's commute with A, the previous equality reduces to

$$\left.\frac{\partial x(t; \mathbf{k})}{\partial k_i}\right|_{\mathbf{k}=0} = -A_i A t e^{At} x_{00}. \tag{8}$$

Theorem 2.1 is a generalization to any value of $\mathbf{k} \geq 0$ with non-smooth initial data of the differentiability result at the point zero for the single delay case by Dontchev [7, Theorem 2] and Clarke and Wolenski [5, Proposition 8.1]. The Remark 2.2 shows that their derivative at 0 for the one delay case (formula 6) may be extended to rougher data since we do not assume the existence of $\|\dot{x}_0\|_\infty$. However, Clarke and Wolenski's approach is more general because they consider possibly nonlinear systems.

3. APPLICATION TO SENSITIVITY OF A LINEAR QUADRATIC CONTROLLER TO SMALL DELAYS

In this section, we set $p = 2$ and consider the following retarded controlled system

$$\dot{x}(t) = A_0 x(t) + \sum_{i \in \mathscr{I}'} A_i x(t - k_i) + B_0 u(t) + \sum_{i \in \mathscr{I}''} B_i u(t - k_i), \quad \text{a.e. } t \geq 0$$

$$x(0) = x_{00} \in \mathbb{R}^n, \qquad x = x_0 \quad \text{a.e. } t \in [-K, 0], \qquad x_0 \in L^2(-K, 0; \mathbb{R}^n)$$

$$u \in L^2_{\text{loc}}(-K, \infty; \mathbb{R}^m) \tag{9}$$

associated to the quadratic cost

$$J_T(u, x) = \frac{1}{2} \int_0^T \left\{ |Cx|^2_{\mathbb{R}^p} + \langle u, Ru \rangle_{\mathbb{R}^m} \right\} dt \tag{10}$$

with $\mathscr{I}' \cup \mathscr{I}'' = \{1, \ldots, I\}$; $A_i \in \mathscr{L}(\mathbb{R}^n)$, $i = 0$, $i \in \mathscr{I}'$; $B_i \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$, $i = 0$, $i \in \mathscr{I}''$; $C \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^p)$, $R = R^T > 0 \in \mathscr{L}(\mathbb{R}^m)$.

Here u is the control input used to control x over $[0, T]$, $0 \leq T \leq +\infty$. We consider the case where u is chosen in feedback form for the system without delay and $u(t)$ for $t < 0$ is arbitrary:

$$u(t) = \begin{cases} u_0(t), & \text{a.e. } t \in [-K, 0], \qquad u_0 \in L^2(-K, 0; \mathbb{R}^m) \\ Fx(t), & \text{a.e. } t \geq 0, \qquad F \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m). \end{cases} \tag{11}$$

Let $(x_F(\cdot; \mathbf{k}), u_F(\cdot; \mathbf{k}))$ be the solution of Eqs. (9) and (11), where, as before, $\mathbf{k} = (k_1, k_2, \ldots, k_I)$, $k = \min_{i=1,\ldots,I} k_i$, and $K = \max_{i=1,\ldots,I} k_i$; $A_i = B_i F_i$, $i \in \mathscr{I}''$.

We are interested in the following quantities used to measure the sensitivity of $J_T(u_F, x_F)$ to small delays.

$$\sigma_{i,T} = \frac{\partial J_T}{\partial k_i}(u_F, x_F)\bigg|_{\mathbf{k}=0}, \qquad i = 1, \dots, I. \tag{12}$$

We consider the cases:

- $T < +\infty$ and F is arbitrary.
- $T = +\infty$ and F is a stabilizing feedback gain when $\mathbf{k} = 0$, i.e., $\lim_{t \to 0} e^{At} = 0$ in $\mathscr{L}(\mathbb{R}^n)$ where, as before, $A = \sum_{i=0}^{I} A_i$. Of course we suppose that in that case $(A_0 + \sum_{i \in \mathscr{I}'} A_i, B_0 + \sum_{i \in \mathscr{I}''} B_i)$ is stabilizable.
- $T = +\infty$ and F is the optimal feedback gain minimizing $J_\infty(u, x)|_{\mathbf{k}=0}$. In that case F is given by

$$\begin{aligned} &P\Big(A_0 + \sum_{i \in \mathscr{I}} A_i\Big) + \Big(A_0 + \sum_{i \in \mathscr{I}} A_i\Big)^T P \\ &\qquad - P\Big(B_0 + \sum_{i \in \mathscr{I}''} B_i\Big) R^{-1} \Big(B_0 + \sum_{i \in \mathscr{I}''} B_i\Big)^T P + C^T C = 0, \\ &\qquad\qquad P = P^T \geq 0, \\ &\qquad\qquad F = -R^{-1}\Big(B_0 + \sum_{i \in \mathscr{I}''} B_i\Big)^T P. \end{aligned} \tag{13}$$

As usual, F is defined by (13) as soon as

$$\begin{aligned} &\Big(A_0 + \sum_{i \in \mathscr{I}'} A_i, B_0 + \sum_{i \in \mathscr{I}''} B_i\Big) \text{ is stabilizable and} \\ &\qquad \Big(C, A_0 + \sum_{i \in \mathscr{I}'} A_i\Big) \text{ detectable.} \end{aligned} \tag{14}$$

Remark 3.1. The computation of $\sigma_{i,T}$, $i = 1, \dots, I$, leads to the consideration of the systems with only one delay,

$$\begin{gathered} \dot{x}(t) = \mathscr{A}_i x(t) + A_i x(t - k_i) \\ x(0) = x_{00} \in \mathbb{R}^n, \qquad x = x_0 \quad \text{a.e. } t \in [-K, 0], \qquad x_0 \in L^2(-K, 0; \mathbb{R}^n), \end{gathered} \tag{15}$$

where $\mathscr{A}_i = \sum_{j=1, j \neq i}^{I} A_j$.

For $i \in \mathscr{I}''$ (systems without open-loop state delay), it is possible to choose x_0 such that $u_0 = Fx_0$, $x_0 \in L^2(-K, 0; \mathbb{R}^n)$, as soon as rank $F = m$. For example, if $(C, A_0 + \sum_{i \in \mathscr{I}'} A_i)$ is observable then, in Eq. (13), $P > 0$, and rank $F = \operatorname{rank}(B_0 + \sum_{i \in \mathscr{I}''} B_i)^T$. In all usual situations, the columns of $(B_0 + \sum_{i \in \mathscr{I}''} B_i)^T$ being independent, rank $F = m$.

Denoting $x_i(\cdot, k_i)$ the solution of (15), $i = 1, \dots, I$, then

$$\sigma_{i,T} = \frac{1}{2}\frac{d}{dk_i} \times \left\{ \int_0^T \left\{ |Cx_i(t;k_i)|^2_{\mathbb{R}^p} + \langle F^T RFx_i(t-k_i;k_i), x_i(t-k_i;k_i)\rangle_{\mathbb{R}^n} \right\} dt \right\}\Bigg|_{k_i=0} \quad \text{for } i \in \mathscr{I}'',$$

and (16)

$$\sigma_{i,T} = \frac{1}{2}\frac{d}{dk_i}\left\{ \int_0^T \langle (C^T C + F^T RF) x_i(t;k_i), x_i(t;k_i)\rangle_{\mathbb{R}^n}\, dt \right\}\Bigg|_{k_i=0} \quad \text{for } i \in \mathscr{I}'.$$

COROLLARY 3.1. *Denote $Q = C^T C + F^T RF$, and suppose that x_0 has a derivative on the left hand of* 0 *with $x_0(0) = x_{00}$. Then (x_F, u_F) being the solution of Eqs.* (9) *and* (11).

(i) *For any finite T, the mapping $\mathbf{k} \to J_T(u_F, x_F)$ has partial derivatives at the point $\mathbf{k} = 0$ which are given by*

$$\sigma_{i,T} = \frac{1}{2}\langle F^T RFx_{00}, x_{00}\rangle_{\mathbb{R}^n} - \langle F^T RFe^{AT}x_{00}, x_{00}\rangle_{\mathbb{R}^n} - \int_0^T\int_0^t \langle Qe^{A(t-s)}A_i Ae^{As}x_{00}, e^{At}x_{00}\rangle_{\mathbb{R}^n}\, ds\, dt, \qquad i \in \mathscr{I}'' \quad (17)$$

$$\sigma_{i,T} = -\int_0^T\int_0^t \langle Qe^{A(t-s)}A_i Ae^{As}x_{00}, e^{At}x_{00}\rangle_{\mathbb{R}^n}\, ds\, dt, \qquad i \in \mathscr{I}'.$$

(ii) *If, in addition F is a stabilizing feedback gain when $\mathbf{k} = 0$, the mapping $\mathbf{k} \to J_\infty(u_F, x_F)$ is well-defined for $|\mathbf{k}|$ small and* (i) *still holds for $T = +\infty$ (with $e^{A\infty} = 0$).*

(iii) *If, furthermore, condition* (14) *holds and the optimal feedback gain given by Eq.* (13) *is used, then*

$$\sigma_{i,\infty} = \frac{1}{2}\langle F^T RFx_{00}, x_{00}\rangle_{\mathbb{R}^n} - \int_0^\infty \langle Ae^{At}x_{00}, A_i^T Pe^{At}x_{00}\rangle_{\mathbb{R}^n}\, dt, \qquad i \in \mathscr{I}''$$

$$\sigma_{i,\infty} = -\int_0^\infty \langle Ae^{At}x_{00}, A_i^T Pe^{At}x_{00}\rangle_{\mathbb{R}^n}\, dt, \qquad i \in \mathscr{I}'. \quad (18)$$

Remark 3.2. (1) It is easy to see that $\sigma_{i,\infty}$ in Eq. (18) can be computed as follows.

For $i \in \mathscr{I}''$, $\sigma_{i,\infty} = \langle(\frac{1}{2}F^T RF - \Sigma_i)x_{00}, x_{00}\rangle_{\mathbb{R}^n}$ with Σ_i solution in $\mathbb{R}^n$ of $A^T\Sigma_i + \Sigma_i A = PA_i A$, a well-posed matrix equation as the eigenvalues of A have strictly negative real parts. For $i \in \mathscr{I}'$, $\sigma_{i,\infty} = -\langle \Sigma_i x_{00}, x_{00}\rangle_{\mathbb{R}^n}$.

(2) In practice x_{00} is often not known when the $\sigma_{i,\infty}$ are needed. One can instead use some worst-case sensitivity measure, for example take $\tilde{\sigma}_{i,\infty} = \lambda_{\max}^{1/2}(\Sigma_i^T\Sigma_i)$, where $\lambda_{\max}(M)$ is the largest eigenvalue of $M = M^T$.

(3) In general, $\sigma_{i,\infty}$ given by Eq. (17) may be positive, negative, or nil. For the latter case consider, e.g., the system $\dot{x}(t) = A_0 x(t) + B_1 u(t - k_1)$. The delayed optimal feedback is no more optimal: $J_\infty(u_F, x_F) \geq J_\infty(u_F, x_F)|_{k_1=0}$. Even more, one can say that $k_1 \to u_F(t-k_1)$ is optimal for $k_1 = 0$, so that $\sigma_{i,\infty} = 0$. Equation (18) leads to the same conclusion: $A_1 = -BR^{-1}B^T P$, so that

$$\begin{aligned}\sigma_{i,\infty} &= \frac{1}{2}\langle F^T RFx_{00}, x_{00}\rangle_{\mathbb{R}^n} + \int_0^\infty \langle Ae^{At}x_{00}, PBR^{-1}B^T Pe^{At}x_{00}\rangle_{\mathbb{R}^n}\, dt \\ &= \frac{1}{2}\langle F^T RFx_{00}, x_{00}\rangle_{\mathbb{R}^n} + \frac{1}{2}\int_0^\infty \frac{d}{dt}\langle e^{At}x_{00}, F^T RFe^{At}x_{00}\rangle_{\mathbb{R}^n}\, dt \\ &= 0. \end{aligned} \tag{19}$$

(4) Dontchev [7] has obtained a sensitivity formula similar to (17) in the particular case $\dot{x}(t) = A_0 x(t) + A_1 u(t - k_1) + B_0 u(t)$.

4. PROOFS

We start with Proposition 2.1 which then is used to prove Theorem 2.1. Next we prove Corollary 3.1.

4.1. *Proof of Proposition* 2.1

Take some fixed $T > 0$. We aim to prove that $\lim_{\mathbf{h}\to\mathbf{k}}\|x(\cdot;\mathbf{k}) - x(\cdot;\mathbf{h})\|_{W^{1,p}(0,T)} = 0$ for any delay-vectors $\mathbf{k} \geq 0$, $\mathbf{h} \geq 0$. To this end we remark that the function Δ defined by $\Delta(t;\mathbf{k},\mathbf{h}) = x(\cdot;\mathbf{k}) - x(\cdot;\mathbf{h})$ satisfies the following retarded equation (recall that $x(t,\mathbf{k}) = x_0$ a.e. $t \in [-K, 0]$):

$$\begin{aligned} &\dot{\Delta}(t) = A_0\Delta(t) + \sum_{i=1}^{I} A_i\Delta(t-h_i) + \sum_{i=1}^{I} A_i g_i(t) \qquad \text{a.e. } t \in [0,T], \\ &\Delta(t) = 0 \qquad \text{a.e. } t \in [-K,0], \\ &g_i(t) = x(t-k_i;\mathbf{k}) - x(t-h_i;\mathbf{k}) \qquad \text{a.e. } t \in [0,T]. \end{aligned} \tag{4.1.1}$$

Recall that $h_i, k_i \leq K < +\infty$. Then from Remark 2.1, there exits $\overline{C}$ independent of $\mathbf{k}$ and $\mathbf{h}$ such that $\|\Delta\|_{W^{1,p}(0,T)} \leq \overline{C}\sum_{i=1}^{I}\|A_i\| \times \|g_i\|_{L^p(0,T)}$, so to complete the proof, it will be enough to check that $\lim_{h_i \to k_i}\|g_i\|_{L^p(0,T)} = 0$. By continuity of the translation in L^p we get $\lim_{h_i \to k_i}\|g_i\|_{L^p(0,T)} = 0$, $\forall p \in [1, \infty[$. For $p = +\infty$, set $\tilde{k}_i = \min(k_i, h_i)$ and $\tilde{h}_i = \max(k_i, h_i)$. Then

$$|g_i(t)| \leq \int_{t-\tilde{h}_i}^{t-\tilde{k}_i} |\dot{x}(s; \mathbf{k})|\, ds \quad \textit{if } t \in \left[\tilde{h}_i, T\right]$$

$$|g_i(t)| \leq \left| x\left(t - \tilde{k}_i\right) - x_0\left(t - \tilde{h}_i\right)\right| \quad \textit{if } t \in [\tilde{k}_i, \tilde{h}_i[\tag{4.1.2}$$

$$|g_i(t)| = 0 \quad \textit{if } t \leq \tilde{k}_i.$$

So with the condition $\lim_{s \to 0} x_0(s) = x_{00}$, we get $\lim_{h_i \to k_i}\|g_i\|_{L^\infty(0,T)} = 0$. ∎

4.2. *Proof of Theorem* 2.1

Take some fixed $T > 0$. To prove that $\mathbf{k} \mapsto x(\cdot; \mathbf{k})$ from $\mathbb{R}_+^I \to C^1(0, T; \mathbb{R}^n)$ is differentiable at point $\mathbf{k}$, it is sufficient to prove that, for each $i \in \{1, \dots, I\}$, we have $\partial x(\cdot; \mathbf{k})/\partial k_i \in C^1(0, T; \mathbb{R}^n)$ and that function $\mathbf{k} \to \partial x(\cdot; \mathbf{k})/\partial k_i$ is continuous.

To this end, denoting $\delta k_i(\delta_{1i}, \delta_{2i}, \dots, \delta_{Ii})$ (the only non-zero component is the ith one that is equal δk_i), we are going to prove that when $\delta k_i \to 0$, the ratio $\Delta(t; \mathbf{k} + \delta \mathbf{k}_i) = (x(t; \mathbf{k} + \delta k_i) - x(t; \mathbf{k}))/\delta k_i$ tends, in $C^1(0, T; \mathbb{R}^n)$, to a limit which is the solution of Eq. (5). Since by Proposition 2.1 the solution of Eq. (5) depends continuously on $\mathbf{k}$, Theorem 2.1 will therefore be proven.

The function Δ satisfies the retarded equation (assuming $x(t, 0) = x_0$ when $t < 0$):

$$\begin{gathered}
\dot{\Delta}(t) = A_0\Delta(t) + \sum_{j=1}^{I} A_j\Delta(t - k_j) \\
+ A_i\left[\Delta(t - k_i - \delta k_i) - \Delta(t - k_i)\right] - A_i g_i(t), \\
\text{a.e. } t \in [0, T] \\
\Delta(t; \mathbf{k}_i) = 0 \text{ on } [-K, 0], \\
g_i(t) = \frac{x(t - k_i; \mathbf{k}) - x(t - k_i - \delta k_i; \mathbf{k})}{\delta k_i}, \quad t \in [0, T], \\
g_i \in C^1(0, T; \mathbb{R}^n).
\end{gathered} \tag{4.2.1}$$

Solution of the previous equation may be rewritten as

$$\Delta(\cdot; \mathbf{k} + \delta \mathbf{k}_i) = \mathscr{L}(\cdot; \mathbf{k} + \delta \mathbf{k}_i) A_i g_i, \tag{4.2.2}$$

where, given $\mathbf{k} + \delta\mathbf{k}_i$, the application $\mathscr{L}(\cdot;\mathbf{k}_i + \delta\mathbf{k}_i)$ is linear from $C^1(0, T; \mathbb{R}^n)$ to $C^1(0, T; \mathbb{R}^n)$ and defined by

$$\mathscr{L}(t;\mathbf{k} + \delta\mathbf{k}_i) A_i g_i = \int_0^t e^{A_0(t-s)} \Bigg\{ \sum_{j=1}^{I} A_j \Delta(s - k_j) + A_i(\Delta(s - k_i - \delta k_i) - \Delta(s - k_i)) - A_i g_i(s) \Bigg\}\, ds. \quad (4.2.3)$$

Since the solution of Eq. (5) may also be written as $\mathscr{L}(\cdot;\mathbf{k}) A_i g$ with $g(t) = \dot{x}(t - k_i)$, we are led to prove that $\lim_{\delta k_i \to 0} \|\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) A_i g_i - \mathscr{L}(\cdot;\mathbf{k}) A_i g\|_{C^1(0,T)} = 0$, or equivalently, since we have continuously differentiable functions, that

$$\lim_{\delta k_i \to 0} \|\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) A_i g_i - \mathscr{L}(\cdot;\mathbf{k}) A_i g\|_{W^{1,\infty}(0,T)} = 0. \quad (4.2.4)$$

But we may write

$$\begin{aligned} &\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) A_i g_i - \mathscr{L}(\cdot;\mathbf{k}) A_i g \\ &\quad = \mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) A_i(g_i - g) + (\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) - \mathscr{L}(\cdot;\mathbf{k})) A_i g, \end{aligned} \quad (4.2.5)$$

and recall that by Proposition 2.1: $\lim_{\delta k_i \to 0} \|(\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) - \mathscr{L}(\cdot;\mathbf{k})) A_i g\|_{W^{1,\infty}(0,T)} = 0$. Therefore, it will be sufficient to prove that $\lim_{\delta k_i \to 0} \|g_i - g\|_{L^\infty(0,T)} = 0$ in order to conclude, with Eq. (3), that $\lim_{\delta k_i \to 0} \|\mathscr{L}(\cdot;\mathbf{k} + \delta\mathbf{k}_i) A_i(g_i - g)\|_{W^{1,\infty}(0,T)} = 0$, and (4.2.4) will follow. Since $g_i(t) = (x(t - k_i;\mathbf{k}) - x(t - k_i - \delta k_i;\mathbf{k}))/\delta k_i$, we have (assuming $\delta k_i > 0$)

$$\begin{aligned} |g_i(t) - g(t)| &= \frac{1}{\delta k_i} \int_{t-k_i-\delta k_i}^{t-k_i} |\dot{x}(s;\mathbf{k}) - \dot{x}(t - k_i;\mathbf{k})|\, ds \\ &\leq \sup_{s \in [0,\, \delta k_i]} |\dot{x}(t - k_i - s;\mathbf{k}) - \dot{x}(t - k_i;\mathbf{k})|. \end{aligned} \quad (4.2.6)$$

Therefore it is clear that $\lim_{\delta k_i \to 0} \sup_{s \in [0,\, \delta k_i]} |\dot{x}(t - k_i - s;\mathbf{k}) - \dot{x}(t - k_i;\mathbf{k})| = 0$ as soon as $\dot{x}(\cdot,\mathbf{k})$ is continuous on $[-K, T]$, which is guaranteed by the hypothesis f and $\dot{x}_0$ continuous with $(x_0(0), \dot{x}_0(0)) = (x_{00}, \dot{x}(0,\mathbf{k}))$. So $\lim_{\delta k_i \to 0} \|g_i - g\|_{L^\infty(0,T)} = 0$, which is the desired result. ∎

Justification of Remark 2.2

Assuming $\mathbf{k} = 0$ we have to prove that for any fixed p in $[1, \infty]$, we have

$$\lim_{\delta k_i \to 0} \|\mathscr{L}(\cdot; \delta \mathbf{k}_i) A_i g_i - \mathscr{L}(\cdot; 0) A_i g\|_{W^{1,p}(0,T)} = 0. \qquad (4.2.7)$$

Then it suffices to prove that $\lim_{k_i \to 0} \|g_i - g\|_{L^p(0,T)} = 0$. We shall do it considering two cases.

Case 1. $t \in [\delta k_i, T]$. We have

$$\begin{aligned} |g_i(t) - g(t)| &= \frac{1}{\delta k_i} \int_{t-\delta k_i}^{t} |\dot{x}(s;0) - \dot{x}(t;0)|\, ds \\ &\le \sup_{s \in [0,\, \delta k_i]} |\dot{x}(t-s;0) - \dot{x}(t;0)| \end{aligned} \qquad (4.2.8)$$

Therefore, it follows from continuity of f that $\lim_{\delta k_i \to 0} \sup_{s \in [0,\, \delta k_i]} |\dot{x}(t - s;0) - \dot{x}(t;0)| = 0$, so we get $\|g_i - g\|_{L^p(\delta k_i, T)} = 0$.

Case 2. $t \in [0, \delta k_i[$. Then

$$g_i(t) - g(t) = \frac{1}{\delta k_i} \int_0^t \dot{x}(s;0)\, ds + \frac{x_{00} - x_0(t - \delta k_i)}{\delta k_i} - \dot{x}(t;0). \quad (4.2.9)$$

Using $\int_0^t \dot{x}(s;0)\, ds = \int_0^{\delta k_i - t} \dot{x}(s;0)\, ds + \int_{\delta k_i - t}^{t} \dot{x}(s;0)\, ds$, we readily get

$$\begin{aligned} &\|g_i(t) - g(t)\|_{L^p(0,\, \delta k_i)} \\ &\le \delta k_i^{1/p} \Bigg[\sup_{s,t \in [0;\, \delta k_i]} |\dot{x}(s;0) - \dot{x}(t - \delta k_i;0)| \\ &\qquad + \sup_{t \in [0,\, \delta k_i]} \left| \dot{x}(\delta k_i - t;0) - \frac{x_{00} - x_0(t - \delta k_i)}{\delta k_i - t} \right| \\ &\qquad + \sup_{s,t \in [0,\, \delta k_i]} |\dot{x}(s;0) - \dot{x}(t;0)| \Bigg]. \end{aligned} \qquad (4.2.10)$$

The conclusion may be reached in two steps.

Step 1. $p \in [1, \infty[$. We know that continuity of f implies that of $\dot{x}(\cdot;0)$. So the condition $\overline{\lim}_{t \to 0} |(x_0(t) - x_{00})/t|_{\mathbb{R}^n} < +\infty$ ensures that $\lim_{\delta k_i \to 0} \|g_i(t) - \dot{x}(t;0)\|_{L^p(0,\, \delta k_i)} = 0$.

Step 2. $p = \infty$. It is sufficient to add the hypothesis of the existence of $\dot{x}_0(0)$ with $\dot{x}_0(0) = \dot{x}(0,0)$ to get

$$\lim_{\delta k_i \to 0} \sup_{t \in [0, \delta k_i]} \left| \dot{x}(\delta k_i - t; 0) - \frac{x_{00} - x_0(t - \delta k_i)}{\delta k_i - t} \right| < \infty,$$

in which case it follows that $\lim_{\delta k_i \to 0} \|g_i(t) - \dot{x}(t;0)\|_{L^\infty(0, \delta k_i)} = 0$.

The triangle inequality shows then that $\lim_{k_i \to 0} \|g_i - g\|_{L^p(0,T)} = 0$ for any $p \in [1, \infty]$.

4.3. *Proof of Corollary* 3.1

Proof of (i). We rewrite $\sigma_{i,T}$ as

$$\sigma_{i,T} = \frac{1}{2} \frac{d}{dk_i} \left\{ \int_0^T \langle Qx_i(t;k_i), x_i(t;k_i) \rangle \, dt + \int_{-k_i}^0 \langle F^T RF x_0(t), x_0(t) \rangle_{\mathbb{R}^n} \, dt \right.$$
$$\left. \int_{T-k_i}^T \langle F^T RF x_i(t;k_i), x_i(t;k_i) \rangle_{\mathbb{R}^n} \, dt \right\} \Big|_{k_i = 0} \quad \text{for } i \in \mathscr{I}'' \qquad (4.3.1)$$
$$\sigma_{i,T} = \frac{1}{2} \frac{d}{dk_i} \left\{ \int_0^T \langle Qx_i(t;k_i); x_i(t;k_i) \rangle \, dt \right\} \Big|_{k_i = 0} \quad \text{for } i \in \mathscr{I}'.$$

The assumptions on $\dot{x}_0(0)$ exist and $x_0(0) = x_{00}$ ensure $\overline{\lim}_{t \to 0} |(x_0(t) - x_{00})/t|_{\mathbb{R}^n} < +\infty$, so that the mapping $k_i \to x_i(\cdot; k_i)$, being continuously differentiable in view of Remark 2.2 and $T < \infty$, we may differentiate it with respect to k under the integral symbol. So we get

$$\sigma_{i,T} = -\int_0^T \int_0^t \langle Qe^{A(t-s)} A_i A e^{As} x_{00}, e^{At} x_{00} \rangle_{\mathbb{R}^n} \, ds \, dt$$
$$\frac{1}{2} \langle F^T RF x_{00}, x_{00} \rangle_{\mathbb{R}^n} - \frac{1}{2} \langle F^T RF e^{AT} x_{00}, x_{00} \rangle_{\mathbb{R}^n} \quad \text{for } i \in \mathscr{I}'',$$
$$\sigma_{i,T} = -\int_0^T \int_0^t \langle Qe^{A(t-s)} A_i A e^{As} x_{00}, e^{At} x_{00} \rangle_{\mathbb{R}^n} \, ds \, dt \quad \text{for } i \in \mathscr{I}'. \quad (4.3.2)$$

Proof of (ii). $T = +\infty$. F being a stabilizing feedback gain when $\mathbf{k} = 0$, there exists $\omega_0 > 0$ and $M \geq 1$ such that $\|e^{At}\| \leq Me^{-\omega_0 t}$ for all $t \geq 0$. It is known (see, e.g., [8, p. 383]) that the stability property is preserved for any delay-vector $\mathbf{k}$ satisfying the condition $M(\Sigma_{i=1}^I \|A_i\|)\Sigma_{i=1}^I \|A_i\| k_i < \omega_0$.

Moreover, for such a $\mathbf{k}$, we have

$$|x(t;\mathbf{k})|_{\mathbb{R}^n} \le M \sup_{s\in[-K,K]} |x(s)|_{\mathbb{R}^n} e^{-\omega(t-K)}, \tag{4.3.3}$$

where $\omega \in]0, \omega_0]$ is the unique solution of equation

$$\omega_0 - \omega = \frac{M}{\omega}\left\{\sum_{i=1}^{I} \|A_i\|(e^{\omega k_i} - 1)\right\} \times \left\{\|A_0\| + \sum_{i=1}^{I} \|A_i\| e^{\omega k_i}\right\}. \tag{4.3.4}$$

So the mapping $\mathbf{k} \to J_\infty(u_F, x_F)$ is well-defined for $|\mathbf{k}|$ small. Moreover, it now becomes clear from Theorem 2.1 that there exists some $g \in L^2(0,\infty;\mathbb{R}_+)$ and independent of $\mathbf{k}$ such that and $|\partial x(\cdot;\mathbf{k})/\partial k_i| \le g$, $i \in \{1,\ldots,I\}$. So we may permute the symbols d/dk_i and $\int_0^\infty$ for $|\mathbf{k}|$ small enough. This proves item (ii). ∎

Proof of (iii). $T = +\infty$ and F is the optimal feedback gain given by Eq. (13). Take first $i \in \mathscr{I}''$. Then

$$\sigma_{i,\infty} = \tfrac{1}{2}\langle F^T R F x_{00}, x_{00}\rangle_{\mathbb{R}^n} + J \tag{4.3.5}$$

with

$$\begin{aligned} J &= -\int_0^\infty \int_0^t \langle Q e^{A(t-s)} A_i A e^{As} x_{00}, e^{At} x_{00}\rangle_{\mathbb{R}^n}\, ds\, dt \\ &= \int_0^\infty \left\langle \int_0^t e^{-As} A_i A e^{As} x_{00} ds, e^{A^T t}(PA + A^T P) e^{At} x_{00}\right\rangle_{\mathbb{R}^n} dt \\ &= \int_0^\infty \left\langle \int_0^t e^{-As} A_i A e^{As} x_{00}\, ds, \frac{d}{dt}(e^{A^T t} P e^{At}) x_{00}\right\rangle_{\mathbb{R}^n} dt. \end{aligned} \tag{4.3.6}$$

An integration by parts gives

$$J = \lim_{t\to\infty} S(t) - \int_0^\infty \langle A e^{At} x_{00}, A_i^T P e^{At} x_{00}\rangle_{\mathbb{R}^n}\, dt \tag{4.3.7}$$

with

$$\begin{aligned} S(t) &= \left\langle \int_0^\infty \int_0^t e^{-As} A_i A e^{As} x_{00}\, ds, e^{A^T t} P e^{At} x_{00} e^{A(t-s)} A_i A e^{As} x_{00}\right\rangle_{\mathbb{R}^n} \\ &= \left\langle \int_0^\infty \int_0^t e^{A(t-s)} A_i A e^{As} x_{00}\, ds, P e^{At} x_{00}\right\rangle_{\mathbb{R}^n}. \end{aligned} \tag{4.3.8}$$

Since there exists $M \leq 1$ and ω_0 such that $\|e^{At}\| \leq Me^{-\omega_0 t}$ for all $t \geq 0$, there exists $C > 0$ such that $\|S(t)\| \leq Cte^{-2\omega_0 t}$. Hence $\lim_{t \to \infty} S(t) = 0$ and

$$J = -\int_0^\infty \langle Ae^{At}x_{00}, A_i^T Pe^{At}x_{00} \rangle_{\mathbb{R}^n} \, dt \tag{4.3.9}$$

so that

$$\sigma_{i,\infty} = \frac{1}{2}\langle F^T RFx_{00}, x_{00} \rangle_{\mathbb{R}^n} - \int_0^\infty \langle Ae^{At}x_{00}, A_i^T Pe^{At}x_{00} \rangle_{\mathbb{R}^n} \, dt. \tag{4.3.10}$$

For $i \in \mathscr{I}'$, we have

$$\sigma_{i,\infty} = J = -\int_0^\infty \langle Ae^{At}x_{00}, A_i^T Pe^{At}x_{00} \rangle_{\mathbb{R}^n} \, dt \tag{4.3.11}$$

that ends this proof. ∎

5. CONCLUDING REMARKS

In Theorem 2.1, we have restricted both statement and computations to first order differentiability, but more general regularity results are reachable for the mapping $\mathbf{k} \mapsto x(\cdot; \mathbf{k})$. Just observe that the first order derivatives are themselves solutions of retarded equations and then the proof of Theorem 2.1 may be repeated to derive conditions of existence and compute partial derivatives of any order.

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