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Semistar Dedekind domains

S. El Baghdadi[a], Marco Fontana[b,*], Giampaolo Picozza[b]

[a]*Department of Mathematics, Faculté des Sciences et Techniques, P.O. Box 523, Beni Mellal, Morocco*
[b]*Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo San Leonardo Murialdo 1, 00146 Roma, Italy*

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Abstract

Let D be an integral domain and $\star$ a semistar operation on D. As a generalization of the notion of Noetherian domains to the semistar setting, we say that D is a $\star$-Noetherian domain if it has the ascending chain condition on the set of its quasi-$\star$-ideals. On the other hand, as an extension the notion of Prüfer domain (and of Prüfer v-multiplication domain), we say that D is a Prüfer $\star$-multiplication domain (P$\star$MD, for short) if D_M is a valuation domain, for each quasi-$\star_f$-maximal ideal M of D. Finally, recalling that a Dedekind domain is a Noetherian Prüfer domain, we define a $\star$-Dedekind domain to be an integral domain which is $\star$-Noetherian and a P$\star$MD. For the identity semistar operation d, this definition coincides with that of the usual Dedekind domains and when the semistar operation is the v-operation, this notion gives rise to Krull domains. Moreover, Mori domains not strongly Mori are $\star$-Dedekind for a suitable spectral semistar operation.

Examples show that $\star$-Dedekind domains are not necessarily integrally closed nor one-dimensional, although they mimic various aspects, varying according to the choice of $\star$, of the "classical" Dedekind domains. In any case, a $\star$-Dedekind domain is an integral domain D having a Krull overring T (canonically associated to D and $\star$) such that the semistar operation $\star$ is essentially "univocally associated" to the v-operation on T.

In the present paper, after a preliminary study of $\star$-Noetherian domains, we investigate the $\star$-Dedekind domains. We extend to the $\star$-Dedekind domains the main classical results and several characterizations proven for Dedekind domains. In particular, we obtain a characterization of a $\star$-Dedekind domain by a property of decomposition of any semistar ideal into a "semistar product" of prime ideals. Moreover, we show that an integral domain D is a $\star$-Dedekind domain

*Corresponding author.
E-mail addresses: baghdadi@fstbm.ac.ma (S. El Baghdadi), fontana@mat.uniroma3.it (M. Fontana), picozza@mat.uniroma3.it (G. Picozza).

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if and only if the Nagata semistar domain $\mathrm{Na}(D, \star)$ is a Dedekind domain. Several applications of the general results are given for special cases of the semistar operation $\star$.

MSC: 13A15; 13G05; 13E99

1. Introduction and background results

Dedekind domains play a crucial role in classical algebraic number theory and their study gave a relevant contribution to a rapid development of commutative ring theory and ideal theory: Noetherian, Krull and Prüfer domains arose in the early stages of these theories, for generalizing different aspects of Dedekind domains.

Star operations provided new insight in multiplicative ideal theory. For instance, the use of the v- and t-operations has produced a common treatment and a deeper understanding of Dedekind and Krull domains. In 1994, Okabe and Matsuda [45] introduced the semistar operations, extending the notion of star operation and the related classical theory of ideal systems, based on the pioneering works by Krull, Noether, Prüfer and Lorenzen (cf. [28,37]). Semistar operations, due to a major grade of flexibility with respect to star operations, provide a natural and general setting for a wide class of questions and for a deeper and comparative study of several relevant classes of integral domains (cf. for instance [16–20,31,41,42,45]).

In this paper, we explore a general theory of Dedekind-type domains, depending on a semistar operation. A first attempt in this direction was done by Aubert [4] and, more extensively, by Halter-Koch [28, Chapter 23], where the author investigated Dedekind domains in the language of finitary ideal systems on commutative monoids. Our approach is based on the classical multiplicative ideal theory on integral domains, as in Gilmer's book [24], extended in a natural way to the semistar case. This approach has already produced a generalized and convenient setting for considering Kronecker function rings [18–20], Nagata rings [20], and Prüfer multiplication domains [17].

Note that the module systems approach on commutative monoids, developed recently by Halter-Koch in [30], provides an alternative general frame for (re)considering semistar operations on integral domains and related topics. More precisely, most of the results contained in this paper are of purely multiplicative nature and remain valid in the more general setting of commutative cancellative monoids (cf. also Remark 1.2).

Recall that a Dedekind domain is a Noetherian Prüfer domain. Let D be an integral domain and $\star$ a semistar operation on D. As a generalization of Noetherian domains to the semistar setting, we define D to be a $\star$-Noetherian domain if it has the ascending chain condition on the set of the ideals of D canonically associated to $\star$ (called quasi-$\star$-ideals); equivalently, a $\star$-Noetherian domain is a domain in which each nonzero ideal is $\star_f$-finite (Lemma 3.3 and Remark 3.6 (1)). For instance, as we will see later, Noetherian, Mori, and strong Mori domains are examples of $\star$-Noetherian domains, for different $\star$-operations.

On the other hand, as an extension of the notion of Prüfer domain (and of Prüfer v-multiplication domain), given a semistar operation $\star$ on an integral domain D, we say that D is a Prüfer semistar multiplication domain (P$\star$MD, for short) if D_M is a valuation domain, for each maximal element M in the nonempty set of the ideals of D associated to the finite type semistar operation canonically deduced from $\star$ (i.e., a quasi-$\star_f$-maximal ideal of D). Finally, we define a $\star$-Dedekind domain ($\star$-DD, for short) to be an integral domain which is $\star$-Noetherian and a P$\star$MD. For the identity semistar operation d, this definition coincides with that of the usual Dedekind domains and when the semistar operation is the v-operation, this notion gives rise to Krull domains. Moreover, Mori domains not strongly Mori are $\star$-Dedekind for a suitable spectral semistar operation (Example 4.22).

In the general semistar setting, $\star$-Dedekind domains are not necessarily integrally closed nor one-dimensional, although they mimic various aspects, varying according to the choice of $\star$, of the "classical" Dedekind domains. In any case, a $\star$-Dedekind domain is an integral domain D having a Krull overring T (canonically associated to D and $\star$) such that the semistar operation $\star$ is essentially "univocally associated" to the v-operation on T (Remark 4.21).

In the present paper we develop a theory which enlightens different facets of the $\star$-Dedekind domains and shows the interest in studying these new classes of integral domains of Dedekind-type, parametrized by semistar operations. After recalling in the present section the main data needed for this work, in Section 2, as a first step to the main goal, we introduce and study the concept of "semistar almost Dedekind domains" ($\star$-ADD, for short), which provides a natural generalization of the classical notion of almost Dedekind domains. Our study, in the particular case of $\star = v$, extends and completes the investigation on t-almost Dedekind domains initiated by Kang [38, Section 4]. Among the main results proven in this section, we have that an integral domain D is a $\star$-ADD if and only if the Nagata semistar domain $\mathrm{Na}(D,\star)$ is an almost Dedekind domain (in particular, in this case, $\mathrm{Na}(D,\star)$ coincides with the Kronecker semistar function ring $\mathrm{Kr}(D,\star)$).

Section 3 is devoted to the study of the semistar Noetherian domains. In particular, we investigate the local–global behaviour of this notion and we obtain several relevant results on $\star$-Noetherian domains, in case of stable semistar operations.

In Section 4, we introduce and study the semistar Dedekind domains. We extend to the $\star$-Dedekind domains the main classical results and several characterizations proven for Dedekind domains. In particular, we obtain a characterization of a $\star$-Dedekind domain by a property of decomposition of any semistar ideal into a "semistar product" of prime ideals. Moreover, we show that an integral domain D is a $\star$-DD if and only if the Nagata semistar domain $\mathrm{Na}(D,\star)$ is a Dedekind domain (in particular, in this case, $\mathrm{Na}(D,\star)$ coincides with the Kronecker semistar function ring $\mathrm{Kr}(D,\star)$, which is in fact a PID).

* * * * *

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\boldsymbol{F}(D)$ represent the nonzero fractional ideals of D (i.e., $\boldsymbol{F}(D) := \{E \in \overline{\boldsymbol{F}}(D) \mid dE \subseteq D$ for some nonzero element $d \in D\}$). Finally, let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K (it is clear that $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$).

A map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^{\star}$, is called a *semistar operation* on D if, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(xE)^{\star} = xE^{\star}$,
$(\star_2)$ $E \subseteq F$ implies $E^{\star} \subseteq F^{\star}$,
$(\star_3)$ $E \subseteq E^{\star}$ and $E^{\star\star} := (E^{\star})^{\star} = E^{\star}$,

cf. for instance [16,41,42,45]. Recall that, given a semistar operation $\star$ on D, for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following basic formulas follow easily from the axioms:

$$(EF)^{\star} = (E^{\star}F)^{\star} = (EF^{\star})^{\star} = (E^{\star}F^{\star})^{\star},$$

$$(E+F)^{\star} = (E^{\star}+F)^{\star} = (E+F^{\star})^{\star} = (E^{\star}+F^{\star})^{\star},$$

$$(E:F)^{\star} \subseteq (E^{\star}: F^{\star}) = (E^{\star}:F) = (E^{\star}:F)^{\star}, \text{ if } (E:F) \neq 0,$$

$$(E \cap F)^{\star} \subseteq E^{\star} \cap F^{\star} = (E^{\star} \cap F^{\star})^{\star}, \text{ if } E \cap F \neq (0),$$

cf. for instance [16, Theorem 1.2 and p. 174].

A (*semi*)*star operation* $\star$ on an integral domain D is a semistar operation, that restricted to the set $\boldsymbol{F}(D)$ of fractional ideals, is a *star operation* on D [24, (32.1)]. It is very easy to see that a semistar operation $\star$ on D is a (semi)star operation (on D) if and only if $D^{\star} = D$.

Example 1.1. (1) The first (trivial) examples of semistar operations are given by d_D (or, simply, d), called the *identity* (*semi*)*star operation on* D, defined by $E^{d_D} := E$, for each $E \in \overline{\boldsymbol{F}}(D)$ and by e_D (or, simply, e), defined by $E^{e_D} := K$, for each $E \in \overline{\boldsymbol{F}}(D)$.

More generally, if T is an overring of D, we can define a semistar operation on D, denoted by $\star_{\{T\}}$ and defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\boldsymbol{F}}(D)$. It is obvious that $d_D = \star_{\{D\}}$, $e_D = \star_{\{K\}}$ and that $\star_{\{T\}}$ is a semistar non-(semi)star operation on D if and only if $D \subsetneq T$.

(2) If $\star$ is a semistar operation on D, then we can consider a map $\star_f : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined, for each $E \in \overline{\boldsymbol{F}}(D)$, as follows:

$$E^{\star_f} := \bigcup \{F^{\star} \mid F \in \boldsymbol{f}(D) \text{ and } \quad F \subseteq E\}.$$

It is easy to see that $\star_f$ is a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Note that, for each $F \in \boldsymbol{f}(D)$, $F^{\star} = F^{\star_f}$. A semistar operation $\star$ is called a *semistar operation of finite type* if $\star = \star_f$. It is easy to see that $(\star_f)_f = \star_f$ (that is, $\star_f$ is of finite type).

For instance, if v_D (or, simply, v) is *the* v-(*semi*)*star operation on* D defined by $E^v := (E^{-1})^{-1}$, for each $E \in \overline{\boldsymbol{F}}(D)$, with $E^{-1} := (D :_K E) := \{z \in K \mid zE \subseteq D\}$, then the semistar operation of finite type $(v_D)_f$ (or, simply, v_f) associated to v_D is denoted by t_D (or, simply, t) and it is called *the* t-(*semi*)*star operation on* D (note that $D^v = D^t = D$).

Note also that, for each overring T of D, the semistar operation $\star_{\{T\}}$ on D is a semistar operation of finite type.

(3) If $\Delta \subseteq \mathrm{Spec}(D)$, the map $\star_{\Delta} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^{\star_{\Delta}} := \bigcap \{ED_P \mid P \in \Delta\}$, is a semistar operation on D [16, Lemma 4.1]. A semistar operation $\star$ is called a *spectral*

semistar operation on D if there exists a subset Δ of $\mathrm{Spec}(D)$ such that $\star = \star_\Delta$. If $\Delta = \{P\}$, then $\star_{\{P\}}$ is the spectral semistar operation on D defined by $E^{\star_{\{P\}}} := ED_P$, for each $E \in \overline{\boldsymbol{F}}(D)$, i.e. $\star_{\{P\}} = \star_{\{D_P\}}$. If $\Delta = \emptyset$, then $\star_\emptyset = e_D$.

We say that a semistar operation is *stable* (*with respect to finite intersections*) if $(E \cap F)^\star = E^\star \cap F^\star$, for each $E, F \in \overline{\boldsymbol{F}}(D)$. For a spectral semistar operation the following properties hold [16, Lemma 4.1]:

(3.a) For each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$, $ED_P = E^{\star_\Delta} D_P$.
(3.b) The semistar operation $\star_\Delta$ is stable.
(3.c) For each $P \in \Delta$, $P^{\star_\Delta} \cap D = P$.
(3.d) For each nonzero integral ideal I of D such that $I^{\star_\Delta} \cap D \neq D$, there exists a prime ideal $P \in \Delta$ such that $I \subseteq P$.

If $\star_1$ and $\star_2$ are two semistar operations on D, we set $\star_1 \leqslant \star_2$, if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$. It is easy to see that $\star_1 \leqslant \star_2$ if and only if $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$. Obviously, if $\star_1 \leqslant \star_2$, then $(\star_1)_f \leqslant (\star_2)_f$; moreover, for each semistar operation $\star$ on D, we have $d_D \leqslant \star_f \leqslant \star \leqslant e_D$. In particular, $t_D \leqslant v_D$; furthermore, it is not difficult to see that, for each (semi)star operation $\star$ on D, we have $\star \leqslant v_D$ and $\star_f \leqslant t_D$ [24, Theorem 34.1(4)].

A *quasi-$\star$-ideal* of D is a nonzero ideal I of D such that $I = I^\star \cap D$. This notion generalizes the notion of *$\star$-ideal* for a star operation on D, which is a nonzero ideal I of D such that $I = I^\star$. More precisely, it is clear that, for a (semi)star operation $\star$, the quasi-$\star$-ideals coincide with the $\star$-ideals.

Note that each nonzero ideal I of D, such that $I^\star \subsetneq D^\star$, is contained in a (non trivial) quasi-$\star$-ideal of D: in fact, the ideal $I^\star \cap D$ is a quasi-$\star$-ideal of D and $I \subseteq I^\star \cap D$.

A *quasi-$\star$-prime* of D is a nonzero prime ideal of D that is also a quasi-$\star$-ideal of D. A *quasi-$\star$-maximal ideal* of D is a (proper) ideal of D, which is maximal in the set of all quasi-$\star$-ideals of D.

If $\star$ is a semistar operation of finite type on D, with $D \neq K$, each quasi-$\star$-ideal of D is contained in a quasi-$\star$-maximal ideal. Moreover, each quasi-$\star$-maximal ideal of D is prime [16, Lemma 4.20]. We denote by $\mathscr{M}(\star)$ the set of all quasi-$\star$-maximal ideals of D. Thus, if $\star = \star_f$ and D is not a field, then $\mathscr{M}(\star) \neq \emptyset$.

Example 1.1. (4) If $\star$ is a semistar operation on D, we denote by $\tilde{\star}$ the spectral semistar operation $\star_{\mathscr{M}(\star_f)}$, induced by the set $\mathscr{M}(\star_f)$ of the quasi-$\star_f$-maximal ideals of D, i.e. for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\tilde{\star}} := \bigcap \{ED_Q \mid Q \in \mathscr{M}(\star_f)\}.$$

The semistar operation $\tilde{\star}$ is stable and of finite type and $\tilde{\star} \leqslant \star_f$ (cf. [18] and also [16, p. 181] for an equivalent definition of $\tilde{\star}$; see [2, Section 2], [25,33] for an analogous construction in the star setting). Note that, when $\star$ is a (semi)star operation on D, then also $\tilde{\star}$ is a (semi)star operation on D.

If $\star = d_D$, then obviously $\tilde{\star} = d_D$. If $\star = v_D$, then $\tilde{\star}$ is the (semi)star operation that we denote by w_D (or, simply, w), following Wang Fanggui and McCasland (cf. [12–14]).

Note that, for $\Delta = \mathscr{M}(\star_f)$, the semistar operation $\tilde{\star}$ satisfies the properties (3.a)–(3.d), stated above for a general spectral semistar operation.

(5) Let D be an integral domain and T an overring of D. Let $\star$ be a semistar operation on D. We can define a semistar operation $\dot{\star}^T : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ on T, by setting:

$$E^{\dot{\star}^T} := E^{\star}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(T)(\subseteq \overline{\boldsymbol{F}}(D)),$$

[18, Proposition 2.8]. When $T = D^{\star}$, we denote simply by $\dot{\star}$ the (semi)star operation $\dot{\star}^{D^{\star}}$ on $D^{\star}$.

Note that $(\dot{\star}_f)^T = (\dot{\star}^T)_f$ [17, Lemma 3.1]. In particular, if $\star = \star_f$ then $\dot{\star}^T$ is a semistar operation of finite type on T.

(6) On the other hand, if $*$ is a semistar operation on an overring T of an integral domain D, we can construct a semistar operation $\ast_D : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ on D, by setting:

$$E^{\ast_D} := (ET)^{*}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D),$$

[18, Proposition 2.9].

For more details on the semistar operations considered in (5) and (6), cf. [18,45].

Remark 1.2. Let $\star$ be a semistar operation on an integral domain D. For each nonzero ideal I of D, define $I_{r(\star)} := I^{\star} \cap D$. Then it is easy to see that the map $I \mapsto I_{r(\star)}$ defines a weak ideal system (=x-system in the sense of Aubert) on D (as a commutative cancellative monoid, disregarding the additive structure), cf. [28, Chapter 2], therefore the theory developed in [28, Part A] applies. In particular, $r(\star_f) = r(\star)_s$ [28, p. 25], $\mathscr{M}(\star) = r(\star)\text{-max}(D)$ [28, p. 57], and $\widetilde{\star} = r(\star)_s[d]$ [29, Definition 3.1 and Proposiition 3.2].

Furthermore, using the more general setting of module systems on monoids, the spectral semistar operations (Example 1.1(3)) and the semistar operations $\dot{\star}^T$ and $\ast_D$, defined in Example 1.1(5) and (6), have a natural correspondent interpretation in terms of module systems, which is described in [30], and so the theory developed in this paper also applies.

Proposition 1.3. *Let D be an integral domain and T an overring of D.*

(1) *Let $*$ be a semistar operation on T. Denote simply by $\star$ the semistar operation $\ast_D$, defined on D, then the semistar operations $\dot{\star}^T$ and $*$ (both defined on T) coincide.*

(2) *Let $\star$ be a semistar operation on D. Denote simply by $*$ the semistar operation $\dot{\star}^T$, defined on T, then $\star \leqslant \ast_D$ (note that both semistar operations are defined on D). Furthermore, if $T = D^{\star}$ then $\star = \ast_D$.*

Proof. (1) and the first statement in (2) are already in [18, Corollary 2.10], [45, Lemma 45]. For the last statement note that, for each $E \in \overline{\boldsymbol{F}}(D)$, $E^{\ast_D} = (ET)^{*} = (ET)^{\dot{\star}^T} = (ET)^{\star} = (ED^{\star})^{\star} = (ED)^{\star} = E^{\star}$. $\square$

If R is a ring and X an indeterminate over R, then the ring $R(X) := \{f/g \mid f, g \in R[X]$ and $\boldsymbol{c}(g) = R\}$ (where $\boldsymbol{c}(g)$ is the content of the polynomial g) is called the *Nagata ring* of R [24, Proposition 33.1]. A more general construction of a Nagata ring associated to a semistar operation defined on an integral domain D was considered in [20] (cf. also [28, Chapter 20, Exercise 4], and [38] for the star case).

Proposition 1.4 ([20, Proposition 3.1, Proposition 3.4, Corollary 3.5, Theorem 3.8]). *Let* $\star$ *be a semistar operation on an integral domain* D *and set* $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0$ *and* $(\boldsymbol{c}(h))^\star = D^\star\}$. *Let* $\tilde{\star}$ (*respectively*, $\dot{\tilde{\star}}$) *be the spectral semistar operation defined on* D (*respectively*, $D^{\tilde{\star}}$), *introduced in Example* 1.1((4) *and* (5)). *Then*:

(1) $N(\star)$ *is a saturated multiplicative subset of* $D[X]$ *and* $N(\star) = N(\star_f) = D[X] \setminus \bigcup\{Q[X] \mid Q \in \mathscr{M}(\star_f)\}$.

Set $\mathrm{Na}(D,\star) := D[X]_{N(\star)}$ *and call this integral domain* the Nagata ring of D with respect to the semistar operation $\star$.

(2) $\mathrm{Max}(\mathrm{Na}(D,\star)) = \{Q[X]_{N(\star)} \mid Q \in \mathscr{M}(\star_f)\}$ *and* $\mathscr{M}(\star_f)$ *coincides with the canonical image in* $\mathrm{Spec}(D)$ *of* $\mathrm{Max}(\mathrm{Na}(D,\star))$.
(3) $\mathrm{Na}(D,\star) = \bigcap\{D_Q(X) \mid Q \in \mathscr{M}(\star_f)\}$.
(4) $E^{\tilde{\star}} = E\mathrm{Na}(D,\star) \cap K$, *for each* $E \in \overline{\boldsymbol{F}}(D)$.
(5) $\mathscr{M}(\star_f) = \mathscr{M}(\tilde{\star})$.
(6) $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}},\dot{\tilde{\star}}) \supseteq D^{\tilde{\star}}(X)$.

It is clear that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f)$ and, when $\star = d$ (the identity (semi)star operation) on D, then $\mathrm{Na}(D,d) = D(X)$.

Given a semistar operation $\star$ on an integral domain D, we say that $\star$ is an *e.a.b.* (*endlich arithmetisch brauchbar*) *semistar operation* of D if, for all $E,F,G \in \boldsymbol{f}(D)$, $(EF)^\star \subseteq (EG)^\star$ implies that $F^\star \subseteq G^\star$ [18, Definition 2.3 and Lemma 2.7].

We recall next the definition of two relevant semistar operations, associated to a given semistar operation.

Example 1.1. (7) Given a semistar operation $\star$ on an integral domain D, we call *the semistar integral closure* $[\star]$ *of* $\star$, the semistar operation on D defined by setting:

$$F^{[\star]} := \cup\{((H^\star : H)F)^{\star_f} \mid H \in \boldsymbol{f}(D)\}, \quad \text{for each } F \in \boldsymbol{f}(D),$$

and

$$E^{[\star]} := \cup\{F^{[\star]} \mid F \in \boldsymbol{f}(D),\ F \subseteq E\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D).$$

It is not difficult to see that the operation $[\star]$ is a semistar operation of finite type on D, that $\star_f \leqslant [\star]$, hence $D^\star \subseteq D^{[\star]}$, and that $D^{[\star]}$ is integrally closed [18, Definition 4.2, Propositions 4.3 and 4.5(3)]. Therefore, it is obvious that if $D^\star = D^{[\star]}$ then $D^\star$ is integrally closed.

(8) Given an arbitrary semistar operation $\star$ on an integral domain D, it is possible to associate to $\star$, an e.a.b. semistar operation of finite type $\star_a$ on D, called *the e.a.b. semistar operation associated to* $\star$, defined as follows:

$$F^{\star_a} := \cup\{((FH)^\star : H) \mid H \in \boldsymbol{f}(D)\}, \quad \text{for each } F \in \boldsymbol{f}(D),$$

and

$$E^{\star_a} := \cup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D),$$

[18, Definition 4.4]. Note that $[\star] \leqslant \star_a$, that $D^{[\star]} = D^{\star_a}$ and if $\star$ is an e.a.b. semistar operation of finite type then $\star = \star_a$ [18, Proposition 4.5].

More information about the semistar operations $[\star]$ and $\star_a$ can be found in [19,23,28,31,37,44,45].

Let $\star$ be a semistar operation on D and let V be a valuation overring of D. We say that V is a *$\star$-valuation overring of D* if, for each $F \in \boldsymbol{f}(D)$, $F^\star \subseteq FV$ (or equivalently, $\star_f \leqslant \star_{\{V\}}$ (Example 1.1(1)).

Note that a valuation overring V of D is a $\star$-valuation overring of D if and only if $V^{\star_f} = V$. (The "only if" part is obvious; for the "if" part recall that, for each $F \in \boldsymbol{f}(D)$, there exists a nonzero element $x \in K$ such that $FV = xV$, thus $F^\star \subseteq (FV)^{\star_f} = (xV)^{\star_f} = xV^{\star_f} = xV = FV$).

More details on semistar valuation overrings can be found in [19,20] (cf. also [27,31,37]).

We recall next the construction of the Kronecker function ring with respect to a semistar operation (the star case is studied in detail in [24, Section 32] and [28, Chapter 20, Example 6]).

Proposition 1.5 ([[18, Proposition 3.3, Theorem 3.11, Theorem 5.1, Corollary 5.2, Corollary 5.3], [19, Theorem 3.5]]). *Let $\star$ be any semistar operation defined on an integral domain D with quotient field K and let $\star_a$ be the e.a.b. semistar operation associated to $\star$ (Example 1.1(8)). Consider the e.a.b. (semi)star operation $\dot{\star}_a := \dot{\star}_a^{D^{\star_a}}$ (defined in Example 1.1(5)) on the integrally closed integral domain $D^{\star_a} = D^{[\star]}$ (Example 1.1((7) and (8))). Set*

$$\mathrm{Kr}(D,\star) := \{f/g \mid f,g \in D[X]\setminus\{0\} \text{ and there exists } h \in D[X]\setminus\{0\}$$
$$\text{such that } (\boldsymbol{c}(f)\boldsymbol{c}(h))^\star \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^\star\} \cup \{0\}.$$

Then we have:

(1) $\mathrm{Kr}(D,\star)$ *is a Bézout domain with quotient field $K(X)$, called* the Kronecker function ring of D with respect to the semistar operation $\star$.
(2) $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$.
(3) $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D^{\star_a},\dot{\star}_a)$.
(4) $E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K$, *for each $E \in \overline{\boldsymbol{F}}(D)$.*
(5) $\mathrm{Kr}(D,\star) = \bigcap\{V(X) \mid V$ *is a $\star$-valuation overring of D*$\}$.
(6) *If $F := (a_0,a_1,\ldots,a_n) \in \boldsymbol{f}(D)$ and $f(X) := a_0 + a_1X + \cdots + a_nX^n \in K[X]$, then:*

$$F\mathrm{Kr}(D,\star) = f(X)\mathrm{Kr}(D,\star) = \boldsymbol{c}(f)\mathrm{Kr}(D,\star).$$

If D is an integral domain and $\star$ is a semistar operation on D, we say that a nonzero ideal I is *$\star$-invertible*, if $(II^{-1})^{\star} = D^{\star}$. We define D to be a P$\star$MD if each nonzero finitely generated ideal of D is $\star_f$-invertible (cf. [17] and also [23,26,35,38,43]). In particular, note that, if $\star$ is a star operation, then D is a P$\star$MD if and only if D is $\star$-Prüfer in the sense of [28, Chapter 17].

By using $\mathrm{Na}(D,\star)$ and $\mathrm{Kr}(D,\star)$, we have the following characterization of a P$\star$MD.

Proposition 1.6 ([17, Theorem 3.1, Remark 3.1]). *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(i) *D is a* P$\star$MD,
(ii) *D_Q is a valuation domain, for each $Q \in \mathscr{M}(\star_f)$*,
(iii) $\mathrm{Na}(D,\star)$ *is a Prüfer domain*,
(iv) $\mathrm{Na}(D,\star) = \mathrm{Kr}(D,\star)$,
(v) *$\tilde{\star}$ is an e.a.b. semistar operation*,
(vi) *$\star_f$ is stable and e.a.b.*

In particular, *D is a* P$\star$MD *if and only if it is a* P$\tilde{\star}$MD. *Moreover*, *in a* P$\star$MD, $\tilde{\star} = \star_f$.

Let D be an integral domain and T an overring of D. Let $\star$ be a semistar operation on D and $\star'$ a semistar operation on T. Then, we say that T is *$(\star,\star')$-linked to D*, if

$$F^{\star} = D^{\star} \;\Rightarrow\; (FT)^{\star'} = T^{\star'},$$

for each nonzero finitely generated ideal F of D. Finally, recall that we say that T is *$(\star,\star')$-flat over D* if it is $(\star,\star')$-linked to D and, in addition, $D_{Q\cap D} = T_Q$, for each quasi-$\star'_f$-maximal ideal Q of T. More details on these notions can be found in [9] (cf. also [32,40]).

2. Semistar almost Dedekind domains

Let D be an integral domain and $\star$ a semistar operation on D. We say that D is a *semistar almost Dedekind domain* (for short, a *$\star$-ADD*) if D_M is a rank-one discrete valuation domain (for short, DVR), for each quasi-$\star_f$-maximal ideal M of D.

Note that, by definition, $\star$-ADD=$\star_f$-ADD and that, if $\star$=d (=the identity (semi)star operation), we obtain the classical notion of "almost Dedekind domain" (for short, ADD) as in [24, Section 36]. Note that, If $\star = v$, the v-ADDs coincide with the t-almost Dedekind domains studied by Kang [38, Section 4]; more generally, if $\star$ is a star operation, then D is a $\star$-ADD if and only if D is a $\star$-almost Dedekind domain in the sense of [28, Chapter 23]. Note also that, a field has only the identity (semi)star operation and thus a field is, by convention, a trivial example of a (d-)ADD (since, in this case, $\mathscr{M}(d) = \emptyset$).

An analogous notion of generalized almost Dedekind domain was considered in the language of ideal systems on commutative monoids in [28, Chapter 23].

Remark 2.1. Let $\star_1, \star_2$ be two semistar operations on D such that $(\star_1)_f \leqslant (\star_2)_f$. If D is a $\star_1$-ADD, then D is a $\star_2$-ADD. In particular:

- D is a ADD $\Rightarrow$ D is a $\star$-ADD, for each semistar operation $\star$ on D;
- if $\star$ is a (semi)star operation on D (so, $\star \leqslant v$), then:
  D is a $\star$-ADD $\Rightarrow$ D is a v-ADD (and, hence, D is integrally closed).

Note that, in general, for a semistar operation $\star$, a $\star$-ADD may be not integrally closed. For instance, let K be a field and $T := K[\![X]\!] = K + M$, where $M := XT$ is the maximal ideal of the discrete valuation domain T. Set $D := R + M$, where R is a nonintegrally closed integral domain with quotient field K (hence, D is not integrally closed [15, Proposition 2.2(10)]). Take $\star := \star_{\{T\}}$ on D. Then, we have $\star = \star_f$, $\dot{\star}^T = d_T$ is the identity (semi)star operation on T and $\mathscr{M}(\star_f) = \{M\}$ (by [20, Lemma 2.3(3)]) and $D_M = T$ [15, Proposition 1.9]. So D is a $\star$-ADD which is not integrally closed (hence, in particular, D is not an ADD).

Proposition 2.2. *Let D be an integral domain, which is not a field, and $\star$ a semistar operation on D. Then*:

(1) *D is a $\star$-ADD if and only if D_P is a DVR, for each quasi-$\star_f$-prime ideal P of D.*
(2) *If D is a* $\star$-ADD, *then D is a* P$\star$MD *and each quasi-$\star_f$-prime of D is a quasi-$\star_f$-maximal of D.*
(3) *Let T be an overring of D and $\star'$ a semistar operation on T. Assume that $D \subseteq T$ is a $(\star, \star')$-linked extension. If D is a $\star$-ADD, then T is a* $\star'$-ADD.
(4) *If D is a* $\star$-ADD, *then $D^\star$ is a $\dot{\star}$-ADD.*

Proof. (1) It follows easily from the fact that each quasi-$\star_f$-prime is contained in a quasi-$\star_f$-maximal [20, Lemma 2.3(1)].

(2) is a straightforward consequence of (1) and of Proposition 1.6 ((i) $\Leftrightarrow$ (ii)).

(3) Let $N \in \mathscr{M}(\star'_f)$, then $(N \cap D)^{\star_f} \neq D^\star$ [9, Proposition 3.2]. Let $M \supseteq N \cap D$ be a quasi-$\star_f$-maximal ideal of D. We have $D_M \subseteq D_{N\cap D} \subseteq T_N$. So $T_N = D_{N\cap D} = D_M$, because D_M is a DVR (by assumption D is a $\star$-ADD). From this proof we deduce also that $N \cap D$ $(=M)$ is a quasi-$\star_f$-maximal ideal of D, for each quasi-$\star'_f$-maximal ideal N of T.

(4) It follows from [9, Lemma 3.1(e)] and (3). $\square$

Remark 2.3. (1) We will show that, for a converse of Proposition 2.2(2), we will need additional conditions (cf. Theorem 2.14 ((1) $\Leftrightarrow$ (3), (4))).

(2) The converse of Proposition 2.2(4) is not true in general. Indeed, let K be a field and $k \subset K$ a proper subfield of K. Let $T := K[\![X]\!]$ and $D := k + M$, where $M := XT$ is the maximal ideal of T. Take $\star := \star_{\{T\}}$ on D. Note that $\star = \star_f$ and that $\dot{\star}^T = d_T$

is the identity (semi)star operation on T. We have that $T = D^{\star}$ is a $\dot{\star}^{T}$-ADD = ADD (since T is a DVR), but D is not a $\star$-ADD, since M is a quasi-$\star_f$-maximal ideal of D and (by [15, Proposition 1.9]) $D_M = D$ is not a valuation domain.

Proposition 2.4. *Let D be an integral domain and $\star$ a (semi)star operation on D. Then the following are equivalent*:

(1) *D is a $\star$-ADD.*
(2) *D is a t-ADD and $\star_f = t$.*

Proof. (1) $\Rightarrow$ (2) By Remark 2.1, if D is a $\star$-ADD, then D is a v-ADD or, equivalently, a t-ADD. Moreover, by Proposition 2.2(2) and [17, Proposition 3.4], $\star_f = t$. The converse is clear. □

Theorem 2.5. *Let D be an integral domain, which is not a field, and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-ADD.*
(2) $\mathrm{Na}(D,\star)$ *is an ADD (i.e.* $\mathrm{Na}(D,\star)$ *is a* 1*-dimensional Prüfer domain and contains no idempotent maximal ideals).*
(3) $\mathrm{Na}(D,\star) = \mathrm{Kr}(D,\star)$ *is an ADD and a Bézout domain.*

Proof. (1) $\Leftrightarrow$ (2). By Proposition 1.4(2), the maximal ideals of $\mathrm{Na}(D,\star)$ are of the form $M\mathrm{Na}(D,\star)$, where $M \in \mathscr{M}(\star_f)$. Also, for each $M \in \mathscr{M}(\star_f)$, we have $\mathrm{Na}(D,\star)_{M\mathrm{Na}(D,\star)} = D_M(X)$. Moreover, it is well-known that, for $M \in \mathscr{M}(\star_f)$, D_M is a DVR if and only if $D_M(X)$ is a DVR [24, Theorem 19.16 (c), Proposition 33.1 and Theorem 33.4 ((1) $\Leftrightarrow$ (3))]. From these facts we conclude easily.

(1) $\Rightarrow$ (3). If D is a $\star$-ADD, in particular D is a P$\star$MD (Proposition 2.2(2)), then $\mathrm{Na}(D,\star)=\mathrm{Kr}(D,\star)$, by Proposition 1.6 ((i)$\Leftrightarrow$(iv)). Therefore, we deduce that $\mathrm{Na}(D,\star)$ is a Bézout domain (Proposition 1.5(1)) and an ADD by (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (2) is trivial. □

Corollary 2.6. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-ADD.*
(2) *D is a $\tilde{\star}$-ADD.*
(3) *$D^{\tilde{\star}}$ is a $\dot{\tilde{\star}}$-ADD.*
(4) *$D^{\tilde{\star}}$ is a t-ADD and $\dot{\tilde{\star}} = t_{D^{\tilde{\star}}}$.*

Proof. Note that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}},\dot{\tilde{\star}})$ (Proposition 1.4(6)), then apply Theorem 2.5((1) $\Leftrightarrow$ (2)) to obtain the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). The equivalence (3) $\Leftrightarrow$ (4) follows from Proposition 2.4. □

Next goal is a characterization of $\star$-ADD's in terms of valuation overrings, in the style of [24, Theorem 36.2]. For this purpose, we prove preliminarily the following:

Lemma 2.7. *Let D be an integral domain and $\star$ a semistar operation on D. Let V be a valuation overring of D. Then the following are equivalent*:

(1) *V is a $\tilde{\star}$-valuation overring of D.*
(2) *V is $(\tilde{\star}, d_V)$-linked to D.*

Proof. (1) $\Rightarrow$ (2): Since V is a $\tilde{\star}$-valuation overring, then $\tilde{\star} \leqslant \star_{\{V\}}$. Thus, the present implication follows from the fact that $\dot{\star}^{V}_{\{V\}} = d_V$ (so $\dot{\tilde{\star}}^{V} = d_V$) and from [9, Lemma 3.1(e)].

(2) $\Rightarrow$ (1): Let N be the maximal ideal of V (which is $(\tilde{\star}, d_V)$-linked to D). Then $(N \cap D)^{\tilde{\star}} \neq D^{\tilde{\star}}$ by [9, Proposition 3.2 ((i)$\Rightarrow$(v))]. Thus, there exists $M \in \mathscr{M}(\star_f) = \mathscr{M}(\tilde{\star})$ (Proposition 1.4(5)) such that $N \cap D \subseteq M$. Hence $D_M \subseteq D_{N\cap D} \subseteq V$. So, if $F \in \boldsymbol{f}(D)$, then $F^{\tilde{\star}} \subseteq FD_M \subseteq FV$. Therefore, V is a $\tilde{\star}$-valuation overring of D. $\square$

Theorem 2.8. *Let D be an integral domain, which is not a field, and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is $\star$-ADD.*
(2) *$D^{\tilde{\star}}$ is integrally closed and each $\tilde{\star}$-valuation overring of D is a DVR.*
(3) *$D^{\tilde{\star}}$ is integrally closed and each valuation overring V of D, which is $(\tilde{\star}, d_V)$-linked to D, is a DVR.*
(4) *$D^{\tilde{\star}}$ is integrally closed and each valuation overring V of D, which is $(\star, d_V)$-linked to D, is a DVR.*

Proof. (1) $\Rightarrow$ (2). Since $D^{\tilde{\star}} = \bigcap\{D_M \mid M \in \mathscr{M}(\star_f)\}$ and D_M is a DVR, for each $M \in \mathscr{M}(\star_f)$, then $D^{\tilde{\star}}$ is integrally closed. Now, let V be a $\tilde{\star}$-valuation overring of D, then $V \supseteq D_M$ for some $M \in \mathscr{M}(\star_f)$ [20, Theorem 3.9]. Since D_M is a DVR, then $V = D_M$ (is a DVR).

(2) $\Leftrightarrow$ (3). Follows immediately from Lemma 2.7.

(3) $\Rightarrow$ (4). It is an immediate consequence of the fact that $\tilde{\star} \leqslant \star$ (cf. [9, Lemma 3.1(h)]).

(4) $\Rightarrow$ (1). Let $M \in \mathscr{M}(\star_f)$ and V be valuation overring of D_M. Then $V = V_{D\setminus M}$ is $(\star, d_V)$-linked to D (cf. [9, Example 3.4(1)]). Hence, by assumption, V is a DVR. Furthermore, D_M is integrally closed, since $D^{\tilde{\star}} \subseteq D_M$ and thus $D_M = D^{\tilde{\star}}_{MD_M \cap D^{\tilde{\star}}}$. So D_M is an ADD, by [24, Theorem 36.2], that is, D_M is a DVR. Therefore D is a $\star$-ADD. $\square$

Corollary 2.9. *Let D be an integral domain, which is not a field. Then the following are equivalent*:

(1) *D is t-almost Dedekind domain.*

(2) *D is integrally closed and each w-valuation overring of D is a DVR.*
(3) *D is integrally closed and each t-linked valuation overring of D is a DVR.*

Proof. This is an immediate consequence of Theorem 2.8 and of the well-known fact that for a valuation domain V, $d_V = w_V = t_V$ (cf. also [9, Section 3] for the t-linkedness). $\square$

Remark 2.10. If D is a $\star$-ADD, which is not a field, then, by Theorem 2.8 and by the fact that a $\star$-valuation overring is a $\tilde{\star}$-valuation overring, each $\star$-valuation overring of D is a DVR. Note that the converse is not true, even if $D^{\star}$ is integrally closed. Let D and T be as in Remark 2.3(2). Assume that k is algebraically closed in K. Since $\star = \star_{\{T\}}$, then $\star = \star_f$, $\mathscr{M}(\star_f) = \{M\}$ and $D = D_M = D^{\tilde{\star}}$ is integrally closed, where $\tilde{\star} = d_D$. Moreover, each $\star$-valuation overring of D is necessarily a valuation overring of T (since $T = D^{\star_f} = D^{\star} \subseteq V = V^{\star_f} = V^{\star}$). This implies that each $\star$-valuation overring of D is a DVR (since the only non trivial valuation overring of T is T, which is a DVR). Therefore, by Proposition 1.4(6) and 1.5(5), $\mathrm{Na}(D,\star) = \mathrm{Na}(D^{\tilde{\star}}, \dot{\tilde{\star}}) = \mathrm{Na}(D, d_D) = D(Z) \subsetneq \mathrm{Kr}(D,\star) = \mathrm{Kr}(T, d_T) = T(Z)$ (where Z is an indeterminate over T and D). On the other hand, since $\mathrm{t.deg}_k(K) \geqslant 1$, it is possible to find ($\tilde{\star}$-) valuation overrings of D (of rank $\geqslant 2$) contained in T [24, Theorem 20.7].

Let D be an integral domain and $\star$ a semistar operation on D. For each quasi-$\star$-prime P of D, we define the *$\star$-height* of P (for short, *$\star$-ht(P)*) the supremum of the lengths of the chains of quasi-$\star$-prime ideals of D, between prime ideal (0) (included) and P. Obviously, if $\star = d$ is the identity (semi)star operation on D, then d-ht(P)= ht(P), for each prime ideal P of D. If the set of quasi-$\star$-primes of D is not empty, the $\star$-dimension of D is defined as follows:

$$\star\text{-dim}(D) := \mathrm{Sup}\{\star\text{-ht}(P) \mid P \text{ is a quasi-}\star\text{-prime of } D\}.$$

If the set of quasi-$\star$-primes of D is empty, then we pose $\star$-dim$(D) := 0$.

Note that, if $\star_1 \leqslant \star_2$, then $\star_2$-dim$(D) \leqslant \star_1$-dim(D). In particular, $\star$-dim$(D) \leqslant d$-dim(D)= dim(D) (=Krull dimension of D), for each semistar operation $\star$ on D. Note that, recently, the notions of t-dimension and of w-dimension have been received a considerable interest by several authors (cf. for instance, [10,11,34]).

Lemma 2.11. *Let D be an integral domain and $\star$ a semistar operation on D, then*

$$\tilde{\star}\text{-dim}(D) = \mathrm{Sup}\{\mathrm{ht}(M) \mid M \in \mathscr{M}(\star_f) = \mathscr{M}(\tilde{\star})\}$$
$$= \mathrm{Sup}\{\mathrm{ht}(P) \mid P \text{ is a quasi-}\tilde{\star}\text{-prime of } D\}.$$

Proof. Let $M \in \mathscr{M}(\star_f)$ and $P \subseteq M$ be a nonzero prime ideal of D. Since $\mathscr{M}(\star_f) = \mathscr{M}(\tilde{\star})$ (Proposition 1.4(5)) we have $P \subseteq P^{\tilde{\star}} \cap D \subseteq PD_M \cap D = P$. So P is a quasi-$\tilde{\star}$-prime ideal of D. Hence ht$(M) = \tilde{\star}$-ht(M), so we get the Lemma. $\square$

Remark 2.12. Note that, in general,

$$\star_f\text{-dim}(D) \leqslant \mathrm{Sup}\{\mathrm{ht}(P) \mid P \text{ is a quasi-}\star_f\text{-prime of } D\}.$$

Moreover, it can happen that $\star_f$-dim$(D) \lneq$ Sup$\{$ht$(P)\,|\,P$ is a quasi-$\star_f$-prime of $D\}$, as the following example shows.

Let T be a DVR, with maximal ideal N, dominating a two-dimensional local Noetherian domain D, with maximal ideal M [8] (or [7, Theorem]), and let $\star := \star_{\{T\}}$. Then, clearly, $\star = \star_f$ and the only quasi-$\star_f$-prime ideal of D is M, since if P is a nonzero prime ideal of D, then $P^\star = PT = N^k$, for some integer $k \geqslant 1$. Therefore, if we assume that P is quasi-$\star_f$-ideal of D, then we would have $P = PT \cap D = N^k \cap D \supseteq M^k$, which implies that $P = M$. Therefore, in this case, $1 = \star_f$-dim$(D) = \star_f$-ht$(M) \lneq$ Sup$\{$ht$(P)\,|\,P$ is a quasi-$\star_f$-prime of $D\} =$ ht$(M) =$ dim$(D) = 2$. Note that, in the present example, $\tilde{\star}$ coincides with the identity (semi)star operation on D.

It is already known that, when $\star = v$, it may happen that t-dim$(D) < w$-dim(D), [11, Remark 2].

The following lemma generalizes [24, Theorem 23.3, the first statement in (a)].

Lemma 2.13. *Let D be a* P$\star$MD. *Let Q be a nonzero P-primary ideal of D, for some prime ideal P of D, and let $x \in D \setminus P$. Then $Q^{\tilde{\star}} = (Q(Q + xD))^{\tilde{\star}}$.*

Proof. Let $M \in \mathcal{M}(\star_f)$. If $Q \nsubseteq M$, then $QD_M = Q^2D_M = Q(Q + xD)D_M (= D_M)$. If $Q \subseteq M$, then QD_M is PD_M-primary and $x \in D_M \setminus PD_M$; so $QD_M = QxD_M$, by [24, Theorem 17.3(a)], since D_M is a valuation domain. Thus $QD_M = (Q^2 + Qx)D_M$, hence $Q^{\tilde{\star}} = (Q(Q + xD))^{\tilde{\star}}$. $\square$

Let $\star$ be a semistar operation on an integral domain D. We say that D has *the $\star$-cancellation law* (for short, *$\star$-CL*) if $A, B, C \in \boldsymbol{F}(D)$ and $(AB)^\star = (AC)^\star$ implies that $B^\star = C^\star$. The following theorem provides several characterizations of the semistar almost Dedekind domains and, in particular, it generalizes [24, Theorem 36.5] and [38, Theorem 4.5].

Theorem 2.14. *Let D be an integral domain which is not a field and let $\star$ be a semistar operation on D. The following are equivalent*:

(1) *D is $\star$-ADD.*
(2) *D has the $\tilde{\star}$-cancellation law.*
(3) *D is a* P$\star$MD, *$\star_f$-dim$(D) = 1$ and $(M^2)^{\star_f} \neq M^{\star_f}$, for each $M \in \mathcal{M}(\star_f)$ $(= \mathcal{M}(\tilde{\star}))$.*
(4) *D is a* P$\star$MD *and $\bigcap_{n \geqslant 1}(I^n)^{\star_f} = 0$ for each proper quasi-$\star_f$-ideal I of D.*
(5) *D is a* P$\star$MD *and it has the $\star_f$-cancellation law.*

Proof. (1) $\Rightarrow$ (2). Let A, B, C be three nonzero (fractional) ideals of D such that $(AB)^{\tilde{\star}} = (AC)^{\tilde{\star}}$. Let $M \in \mathcal{M}(\star_f)$. Then, we have $ABD_M = (AB)^{\tilde{\star}}D_M = (AC)^{\tilde{\star}}D_M = ACD_M$ (we used twice the fact that $\tilde{\star}$ is spectral, defined by $\mathcal{M}(\star_f)$). Moreover, since D_M is a DVR then, in particular, AD_M is principal, thus $BD_M = CD_M$. Hence $B^{\tilde{\star}} = C^{\tilde{\star}}$.

(2) $\Rightarrow$ (3). If D has $\tilde{\star}$-CL, then in particular, $\tilde{\star}$ is an e.a.b. semistar operation on D [18, Lemma 2.7], thus D is a P$\star$MD (Proposition 1.6 ((v)$\Rightarrow$(i))). Let $M \in \mathcal{M}(\star_f)$.

Clearly, by $\tilde{\star}$-CL, $(M^2)^{\tilde{\star}} \neq M^{\tilde{\star}}$, and hence $(M^2)^{\star_f} \neq M^{\star_f}$ (since $\tilde{\star} = \star_f$ by Proposition 1.6). Next we show that ht$(M) = 1$, for each $M \in \mathscr{M}(\star_f)$. Deny, let $P \subset M$ be a nonzero prime ideal of D and let $x \in M \setminus P$. By Lemma 2.13, $P^{\tilde{\star}} = (P(P+xD))^{\tilde{\star}}$. Hence $D^{\tilde{\star}} = (P + xD)^{\tilde{\star}}$, by $\tilde{\star}$-CL. So $P + xD \nsubseteq M$, which is impossible. Hence ht$(M) = 1$, for each $M \in \mathscr{M}(\star_f)$. Therefore, we conclude that $\star_f$-dim$(D) = \tilde{\star}$-dim$(D) = 1$ (Lemma 2.11).

(3) $\Rightarrow$ (4). Recall that each proper quasi-$\star_f$-ideal is contained in a quasi-$\star_f$-maximal ideal, then it suffices to show that $\cap_{n \geqslant 1}(M^n)^{\star_f} = 0$, for each $M \in \mathscr{M}(\star_f)$. Since, by assumption $(M^2)^{\star_f} \neq M^{\star_f}$, then in particular $(M^2)^{\tilde{\star}} \neq M^{\tilde{\star}}$, and so $M^2 D_M \neq MD_M$. Henceforth $\{M^n D_M\}_{n \geqslant 1}$ is the set of MD_M-primary ideals of D_M [24, Theorem 17.3(b)]. From the assumption we deduce that $\dim(D_M) = 1$ (because $\star_f = \tilde{\star}$ by Proposition 1.6), then $\cap_{n \geqslant 1} M^n D_M = 0$ [24, Theorem 17.3 (c) and (d)]. In particular, we have $\cap_{n \geqslant 1}(M^n)^{\tilde{\star}} \subseteq \cap_{n \geqslant 1}((M^n)^{\tilde{\star}} D_M) = \cap_{n \geqslant 1}(M^n D_M) = 0$, therefore $\cap_{n \geqslant 1}(M^n)^{\star_f} = 0$.

(4) $\Rightarrow$ (1). Let $M \in \mathscr{M}(\star_f)$. It is easy to see that $(M^n)^{\tilde{\star}} = M^n D_M \cap D^{\tilde{\star}}$, for each $n \geqslant 1$. So, $(\cap_{n \geqslant 1} M^n D_M) \cap D^{\tilde{\star}} \subseteq \cap_{n \geqslant 1}(M^n D_M \cap D^{\tilde{\star}}) = \cap_{n \geqslant 1}(M^n)^{\tilde{\star}} \subseteq \cap_{n \geqslant 1}(M^n)^{\star_f} = 0$ (the last equality holds by assumption). Hence $\cap_{n \geqslant 1} M^n D_M = 0$, since D_M is an essential valuation overring of $D^{\tilde{\star}}$. It follows that D_M is a DVR [24, p. 192 and Theorem 17.3(b)].

(2) $\Leftrightarrow$ (5) is a consequence of the fact that in a P$\star$MD, $\tilde{\star} = \star_f$ and that the $\tilde{\star}$-CL implies P$\star$MD. □

Remark 2.15. As a comment to Theorem 2.14 ((1)$\Leftrightarrow$(5)), note that D may have the $\star_f$-CL without being a $\star$-ADD. It is sufficient to consider the example in Remark 2.3(2). In that case, $\star = \star_f$ and $\tilde{\star} = d_D$, since $\mathscr{M}(\star_f) = \{M\}$. Clearly, D has the $\star$-cancellation law (because T is a DVR), but, as we have already remarked, D is not a $\star$-ADD, hence, equivalently, D has not the ($\tilde{\star}$-)cancellation law.

Next result provides a generalization to the semistar case of [24, Theorem 36.4 and Proposition 36.6].

Proposition 2.16. *Let D be an integral domain, which is not a field, and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-ADD.*
(2) *For each nonzero ideal I of D, such that $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} =: P$ is a prime ideal of D, then $I^{\tilde{\star}} = (P^n)^{\tilde{\star}}$, for some $n \geqslant 1$.*
(3) *$\tilde{\star}$-dim$(D) = 1$ and, for each primary quasi-$\tilde{\star}$-ideal Q of D, then $Q^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, for some $M \in \mathscr{M}(\star_f)$ and for some $n \geqslant 1$.*

Proof. (1) $\Rightarrow$ (2) and (3). Let I be a nonzero ideal of D with $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} = P$ is prime. Let M be a quasi-$\star_f$-maximal ideal of D such that $I \subseteq M$. So $\sqrt{I} = P \subseteq M$, and hence $P = M$, since D_M is a DVR. Thus $ID_M = M^n D_M$ for some $n \geqslant 1$. On the other hand, if $N \in \mathscr{M}(\star_f)$ and $N \neq M$, then $ID_N = D_N = M^n D_N$. Hence $I^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, i.e. $I^{\tilde{\star}} = (P^n)^{\tilde{\star}}$. The fact that $\tilde{\star}$-dim$(D) = 1$ follows from Theorem 2.14((1)⇒(3)) (since, in the present situation, $\star_f = \tilde{\star}$).

(2) ⇒ (1). Let $M \in \mathscr{M}(\star_f)$. Let A be an ideal of D_M and assume that $\sqrt{A} = PD_M$, for some prime ideal P of D, $P \subseteq M$. Set $B := A \cap D$. We have $\sqrt{B} = P$ and hence $B^{\star_f} \subseteq M^{\star_f} \subset D^{\star}$. By assumption, $B^{\tilde{\star}} = (P^n)^{\tilde{\star}}$, for some $n \geqslant 1$, hence $A = (A \cap D)D_M = BD_M = B^{\tilde{\star}}D_M = (P^n)^{\tilde{\star}}D_M = P^nD_M$. It follows from [24, Proposition 36.6] that D_M is an ADD. Hence D_M is a DVR.

(3) ⇒ (1). We can assume $\star = \star_f$, since $\star$-ADD and $\star_f$-ADD coincide. Let $M \in \mathscr{M}(\star_f)$ $(=\mathscr{M}(\tilde{\star}))$ (Proposition 1.4(5)). Since $\tilde{\star}$-dim$(D)=1$, then ht$(M)=$dim(D_M) $=1$ (Lemma 2.11). We can now proceed and conclude as in the proof of (2) ⇒ (1). (In this case, we have $\sqrt{A} = MD_M$ and so B is a M-primary quasi-$\tilde{\star}$-ideal of D. Therefore, by assumption, $B^{\tilde{\star}} = (M^n)^{\tilde{\star}}$, for some $n \geqslant 1$.) □

Remark 2.17. Note that, if D is a $\star$-ADD, which is not a field, then necessarily D satisfies the following conditions (obtained from the statements (2) and (3) of Proposition 2.16; recall that, in this case, $\star_f = \tilde{\star}$, by Proposition 2.2(2) and Proposition 1.6):

(2_f) For each nonzero ideal I of D, such that $I^{\star_f} \neq D^{\star}$ and $\sqrt{I} =: P$ is a prime ideal of D, then $I^{\star_f} = (P^n)^{\star_f}$, for some $n \geqslant 1$.

(3_f) $\star_f$-dim$(D)=1$ and, for each primary quasi-$\star_f$-ideal Q of D, then $Q^{\star_f} = (M^n)^{\star_f}$, for some $M \in \mathscr{M}(\star_f)$ and for some $n \geqslant 1$.

On the other hand, D may satisfy either (2_f) or (3_f) without being a $\star$-ADD. It is sufficient to consider the example in Remark 2.3(2). In that case, $\star = \star_f$ and $\mathscr{M}(\star_f) = \{M\}$. Clearly, since D is a local one-dimensional domain (in fact, $\tilde{\star}$-dim$(D)=$ $\star_f$-dim$(D)=$dim$(D)=1$), for each nonzero ideal I of D, with $I^{\star_f} \neq D^{\star}$, then $\sqrt{I} = M$ and $I^{\star_f} = (M^n)^{\star_f}$, for some $n \geqslant 1$, since T is a DVR. But, as we have already remarked, D is not a $\star$-ADD.

3. Semistar Noetherian domains

Let D be an integral domain and $\star$ a semistar operation on D. We say that D is a $\star$-*Noetherian domain* if D has the ascending chain condition on quasi-$\star$-ideals.

Note that, if d $(=d_D)$ is the identity (semi)star operation on D, the d-Noetherian domains are just the usual Noetherian domains and the notions of v-Noetherian [respectively, w-Noetherian] domain and Mori [respectively, strong Mori] domain coincide [5, Theorem 2.1] [respectively, [13]].

Recall that the concept of *star* Noetherian domain has already been introduced, see for instance [1,23,46]. Using ideal systems on commutative monoids, a similar general notion of noetherianity was considered in [28, Chapter 3].

Lemma 3.1. *Let D be an integral domain.*

(1) *Let $\star \leqslant \star'$ be two semistar operations on D, then D is $\star$-Noetherian implies D is $\star'$-Noetherian.*
In particular:
(1a) *A Noetherian domain is a $\star$-Noetherian domain, for any semistar operation $\star$ on D.*
(1b) *If $\star$ is a (semi)star operation and if D is a $\star$-Noetherian domain, then D is a Mori domain.*

(2) *Let T be an overring of D and $*$ a semistar operation on T. If T is $*$-Noetherian, then D is $*_D$-Noetherian. In particular, if $\star$ is a semistar operation on D, such that $D^\star$ is a $\dot{\star}$-Noetherian domain, then D is a $\star$-Noetherian domain.*

Proof. (1) The first statement holds because each quasi-$\star'$-ideal is a quasi-$\star$-ideal. (1a) and (1b) follow from (1) since, for each semistar operation $\star$, $d \leqslant \star$ and, if $\star$ is a (semi)star operation, then $\star \leqslant v$.

(2) If we have a chain of quasi-$\underset{\cdot}{*}$-ideals $\{I_n\}_{n\geqslant 1}$ of D that does not stop then, by considering $\{(I_nT)^*\}_{n\geqslant 1}$, we get a chain of quasi-$*$-ideals of T that does not stop, since two distinct quasi-$\underset{\cdot}{*}$-ideals $I \neq I'$ of D are such that $(IT)^* \neq (I'T)^*$. The second part of the statement follows immediately from the fact that, if we set $* := \dot{\star}$, then $\underset{\cdot}{*} = \star$ (Proposition 1.3(2)). $\square$

Remark 3.2. The converse of (2) in Lemma 3.1 does not hold in general. For instance, take $D \subset T$, where D is a Noetherian domain and T is a non-Noetherian overring of D. Let $* := d_T$ and $\star := \star_{\{T\}}$. Note that $\underset{\cdot}{*} = \star$. Then, D is $\star$-Noetherian, by (1a) of Lemma 3.1, but $D^\star = T^* = T$ is not $*$-Noetherian (or, equivalently, $\dot{\star}^T$-Noetherian), because $* = d_T (= \dot{\star}^T = \dot{\star})$ and T is not Noetherian.

However, if $\star = \tilde{\star}$, the last statement of (2) in Lemma 3.1 can be reversed, as we will see in Proposition 3.4.

Lemma 3.3. *Let D be an integral domain and let $\star$ be a semistar operation on D. Then, D is a $\star$-Noetherian domain if and only if, for each nonzero ideal I of D, there exists a finitely generated ideal $J \subseteq I$ of D such that $I^\star = J^\star$. Therefore, D is a $\star$-Noetherian domain if and only if, for each $E \in \boldsymbol{F}(D)$, there exists $F \in \boldsymbol{f}(D)$, such that $F \subseteq E$ and $F^\star = E^\star$. In particular, if $\star$ is a star operation on D and if D is $\star$-Noetherian then $\star$ is a star operation of finite type on D.*

Proof. For the "only if" part, let $x_1 \in I$, $x_1 \neq 0$, and set $I_1 := x_1D$. If $I^\star = I_1^\star$ we are done. Otherwise, it is easy to see that $I \nsubseteq I_1^\star \cap D$. Let $x_2 \in I \setminus (I_1^\star \cap D)$ and set $I_2 := (x_1, x_2)D$. By iterating this process, we construct a chain $\{I_n^\star \cap D\}_{n\geqslant 1}$ of quasi-$\star$-ideals of D. By assumption this chain must stop, i.e., for some $k \geqslant 1$, $I_k^\star \cap D = I_{k+1}^\star \cap D$, and so $I_k^\star = (I_k^\star \cap D)^\star = I^\star$. So, we conclude by taking $J := I_k$. Conversely, let $\{I_n\}_{n\geqslant 1}$ be a chain of quasi-$\star$-ideals in D and set $I := \bigcup_{n\geqslant 1} I_n$. Let $J \subseteq I$ be a finitely generated ideal of D such that $J^\star = I^\star$, so there exists $k \geqslant 1$ such that $J \subseteq I_k$ and $J^\star = I_k^\star = I^\star$. This implies that the chain of quasi-$\star$-ideals $\{I_n\}_{n\geqslant 1}$ stops (in fact, $I_n = I_k = I^\star \cap D$, for each $n \geqslant k$). $\square$

Proposition 3.4. *Let D be an integral domain and let $\star$ be a semistar operation on D.*

(1) *Assume that $\star$ is stable. Then D is $\star$-Noetherian if and only if $D^{\star}$ is $\dot{\star}$-Noetherian.*
(2) *D is $\tilde{\star}$-Noetherian if and only if $D^{\tilde{\star}}$ is $\dot{\tilde{\star}}$-Noetherian.*

Proof. (1) The "if" part follows from Lemma 3.1(2) and Proposition 1.3(2) (without using the hypothesis of stability). Conversely, let I be a nonzero ideal of $D^{\star}$ and set $J := I \cap D$. Then, $J^{\star} = (I \cap D)^{\star} = I^{\star} \cap D^{\star} = I^{\star}$. Therefore, by Lemma 3.3 (applied to D), we can find $F \in \boldsymbol{f}(D)$ such that $F \subseteq J$ and $F^{\star} = J^{\star}$. Hence, $(FD^{\star})^{\dot{\star}} = F^{\star} = J^{\star} = I^{\star} = I^{\dot{\star}}$. The conclusion follows from Lemma 3.3 (applied to $D^{\star}$, since $FD^{\star} \subseteq I$ and $FD^{\star} \in \boldsymbol{f}(D^{\star})$).

(2) is a straightforward consequence of (1). □

Proposition 3.5. *Let D be an integral domain and $\star$ a semistar operation on D. Then, D is $\star$- Noetherian if and only if D is $\star_f$-Noetherian.*

Proof. The "if" part follows from Lemma 3.1(1), since $\star_f \leqslant \star$. The converse follows immediately from Lemma 3.3. □

Remark 3.6. Let D be an integral domain and $\star$ a semistar operation on D.

(1) Let $E \in \overline{\boldsymbol{F}}(D)$, we say that E is *$\star$-finite* if there exists $F \in \boldsymbol{f}(D)$ such that $E^{\star} = F^{\star}$. From Lemma 3.3 it follows that if D is a $\star$-Noetherian domain, then each nonzero fractional ideal is $\star$-finite. The converse does not hold in general [23, Example 18]. However, when $\star = \star_f$, $E \in \overline{\boldsymbol{F}}(D)$ is $\star$-finite if and only if there exists $F \in \boldsymbol{f}(D)$ such that $E^{\star} = F^{\star}$, with $F \subseteq E$ [21, Lemma 2.3] (note that the star operation case was investigated in [1]). From the previous considerations, from Lemma 3.3 and from Proposition 3.5, we deduce easily that *D is a $\star$-Noetherian domain if and only if every nonzero fractional ideal of D is $\star_f$-finite*.

(2) Note that:

$\tilde{\star}$-Noetherian $\Rightarrow$ $\star$-Noetherian,

because $\tilde{\star} \leqslant \star$ (Lemma 3.1(1)). The converse is not true in general. Indeed, if $\star := v$, then $\star_f = t$ and $\tilde{\star} = w$ and we know that v-Noetherian ($=t$-Noetherian) is Mori and that w-Noetherian is strong Mori [13, Section 4]. Since it is possible to give examples of Mori domains that are not strong Mori [14, Corollary 1.11], we deduce that $\star$-Noetherian does not imply $\tilde{\star}$-Noetherian.

In the next result, we provide a sufficient condition for the transfer of the semistar Noetherianity to overrings.

Proposition 3.7. *Let D be an integral domain and let T be an overring of D. Let $\star$ be a semistar operation on D and $\star'$ a semistar operation on T. Assume that T is $(\star, \star')$-flat over D. If D is $\tilde{\star}$-Noetherian, then T is $\widetilde{\star'}$-Noetherian.*

Proof. Let A be a nonzero ideal of T. Let $N \in \mathcal{M}(\widetilde{\star'}) = \mathcal{M}(\star'_f)$ (Proposition 1.4(5)). From the $(\star,\star')$-flatness, it follows that $T_N = D_{N\cap D}$. Then, $A^{\widetilde{\star'}} = \cap\{AT_N \mid N \in \mathcal{M}(\star'_f)\} = \cap\{AD_{N\cap D} \mid N \in \mathcal{M}(\star'_f)\}$. Now, $N \cap D$ is a prime of D such that $(N\cap D)^{\tilde{\star}} \neq D^{\tilde{\star}}$ (by [9, Proposition 3.2], since T is $(\star,\star')$-linked to D, by definition of $(\star,\star')$-flatness). Hence, $N \cap D$ is a quasi-$\tilde{\star}$-ideal. Consider the ideal $A \cap D$ of D. Since D is $\tilde{\star}$-Noetherian, it follows by Lemma 3.3 that there exists a finitely generated ideal C of D, such that $C \subseteq A\cap D$ and $C^{\tilde{\star}} = (A\cap D)^{\tilde{\star}}$. Then, $AT_N = AD_{N\cap D} = (A\cap D)D_{N\cap D} = (A\cap D)^{\tilde{\star}}D_{N\cap D} = C^{\tilde{\star}}D_{N\cap D} = CD_{N\cap D} = (CT)T_N$. Thus, $A^{\widetilde{\star'}} = (CT)^{\widetilde{\star'}}$, with CT finitely generated ideal of T, such that $CT \subseteq A$. Hence, T is $\widetilde{\star'}$-Noetherian. $\square$

Let D be an integral domain and $\star$ a semistar operation on D. We say that D has the *$\star$-finite character property* (for short, *$\star$-FC property*) if each nonzero element x of D belongs to only finitely many quasi-$\star$-maximal ideals of D. Note that the $\star_f$-FC property coincides with the $\tilde{\star}$-FC property, because $\mathcal{M}(\star_f)=\mathcal{M}(\tilde{\star})$ (Proposition 1.4(5)).

Proposition 3.8. *Let D be an integral domain and $\star$ a semistar operation on D. If D is $\tilde{\star}$-Noetherian, then D_M is Noetherian, for each $M \in \mathcal{M}(\star_f)$. Moreover, if D has the $\star_f$-FC property, then the converse holds.*

Proof. Let $M \in \mathcal{M}(\star_f)$, A an ideal of D_M and $I := A \cap D$. Since D is $\tilde{\star}$-Noetherian, there exists a finitely generated ideal $J \subseteq I$ of D with $J^{\tilde{\star}} = I^{\tilde{\star}}$ (Lemma 3.3). Then, $A = ID_M = I^{\tilde{\star}}D_M = J^{\tilde{\star}}D_M = JD_M$ (we used twice the fact that $\tilde{\star}$ is spectral, defined by $\mathcal{M}(\star_f)$). Then A is a finitely generated ideal of D_M and so D_M is Noetherian. For the converse, assume that the $\star_f$-FC property holds on D. Let I be a nonzero ideal of D and let $0 \neq x \in I$. Let $M_1, M_2, \ldots, M_n \in \mathcal{M}(\star_f)$ be the quasi-$\star_f$-maximal ideals containing x. Since D_{M_i} is Noetherian for each $i = 1,2,\ldots,n$, then $ID_{M_i} = J_iD_{M_i}$, for some finitely generated ideal $J_i \subseteq I$ of D. The ideal $B := xD + J_1 + J_2 + \cdots + J_n$ of D is finitely generated and contained in I. It is clear that, for each $i=1,2,\ldots,n$, $ID_{M_i}=BD_{M_i}$. Moreover, if $M \in \mathcal{M}(\star_f)$ and $M \neq M_i$, for each $i=1,2,\ldots,n$, then $x \notin M$ and this fact implies $ID_M=BD_M=D_M$. Then, $I^{\tilde{\star}}=\bigcap\{ID_M \mid M \in \mathcal{M}(\star_f)\}=\bigcap\{BD_M \mid M \in \mathcal{M}(\star_f)\}=B^{\tilde{\star}}$. Thus, by Lemma 3.3, D is $\tilde{\star}$-Noetherian. $\square$

Remark 3.9. (1) Note that Proposition 3.8, in case of star operations, can be deduced from [29, Proposition 4.6], proven in the context of weak ideal systems on commutative monoids.

(2) Note that strong Mori domains (that is, w-Noetherian domains, where $w := \tilde{v}$) or, more generally, Mori domains satisfy always the t-FC property ($=w$-FC property, since $\mathcal{M}(w) = \mathcal{M}(t)$, for every integral domain) by [6, Proposition 2.2(b)]. But it is not true in general that the $\tilde{\star}$-Noetherian domains satisfy the $\star_f$-FC property (take, for instance, $D := \mathbb{Z}[X]$, $\star := d$, and observe that X is contained in infinitely many maximal ideals of $\mathbb{Z}[X]$).

Note that, from Proposition 3.8 and from the previous considerations, we obtain in particular that *an integral domain D is strong Mori if and only if D_M is Noetherian, for each $M \in \mathcal{M}(t)$, and D has the w-FC property* (cf. also [14, Theorem 1.9]).

4. Semistar Dedekind domains

Let D be an integral domain and $\star$ a semistar operation on D. We recall from Section 1 (or [21, Section 2]) that a nonzero fractional ideal F ($\in \boldsymbol{F}(D)$) of D is *$\star$-invertible* if $(FF^{-1})^\star = D^\star$ and $E \in \overline{\boldsymbol{F}}(D)$ is *quasi-$\star$-invertible* if $(E(D^\star : E))^\star = D^\star$ (note that, the last property implies that $E \in \boldsymbol{F}(D^\star)$). It is clear that a $\star$-invertible ideal is quasi-$\star$-invertible. The converse is not true in general [21, Example 2.9 and Proposition 2.16] but, if $\star$ is stable (e.g., for $\star = \tilde{\star}$), a finitely generated ideal is $\star$-invertible if and only if it is quasi-$\star$-invertible [21, Corollary 2.17(2)].

Proposition 4.1. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-Noetherian domain and a P$\star$MD,*
(1_f) *D is a $\star_f$-Noetherian domain and a P$\star_f$MD,*
(2) *$\boldsymbol{F}^{\tilde{\star}}(D) := \{F^{\tilde{\star}} \mid F \in \boldsymbol{F}(D)\}$ is a group under the multiplication "$\times$", defined by $F^{\tilde{\star}} \times G^{\tilde{\star}} := (F^{\tilde{\star}}G^{\tilde{\star}})^{\tilde{\star}} = (FG)^{\tilde{\star}}$, for all $F, G \in \boldsymbol{F}(D)$,*
(3) *Each nonzero fractional ideal of D is quasi-$\tilde{\star}$-invertible,*
(4) *Each nonzero (integral) ideal of D is quasi-$\tilde{\star}$-invertible.*

Proof. $(1) \Leftrightarrow (1_f)$ is obvious (Proposition 3.5 and Proposition 1.6 ((i) $\Leftrightarrow$ (vi))).

$(1) \Rightarrow (2)$. One can easily check that $\boldsymbol{F}^{\tilde{\star}}(D)$ is a monoid, with $D^{\tilde{\star}}$ as the identity element (with respect to "$\times$"). We next show that each element of $\boldsymbol{F}^{\tilde{\star}}(D)$ is invertible for the monoid structure. Let $F \in \boldsymbol{F}(D)$, then there exists $0 \neq d \in D$ such that $I := dF \subseteq D$. Write $I^{\star_f} = J^{\star_f}$, where $J \subseteq I$ is a finitely generated ideal of D (Lemma 3.3 and Proposition 3.5). Since D is a P$\star$MD, then $\star_f = \tilde{\star}$ (Proposition 1.6). So, $I^{\tilde{\star}} = J^{\tilde{\star}}$. We have $(JJ^{-1})^{\tilde{\star}} = D^{\tilde{\star}}$, since D is a P$\tilde{\star}$MD (Proposition 1.6). Then, $D^{\tilde{\star}} = (J^{\tilde{\star}}J^{-1})^{\tilde{\star}} = (IJ^{-1})^{\tilde{\star}} = (dFJ^{-1})^{\tilde{\star}} = (F^{\tilde{\star}}(dJ^{-1})^{\tilde{\star}})^{\tilde{\star}}$. Thus $F^{\tilde{\star}}$ is invertible in $(\boldsymbol{F}^{\tilde{\star}}(D), \times)$.

$(2) \Rightarrow (3)$. Let $F \in \boldsymbol{F}(D)$. By assumption, there exists $G \in \boldsymbol{F}(D)$ such that $(FG)^{\tilde{\star}} = D^{\tilde{\star}}$. We have $FG \subseteq D^{\tilde{\star}}$, so $G \subseteq (D^{\tilde{\star}} : F)$. Thus $D^{\tilde{\star}} = (FG)^{\tilde{\star}} \subseteq (F(D^{\tilde{\star}} : F)) \subseteq D^{\tilde{\star}}$. Hence $(F(D^{\tilde{\star}} : F))^{\tilde{\star}} = D^{\tilde{\star}}$, that is, F is quasi-$\tilde{\star}$-invertible.

$(3) \Rightarrow (4)$ is straightforward.

$(4) \Rightarrow (1)$ From the previous comments on quasi semistar invertibility for nonzero finitely generated ideals in the stable case, it is clear that the assumption implies that D is a P$\tilde{\star}$MD and hence D is a P$\star$MD (Proposition 1.6). To prove that D is a $\star$-Noetherian domain, since $\tilde{\star} = \star_f$ (Proposition 1.6), it is enough to show, by using Proposition 3.5, that D is $\tilde{\star}$-Noetherian. Let I be a nonzero ideal of D, then, by assumption, $(I(D^{\tilde{\star}} : I))^{\tilde{\star}} = D^{\tilde{\star}}$. By [21, Lemma 2.3 and Proposition 2.15] applied to $\tilde{\star}$, there exists a nonzero finitely generated ideal J of D such that $J \subseteq I$ and $J^{\tilde{\star}} = I^{\tilde{\star}}$. From Lemma 3.3, we deduce that D is $\tilde{\star}$-Noetherian. $\square$

An integral domain D with a semistar operation $\star$ satisfying the equivalent conditions (1)–(4) of Proposition 4.1 is called a *$\star$-Dedekind domain* (*$\star$-DD*, for short). Note that, by definition, the notions of $\star$-DD and $\star_f$-DD coincide.

Remark 4.2. (1) By Proposition 4.1(1), if $\star = d$ we obtain that a d-DD coincides with a classical Dedekind domain [24, Theorem 37.1]; if $\star = v$, we have that a v-DD coincides with a Krull domain (since a Mori PvMD is a Krull domain [39, Theorem 3.2 ((1) $\Leftrightarrow$(3))]; note that a Mori domain verifies the t-FC property by [6, Proposition 2.2(b)]). More generally, if $\star$ is a star operation, then D is a $\star$-DD if and only if D is $\star$-Dedekind in the sense of [28, Chapter 23].

(2) If D is $\star$-DD then D is $\star$-ADD (for a converse, see the following Theorem 4.11). Indeed, a $\star$-DD is a P$\star$MD and so $\tilde{\star} = \star_f$ (Proposition 1.6). This equality implies also that D is $\tilde{\star}$-Noetherian (Proposition 3.5 and Proposition 4.1(1)). Therefore D_M is Noetherian (by Proposition 3.8) and, hence, we conclude that D_M is a DVR, for each $M \in \mathscr{M}(\star_f)$.

Corollary 4.3. *Let D be an integral domain and $\star$ a semistar operation on D. Then D is a $\star$-DD if and only if D is a $\tilde{\star}$-DD.*

Proof. It follows from Proposition 4.1(4) and from the fact that $\tilde{\tilde{\star}} = \tilde{\star}$, since $\mathscr{M}(\widetilde{\star}) = \mathscr{M}(\star_f)$ (cf. also [16, page 182]). $\square$

Theorem 4.4. *Let D be an integral domain.*

(1) *Let $\star \leqslant \star'$ be two semistar operations on D. Then*:

$$D \text{ is a } \star\text{-DD} \quad \Rightarrow \quad D \text{ is a } \star'\text{-DD}.$$

In particular:

(1a) *If D is a Dedekind domain, then D is a $\star$-DD, for any semistar operation $\star$ on D.*

(1b) *Assume that $\star$ is a (semi)star operation on D. Then a $\star$-DD is a Krull domain.*

(2) *Let T be an overring of D. Let $\star$ be a semistar operation on D and $\star'$ a semistar operation on T. Assume that T is a $(\star, \star')$-linked overring of D. If D is a $\star$-DD, then T is a $\star'$-DD. In particular, if D is a $\star$-DD, then $D^\star$ is a $\dot{\star}$-DD.*

Proof. (1) follows from [17, p. 30] and Lemma 3.1(1). (1a) and (1b) are consequence of (1), Remark 4.2(1) and of the fact that $d \leqslant \star$, for each semistar operation $\star$, and if $\star$ is a (semi)star operation, then $\star \leqslant v$.

(2) Note that if T is a $(\star, \star')$-linked overring of D and if D is a P$\star$MD, then T is a $(\star, \star')$-flat over D [9, Theorem 5.7 ((i)$\Rightarrow$(ii))]. By Proposition 4.1(1) and Corollary 4.3, we know that D is $\tilde{\star}$-Noetherian and a P$\star$MD (or, equivalently, a P$\tilde{\star}$MD). Hence, T is $\widetilde{\star'}$-Noetherian (Proposition 3.7) and T is a P$\star'$MD (or, equivalently, a P$\widetilde{\star'}$MD) by [9, Corollary 5.4]. The first statement follows from Proposition 4.1(1) and Corollary 4.3. The last statement is a consequence of [9, Lemma 3.1(e)]. $\square$

Proposition 4.5. *Let D be an integral domain and $\star$ a (semi)star operation on D. Then the following are equivalent*:

(1) *D is a $\star$-DD*
(2) *D is a Krull domain and $\star_f = t$*

Proof. (1) $\Rightarrow$ (2). By Theorem 4.4(1b), if D is a $\star$-DD, then D is a Krull domain, in this case, $\star_f = t$ [17, Proposition 3.4].

(2) $\Rightarrow$ (1). This follows from Remark 4.2(1) and from the fact that v-DD=t-DD=$\star_f$-DD=$\star$-DD. $\square$

Note that Proposition 4.5 has already been proven in [28, Theorem 23.3((a)$\Leftrightarrow$(d))], by using the language of monoids and ideal systems.

Remark 4.6. Note that if D is $\star$-DD, then by Theorem 4.4(2) $D^\star$ is $\dot{\star}$-DD, that is $D^\star$ is a Krull domain and $(\dot{\star})_f = t_{D^\star}$ (Proposition 4.5). However, the converse does not hold in general as the example in Remark 2.3(2) shows. Nevertheless, the converse is true when $\star = \tilde{\star}$ (see the following Corollary 4.20) or when the extension $D \subseteq D^\star$ is flat, as a consequence of Lemma 3.1(2) and [17, Proposition 3.2]. For a more accurate discussion on this problem see the following Remark 4.21.

Next result is a “Cohen-type” Theorem for quasi-$\star$-invertible ideals.

Lemma 4.7. *Let D be an integral domain and $\star$ a semistar operation of finite type on D. The following are equivalent*:

(1) *Each nonzero quasi-$\star$-prime of D is a quasi-$\star$-invertible ideal of D.*
(2) *Each nonzero quasi-$\star$-ideal of D is a quasi-$\star$-invertible ideal of D.*
(3) *Each nonzero ideal of D is a quasi-$\star$-invertible ideal of D.*

Proof. (1) $\Rightarrow$ (2). Let S be the set of the quasi-$\star$-ideals of D that are not quasi-$\star$-invertible. Assume that $S \neq \emptyset$. Since $\star = \star_f$ by assumption, then Zorn's Lemma can be applied, thus we deduce that S has maximal elements. We next show that a maximal element of S is prime. Let P be a maximal element of S and let $r, s \in D$, with $rs \in P$. Suppose $s \notin P$. Let $J := (P :_D rD)$. We claim that $J^\star \cap D = J$. Indeed, since $(P :_D rD)^\star \subseteq (P^\star :_{D^\star} rD)$, then $J^\star \cap D \subseteq (P^\star :_{D^\star} rD) \cap D = (P^\star :_D rD)$. Moreover, if $x \in (P^\star :_D rD)$, then $xr \in P^\star \cap D = P$, and hence $(P^\star :_D rD) \subseteq (P :_D rD) = J$. Thus $J = J^\star \cap D$, i.e. J is a quasi-$\star$-ideal of D. Clearly, J contains properly P (since $s \in J \setminus P$). By the maximality of P in S, it follows that J is quasi-$\star$-invertible, that is $(J(D^\star : J))^\star = D^\star$. We notice that $P(D^\star : J) \in \overline{\boldsymbol{F}}(D)$ is not quasi-$\star$-invertible, since P is not quasi-$\star$-invertible [21, Lemma 2.11]. We deduce that $(P(D^\star : J))^\star \cap D$ is a proper quasi-$\star$-ideal, that is not quasi-$\star$-invertible [21, Remark 2.13(a)] and, obviously, it contains P. From the maximality of P in S, we have $(P(D^\star : J))^\star \cap D = P$. Now, $rJ \subseteq P$ implies $(rJ)^\star \subseteq P^\star$. Then $r \in (rD)^\star = (rJ(D^\star : J))^\star \subseteq (P(D^\star : J))^\star$. Therefore, $r \in (P(D^\star : J))^\star \cap D = P$ and so we have proven that P is a prime ideal of D.

(2) ⇒ (3) is a consequence of [21, Remark 2.13(a)], after remarking that, for each nonzero ideal J of D, then $J \subseteq I := J^{\star} \cap D$, where I is a quasi-$\star$-ideal of D and $J^{\star} = I^{\star}$.

(3) ⇒ (2) ⇒ (1) are trivial. □

Remark 4.8. Note that, in the situation of Lemma 4.7, the statement:

(0) *each nonzero quasi-$\star$-maximal ideal of D is a quasi-$\star$-invertible ideal of D*,

is, in general, strictly weaker than (1). Take, for instance, D equal to a discrete valuation domain of rank $\geqslant 2$, and $\star = d_D$.

The next two theorems generalize [24, Theorem 37.8 ((1)⇔(4)), Theorem 37.2]. Similar results are proven in [28, Theorem 23.3((a)⇔(c), (h))].

Theorem 4.9. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*
(2) *Each nonzero quasi-$\tilde{\star}$-prime ideal of D is quasi-$\tilde{\star}$-invertible.*

Proof. Easy consequence of Lemma 4.7 ((1) ⇔ (3)) and Proposition 4.1 (4). □

From the previous theorem, we deduce the following characterization of Krull domains (cf. [36, Theorem 2.3((1)⇔(3))], [39, Theorem 3.6 ((1)⇔(4))] and [13, Theorem 5.4 ((i)⇔(vi))]).

Corollary 4.10. *Let D be an integral domain. The following are equivalent*:

(1) *D is a Krull domain.*
(2) *Each nonzero w-prime ideal of D is w-invertible.*
(3) *Each nonzero t-prime ideal of D is t-invertible.*

Proof. (1) ⇔ (2) is a direct consequence of Theorem 4.9.

(1) ⇒ (3) is a straightforward consequence of (1) ⇒ (2) and of the fact that, in a Krull domain (which is a particular PvMD), $t = \widetilde{t} = w$ (Proposition 1.6).

(3) ⇒ (2). Note that, by assumption, and by Lemma 4.7 ((1) ⇔ (3)), every nonzero ideal of D is t-invertible. Let Q be a nonzero w-prime. If $(QQ^{-1})^w \neq D$, then $Q \subseteq (QQ^{-1})^w \subseteq M$, for some $M \in \mathscr{M}(w) = \mathscr{M}(t)$ (Proposition 1.4(5)), thus $(QQ^{-1})^t = ((QQ^{-1})^w)^t \subseteq M^t = M$, which is a contradiction. □

Theorem 4.11. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*

(2) *D is a $\star$-ADD and each nonzero element of D is contained in only finitely many quasi-$\star_f$-maximal ideals* (*i.e. D has the $\star_f$-FC property*).
(3) D is a $\star$-Noetherian $\star$-ADD.

Proof. (1) $\Rightarrow$ (2). Clearly D is a $\star$-ADD, by Remark 4.2(2). Since each quasi-$\star_f$-maximal ideal of D is a contraction of a $\dot{\star}_f$-maximal ideal of $D^\star$ [20, Lemma 2.3(3)], in order to show that D has $\star_f$-FC property, it is enough to check that $D^\star$ satisfies the $\dot{\star}_f$-FC property. On the other hand, since (1) implies that $D^\star$ is a $\dot{\star}$-DD (Theorem 4.4(2)), without loss of generality, we can assume that $\star$ is a (semi)star operation on D and D is a $\star$-DD. By Proposition 4.5, D is a Krull domain and $\star_f = t$. Thus, each nonzero element is contained in only finitely many t-maximal ideals ($=\star_f$-maximal ideals) of D.

(2) $\Rightarrow$ (1). We need to show that D is $\star_f$-DD. First, note that D is a P$\star_f$MD and D_M is Noetherian, for each $M \in \mathscr{M}(\star_f)$ (Proposition 2.2 (1) and (2)). The conclusion now follows from Proposition 3.8 and Proposition 4.1(1), after recalling that, in a P$\star_f$MD, $\star_f = \widetilde{\star}$ (Proposition 1.6).

(1) $\Leftrightarrow$ (3) is a consequence of Proposition 2.2(2), Proposition 4.1 and Remark 4.2(2). $\square$

From the previous theorem, we deduce a restatement of a well-known characterization of Krull domains:

Corollary 4.12. *Let D be an integral domain, then the following are equivalent*:

(1) *D is a Krull domain.*
(2) *D is a t-almost Dedekind domain and each nonzero element of D is contained in only finitely many t-maximal ideals* (=*t-FC property*). $\square$

Let D be an integral domain and $\star$ a semistar operation on D. We recall that the *$\star$-integral closure $D^{[\star]}$ of D* (or, the *semistar integral closure with respect to the semistar operation $\star$ of D*) is the integrally closed overring of $D^\star$ defined by $D^{[\star]} := \bigcup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}$ [18, Definition 4.1]. We say that *D is quasi-$\star$-integrally closed* (respectively, *$\star$-integrally closed*) if $D^\star = D^{[\star]}$ (respectively, $D = D^{[\star]}$). It is clear that:

– D is quasi-$\star$-integrally closed if and only if D is quasi-$\star_f$-integrally closed (respectively, D is $\star$-integrally closed if and only if D is $\star_f$-integrally closed);

– D is $\star$-integrally closed if and only if D is quasi-$\star$-integrally closed and $\star$ is a (semi)star operation on D.

Note that when $\star = v$, then the overring $D^{[v]} = D^{[t]}$ was studied in [3] under the name of *pseudo-integral closure of D*.

Lemma 4.13. *Let D be an integral domain and $\star$ a semistar operation on D.*

(1) *If $\star$ is e.a.b., then $D^\star = D^{[\star]}$* (*i.e. D is quasi-$\star$-integrally closed*).
(2) *D is quasi-$\tilde{\star}$-integrally closed if and only if $D^{\tilde{\star}}$ is integrally closed.*

Proof. (1) Note that, in general, $D^{\star} \subseteq D^{[\star]}$. For the converse, let $F \in \boldsymbol{f}(D)$ and let $x \in (F^{\star} : F^{\star})$. Then, $xF^{\star} \subseteq F^{\star}$ and $F^{\star} = F^{\star} + F^{\star}(xD)$. Therefore we have $(F(D+xD))^{\star} = (F^{\star}(D+xD))^{\star} = (F^{\star} + F^{\star}(xD))^{\star} = F^{\star}$. From the fact that F is finitely generated and that $\star$ is e.a.b., we obtain $(D+xD)^{\star} = D^{\star}$. It follows that $x \in D^{\star}$ and so $(F^{\star} : F^{\star}) \subseteq D^{\star}$. Hence, $D^{\star} = D^{[\star]}$.

(2) The "only if" part is clear. For the "if" part, let D' be the integral closure of D, since $D^{\tilde{\star}}$ is integrally closed, then $(D')^{\tilde{\star}} \subseteq D^{\tilde{\star}} \subseteq D^{[\tilde{\star}]}$ hence, by [17, Example 2.1(c2)], $(D')^{\tilde{\star}} = D^{\tilde{\star}} = D^{[\tilde{\star}]}$. Therefore, D is quasi-$\tilde{\star}$-integrally closed. □

Corollary 4.14. *Let $\star$ be a semistar operation on an integral domain D. If D is a* P$\star$MD (*in particular, a $\star$-DD*) *then D is quasi-$\star$-integrally closed.*

Proof. It follows from Lemma 4.13(1) and from the fact that, in a P$\star$MD, $\tilde{\star} = \star_f$ is an e.a.b. semistar operation (Proposition 1.6 ((i)⇒(v), (vi))). □

The following result shows that a semistar version of the "Noether's Axioms" provides a characterization of the semistar Dedekind domains.

Theorem 4.15. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*
(2) *D is $\tilde{\star}$-Noetherian, $\tilde{\star}$-$\dim(D) = 1$ and D is quasi-$\tilde{\star}$-integrally closed.*
(3) *D is $\tilde{\star}$-Noetherian, $\tilde{\star}$-$\dim(D) = 1$ and $D^{\tilde{\star}}$ is integrally closed.*

Proof. The equivalence (2) ⇔ (3) follows from Lemma 4.13(2).

(1) ⇒ (2). Since D is a $\star$-DD, then D is $\star$-ADD (Remark 4.2(2)). Hence $\tilde{\star}$-$\dim(D) = 1$ (Proposition 2.16). Moreover, recall that a $\star$-DD is a $\tilde{\star}$-DD (Corollary 4.3). Then D is $\tilde{\star}$-Noetherian and a P$\tilde{\star}$MD (Proposition 4.1), and so D is quasi-$\tilde{\star}$-integrally closed by Corollary 4.14.

(3) ⇒ (1) For each $M \in \mathscr{M}(\star_f)$, it is well known that $D^{\tilde{\star}} \subseteq D_M$ and $D^{\tilde{\star}}_{MD_M \cap D^{\tilde{\star}}} = D_M$. Since $D^{\tilde{\star}}$ is integrally closed, this implies that D_M is also integrally closed. Therefore D_M is a local, Noetherian (by Proposition 3.8), integrally closed, one dimensional (by Lemma 2.11) domain, that is, a DVR [24, Theorem 37.8]. Hence D is a P$\star$MD. In particular, we have $\tilde{\star} = \star_f$ (Proposition 1.6), thus D is $\star_f$-Noetherian, by the assumption, and so D is $\star$-Noetherian (Proposition 3.5). We conclude that D is a $\star$-DD. □

By taking $\star = v$ in Theorem 4.15, we obtain the following characterization of Krull domains:

Corollary 4.16. *Let D be an integral domain. The following are equivalent*:

(1) *D is a Krull domain.*

(2) *D is a strong Mori domain*, w-dim$(D)=1$ *and* $D=D^{[w]}$.
(3) *D is a strong Mori domain*, w-dim$(D)=1$ *and D is integrally closed.*
(4) *D is a strong Mori domain*, t-dim$(D)=1$ *and D is integrally closed.*

Proof. The only part which needs a justification is the statement on t-dimension and w-dimension (in the equivalence (3) $\Leftrightarrow$ (4)). This follows from the fact that, in every integral domain, $w \leqslant t$ and $\mathscr{M}(t)=\mathscr{M}(w)$. □

Remark 4.17. Note that, if D is a $\star$-DD, then we know that $\tilde{\star}=\star_f$, and so D satisfies the properties:

(2_f) D is $\star_f$-Noetherian, $\star_f$-dim$(D)=1$ and D is quasi-$\star_f$-integrally closed;
(3_f) D is $\star_f$-Noetherian, $\star_f$-dim$(D)=1$ and $D^{\star_f}$ ($=D^{\star}$) is integrally closed

obtained from (2) and (3) of Theorem 4.15, replacing $\tilde{\star}$ with $\star_f$. But, conversely, if D satisfies either (2_f) or (3_f) then D is not necessarily a $\star$-DD. Indeed, let D,T and $\star$ be as in the example of Remark 2.3(2). Then we have already observed that $\star=\star_f$ and $\tilde{\star}=d_D$. Moreover, D is not a $\star$-DD (because it is not a $\star$-ADD), but $D^{\star_f}=T=D^{[\star_f]}$ is integrally closed (since T is a DVR), $\star_f$-dim$(D)=1$ (since $\mathscr{M}(\star_f)=\{M\}$ and $\star_f$-dim$(D)\leqslant$ dim$(D)=1$) and D is $\star_f$-Noetherian (Lemma 3.3, since T is Noetherian).

Note that (3_f) does not imply that D is a $\star$-DD, even if $\star$ is a (semi)star operation on D. Take T and D as in the example described in Remark 2.3(2) and, moreover, assume that k is algebraically closed in K. It is well known that, in this situation, D is integrally closed. Let $\star := v$ on D. It is easy to see that $\mathscr{M}(v)=\mathscr{M}(t)=\{M\}$, thus $w=d$ is the identity (semi)star operation on D (hence, $D^{[w]}=D^{[d]}=D$) and t-dim$(D)=1$ ($=v$-dim$(D)=w$-dim$(D)=$ dim(D)). Moreover, it is known that D is a Mori domain [22, Theorem 4.18] and thus D is a t-Noetherian domain. However, D is not a Krull domain, since D is not completely integrally closed (being T the complete integral closure of D). Note that, in this situation, D is even not a strong Mori domain (by Corollary 4.16).

Note also that, in the previous example, $D \subsetneq D^{[t]}$ (i.e. D is not t-integrally closed, hence does not satisfies condition (2_f) for $\star=v$), since $D^{[t]}=T$ by [3, Theorem 1.8(ii)].

On the other hand, if $\star$ is a (semi)star operation on D, then we know that D is a $\star$-DD if and only if D is a v-DD (i.e. a Krull domain) and $\star_f=t$ (Proposition 4.5). It is interesting to observe that, for $\star=v$, condition (1) of Theorem 4.15 is equivalent to (2_f). More precisely we have the following variation of the equivalence (1) $\Leftrightarrow$ (4) of Corollary 4.16:

D is a Krull domain if and only if D is t-Noetherian, t-dim$(D)=1$ and D is t-integrally closed (i.e. $D=D^{[t]}$).

As a matter of fact, let $F\in \boldsymbol{f}(D)$, then $D=D^{[t]}=D^{[v]}$ implies that $D=(F^v{:}F^v)=(F^{-1}:F^{-1})=(FF^{-1})^{-1}$ and so $(FF^{-1})^v=D$. Moreover, since t-Noetherian is equivalent to v-Noetherian (Proposition 3.5) and v-Noetherian implies that $v=t$ (Lemma 3.3), then $(FF^{-1})^t=D$. Thus D is a PvMD and so D is a v-DD (Proposition 4.1).

Finally, from the previous considerations we deduce that D is a $\star$-*DD* if and only if

$(\bar{2}_f)$ D is $\star_f$-Noetherian, $\star_f$-dim$(D)=1$, D is quasi-$\star_f$-integrally closed and $\star_f=t$.

We conclude with a question: is there an example of an integral (Krull) domain D, equipped with a (semi)star operation $\star$, such that condition (2_f) holds but $(\bar{2}_f)$ does not? Note that if such an example exists then necessarily $d \lneq \star_f$ $(\lneq t)$ [24, Theorem 37.8 ((1)⇔(2))].

Next result generalizes [24, Proposition 38.7].

Theorem 4.18. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*
(2) $\mathrm{Na}(D,\star)$ $(=\mathrm{Kr}(D,\star))$ *is a PID.*
(3) $\mathrm{Na}(D,\star)$ $(=\mathrm{Kr}(D,\star))$ *is a Dedekind domain.*

Proof. (1) ⇒ (2). Since D is a P$\star$MD, then $\mathrm{Na}(D,\star)=\mathrm{Kr}(D,\star)$ is a Bézout domain (Proposition 1.5 ((i)⇒(iv)) and Proposition 1.4(1)). Now, let $\mathfrak{J}$ be a nonzero ideal of $\mathrm{Na}(D,\star)$ and set $I := \mathfrak{J}\cap D$. We claim that $\mathfrak{J}=I\mathrm{Na}(D,\star)$. The inclusion $I\mathrm{Na}(D,\star)\subseteq\mathfrak{J}$ is clear. For the opposite inclusion, since $\mathfrak{J}=(\mathfrak{J}\cap D[X])\mathrm{Na}(D,\star)$, it is enough to show that $\mathfrak{J}\cap D[X]\subseteq I\mathrm{Na}(D,\star)$. Let $f\in\mathfrak{J}\cap D[X]$, then $f\mathrm{Na}(D,\star)=f\mathrm{Kr}(D,\star)=\boldsymbol{c}(f)\mathrm{Kr}(D,\star)=\boldsymbol{c}(f)\mathrm{Na}(D,\star)$ (where the second equality holds by Proposition 1.5(6)). Hence $\boldsymbol{c}(f)\subseteq f\mathrm{Na}(D,\star)\cap D\subseteq\mathfrak{J}\cap D=I$. Therefore we conclude that $f\in\boldsymbol{c}(f)\mathrm{Na}(D,\star)\subseteq I\mathrm{Na}(D,\star)$, which proves our claim. Now, since D is a $\tilde{\star}$-Noetherian domain (as D is a $\star$-DD, cf. Corollary 4.3 and Proposition 4.1), then $I^{\tilde{\star}}=F^{\tilde{\star}}$ for some $F\in\boldsymbol{f}(D)$, with $F\subseteq I$ (Lemma 3.3). Since $E^{\tilde{\star}}=E\mathrm{Na}(D,\star)\cap K$, for each $E\in\bar{F}(D)$ (Proposition 1.4(4)), then we have $\mathfrak{J}=I\mathrm{Na}(D,\star)=I^{\tilde{\star}}\mathrm{Na}(D,\star)=F^{\tilde{\star}}\mathrm{Na}(D,\star)=F\mathrm{Na}(D,\star)$. Hence we conclude that $\mathfrak{J}$ is a principal ideal in $\mathrm{Na}(D,\star)$, because, as we have already remarked, $\mathrm{Na}(D,\star)$ is a Bézout domain.

(2) ⇒ (3) is trivial.

(3) ⇒ (1). Assume that $\mathrm{Na}(D,\star)$ is a Dedekind domain then, obviously, $\mathrm{Na}(D,\star)=\mathrm{Kr}(D,\star)$ (Proposition 1.6 ((i) ⇒ (iv))) and $\mathrm{Na}(D,\star)$ is an ADD, and hence D is a $\star$-ADD (Theorem 2.5). In order to apply Theorem 4.11, it remains to show that D has the $\star_f$-FC property. Let $0\neq x\in D$. Since $\mathrm{Max}(\mathrm{Na}(D,\star))=\{M\mathrm{Na}(D,\star)\mid M\in\mathscr{M}(\star_f)\}$ (Proposition 1.4(2)) and $\mathrm{Na}(D,\star)$ is a Dedekind domain, then there are only finitely many maximal ideals $M\mathrm{Na}(D,\star)$ containing x. Furthermore, $M\mathrm{Na}(D,\star)\cap D=M$, for each $M\in\mathscr{M}(\star_f)=\mathscr{M}(\tilde{\star})$ (Proposition 1.4(4)). Hence x is contained in only finitely many quasi-$\star_f$-maximal ideals of D. Therefore we conclude that D is a $\star$-DD. □

From the previous result, we deduce immediately:

Corollary 4.19. *Let D be an integral domain. The following are equivalent*:

(1) *D is a Krull domain.*

(2) $\mathrm{Na}(D,v)$ ($=\mathrm{Kr}(D,v)$) *is a PID*.
(3) $\mathrm{Na}(D,v)$ ($=\mathrm{Kr}(D,v)$) *is a Dedekind domain*.

Another consequence of Theorem 4.18 is the following:

Corollary 4.20. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*
(2) *D is a $\tilde{\star}$-DD.*
(3) *$D^{\tilde{\star}}$ is a $\dot{\tilde{\star}}$-DD.*
(4) *$D^{\tilde{\star}}$ is a Krull domain and $\dot{\tilde{\star}} = t_{D^{\tilde{\star}}}$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from Theorem 4.18 and the fact that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}}, \dot{\tilde{\star}})$ (Proposition 1.4(6)).

The equivalence (3) $\Leftrightarrow$ (4) follows from Proposition 4.5, using the fact that (1) $\Leftrightarrow$ (3). □

Remark 4.21. From Corollary 4.20 ((1)$\Leftrightarrow$(4)), we have that if D is a $\star$-DD then $T := D^{\tilde{\star}}$ is a Krull domain and $\tilde{\star} = (t_{\dot{T}})_D$ (where t_T is the t-operation of T). Note that it is not true in general that, if T is a Krull overring of an integral domain D, then D is a $(t_{\dot{T}})_D$-Dedekind domain.

For instance, let K be a field and X an indeterminate over K. Set $T := K[\![X]\!]$, $M := XT$ and $D := K[\![X^2, X^3]\!]$. It is easy to see that D is a one-dimensional local Noetherian integral domain with integral closure equal to T and maximal ideal equal to $N := M \cap D$ (with $NT = N$). Therefore, in this case, $t_T = d_T$ is the identity (semi)star operation on T and so the semistar operation $(t_{\dot{T}})_D$ on D coincides with $\star_{\{T\}}$. Clearly $\widetilde{\star_{\{T\}}} = d_D$ ($\lneq \star_{\{T\}}$), since $\mathscr{M}(\star_{\{T\}}) = \{N\}$ and, obviously $D = D_N$ is not a DVR. Therefore D is not a $(t_{\dot{T}})_D$-Dedekind domain.

From the positive side, we have the following answer to the question of when, given a Krull overring T of an integral domain D, D is a $(t_{\dot{T}})_D$-DD:

(4.21.1) *Let T be an overring of an integral domain D. The following are equivalent*:

(1) *D is a $(t_{\dot{T}})_D$-DD.*
(2) *T is a Krull domain and, for each t_T-maximal ideal Q of T, $D_{Q\cap D} = T_Q$.*

The previous characterization is a straightforward consequence of the following "restatement" of the equivalence given in Corollary 4.20 ((1)$\Leftrightarrow$(4)):

(4.21.2) *If D is an integral domain and $\star$ is a semistar operation on D, then the following are equivalent*:

(1) *D is a $\star$-DD.*
(2) *There exists an overring T of D such that T is a Krull domain, $\star_f = (t_{\dot{T}})_D$ and, for each t_T-maximal ideal Q of T, $D_{Q\cap D} = T_Q$.*

To show the previous equivalence, note that in general the set of the quasi-$(t_{T})_D$-maximal ideals in D coincide with the set $\{Q \cap D \mid Q$ is a t_T-maximal ideal in $T\}$ [20, Lemma 2.3(3)]. Therefore the assumption that $\star_f = (t_{T})_D$ and, for each t_T-maximal ideal Q of T, $D_{Q\cap D} = T_Q$ implies that $E^{\tilde{\star}} = (ET)^{t_T}$, for each $E \in \overline{\boldsymbol{F}}(D)$, (in particular, $D^{\tilde{\star}} = T$), and so $\dot{\tilde{\star}}^T = t_T$. Therefore (4.21.2(2)) implies condition (4) of Corollary 4.20.

Conversely, assume that condition (4) of Corollary 4.20 holds and set $T := D^{\tilde{\star}}$. Since $\dot{\tilde{\star}} = t_T$ and D is a $\star$-DD (Corollary 4.20 ((4)⇒(1))), then $\tilde{\star} = \star_f$ (Propositions 1.6 and 4.1), and so $\dot{\star}_f = t_T$. Therefore $\star_f = (t_{T})_D$ (Proposition 1.3(2)). Moreover, by the previous considerations, the set of the quasi-$\star_f$-maximal ideals in D coincide with the set $\{Q \cap D \mid Q$ is a t_T-maximal ideal in $T\}$. Since D is a $\star$-DD and hence, in particular, a $\star$-ADD (and since T is a Krull domain), then $D_{Q\cap D}$ is a DVR, which must coincide with its (DVR) overring T_Q, for each t_T-maximal ideal Q of T.

It is possible to give another proof of (4.21.2) by using Lemma 3.1(2) and showing the following preliminary result of intrinsical interest concerning the P$\star$MDs:

(4.21.3) *Let D be an integral domain and $\star$ a semistar operation on D. Then, the following are equivalent*:

(1) *D is a* P$\star$MD.
(2) *There exists an overring T of D such that T is a* Pv_TMD, $\star_f = (t_{T})_D$ *and, for each t_T-maximal ideal Q of T, $D_{Q\cap D} = T_Q$.*

The proof is based on a variation of the techniques already discussed above and the details are omitted.

Example 4.22. Let D be a Mori domain, let Θ be the set of all the maximal t-ideals of D which are t-invertible and let $\star_\Theta$ be the spectral semistar operation on D associated to Θ (Example 1.1(3)). Assume that $\Theta \neq \emptyset$ (i.e. that D is a Mori nonstrongly Mori domain, accordingly to the terminology introduced by Barucci and Gabelli [6, page 105]), then D is a $\star_\Theta$-DD.

We apply the characterization given in (4.21.1) or in Corollary 4.20 ((1)⇔(4)). Note that by [6, Proposition 3.1 and Theorem 3.3 (a)], $D^{\star_\Theta}$ is a Krull domain such that the map $P \mapsto P^{\star_\Theta}$ defines a bijection between Θ and the set $\mathscr{M}(t_{D^{\star_\Theta}})$ of all the t-maximal ideals of $D^{\star_\Theta}$ and $D_P = (D^{\star_\Theta})_{P^{\star_\Theta}}$. Therefore the (semi)star operation $\dot{\star}_\Theta$ on $D^{\star_\Theta}$ coincides with the t-operation, $t_{D^{\star_\Theta}}$, on $D^{\star_\Theta}$. Moreover, it is easy to see that, on D, the semistar operation $(t_{D^{\star_\Theta}})_D$ coincides with $\star_\Theta$.

Let D be an integral domain and $\star$ a semistar operation on D. We say that two nonzero ideals A and B are $\star$-*comaximal* if $(A + B)^\star = D^\star$. Note that, if $\star$ is a semistar operation of finite type, then A and B are $\star$-comaximal if and only if A and B are not contained in a common quasi-$\star$-maximal ideal.

Lemma 4.23. *Let D be an integral domain and $\star$ a semistar operation on D. Let A and B be two nonzero $\star$-comaximal ideals of D. Then $(A \cap B)^\star = (AB)^\star$.*

Proof. In general $(A+B)(A\cap B)\subseteq AB$. Then, $((A+B)(A\cap B))^\star \subseteq (AB)^\star \subseteq (A\cap B)^\star$. But $((A+B)(A\cap B))^\star = ((A+B)^\star(A\cap B))^\star = (D^\star(A\cap B))^\star = (A\cap B)^\star$. Hence, $(A\cap B)^\star = (AB)^\star$. □

Corollary 4.24. *Let D be an integral domain and $\star$ a semistar operation of finite type. Let $n\geqslant 2$ and let $A_1, A_2, \ldots, A_n$ be nonzero ideals of D, such that $(A_i + A_j)^\star = D^\star$, for $i\neq j$. Then, $(A_1\cap A_2\cap\cdots\cap A_n)^\star = (A_1A_2\cdot\ldots\cdot A_n)^\star$.*

Proof. We prove it by induction on $n\geqslant 2$, using Lemma 4.23 for the case $n=2$. Set $A := A_1\cap A_2\cap\cdots\cap A_{n-1}$ and $B := A_n$. Then, A and B are not contained in a common quasi-$\star$-maximal ideal, otherwise, A_n and A_j (for some $1\leqslant j\leqslant n-1$) would be contained in a common quasi-$\star$-maximal ideal. Hence $(A_1\cap A_2\cap\cdots\cap A_{n-1}\cap A_n)^\star = (A\cap B)^\star = (AB)^\star = (A^\star B)^\star = (A_1A_2\cdot\ldots\cdot A_n)^\star$. □

Theorem 4.25. *Let D be an integral domain and $\star$ a semistar operation. The following are equivalent*:

(1) *D is a $\star$-DD.*

(2) *For each nonzero ideal I of D, there exists a finite family of quasi-$\star_f$-prime ideals $P_1, P_2, \ldots, P_n$ of D, pairwise $\star_f$-comaximals, and a finite family of non negative integers $e_1, e_2, \ldots, e_n$ such that $I^{\tilde{\star}} = (P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})^{\tilde{\star}}$.*

Moreover, if (2) holds and if $I^{\tilde{\star}}\neq D^{\tilde{\star}}$, then we can assume that $P_i^{\tilde{\star}}\neq D^{\tilde{\star}}$, for each $i=1,2,\ldots,n$. In this case, the integers $e_1, e_2, \ldots, e_n$ are positive and the factorization is unique.

Proof. (1) $\Rightarrow$ (2). Let I be a nonzero ideal of D. To avoid the trivial case, we can assume that $I^{\tilde{\star}}\neq D^{\tilde{\star}}$. Let $P_1, P_2, \ldots, P_n$ be the finite (non empty) set of quasi-$\star_f$-maximal ideals such that $I\subseteq P_i$, for $1\leqslant i\leqslant n$ (Theorem 4.11). We have $I^{\tilde{\star}} = \cap\{ID_P \mid P\in\mathscr{M}(\star_f)\} = \cap_{i=1}^{i=n}(ID_{P_i}\cap D^{\tilde{\star}})$. Since D_{P_i} is a DVR, then $ID_{P_i} = P_i^{e_i}D_{P_i}$, for some integers $e_i\geqslant 1$, $i=1,2,\ldots,n$. Therefore, we have $ID_{P_i}\cap D^{\tilde{\star}} = P_i^{e_i}D_{P_i}\cap D^{\tilde{\star}} = (P_i^{e_i})^{\tilde{\star}}$. Hence $I^{\tilde{\star}} = (P_1^{e_1})^{\tilde{\star}}\cap(P_2^{e_2})^{\tilde{\star}}\cap\cdots\cap(P_n^{e_n})^{\tilde{\star}} = (P_1^{e_1}\cap P_2^{e_2}\cap\cdots\cap P_n^{e_n})^{\tilde{\star}} = (P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})^{\tilde{\star}}$, by Corollary 4.24.

For the last statement, let $I^{\tilde{\star}} = (P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})^{\tilde{\star}}$, if $P_i^{\tilde{\star}} = D^{\tilde{\star}}$, for some i, then obviously we can cancel P_i from the factorization of $I^{\tilde{\star}}$.

We prove next the uniqueness of the representation of $I^{\tilde{\star}}$. From (Proposition 1.4(4)), we deduce that $I\mathrm{Na}(D,\star) = P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n}\mathrm{Na}(D,\star) = (P_1\mathrm{Na}(D,\star))^{e_1}(P_2\mathrm{Na}(D,\star))^{e_2}\cdot\ldots\cdot(P_n\mathrm{Na}(D,\star))^{e_n}$ is the unique factorization into primes of the ideal $I\mathrm{Na}(D,\star)$ in the PID $\mathrm{Na}(D,\star)$ (Theorem 4.18). Since $P_i = P_i\mathrm{Na}(D,\star)\cap D$ (because each P_i is a quasi-$\tilde{\star}$-maximal ideal of D), the factorization of $I^{\tilde{\star}}$ is unique.

(2) $\Rightarrow$ (1) Without loss of generality, we can assume that D is not a field. First, we prove that each localization to a quasi-$\star_f$-maximal ideal of D is a DVR. Let $M\in\mathscr{M}(\star_f)$ and let J be a nonzero proper ideal of D_M. Set $I := J\cap D$ $(\subseteq M)$. Then, it is easy to see that $I^{\tilde{\star}}\neq D^{\tilde{\star}}$ thus, by assumption, $I^{\tilde{\star}} = (P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})^{\tilde{\star}}$, for some family of quasi-$\star_f$-prime ideals P_i, with $P_i^{\tilde{\star}}\neq D^{\tilde{\star}}$ and for some family of integers $e_i\geqslant 1$,

$i=1,2,\ldots,n$. It follows that $J=ID_M=I^{\tilde{\star}}D_M=(P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})^{\tilde{\star}}D_M=(P_1^{e_1}P_2^{e_2}\cdot\ldots\cdot P_n^{e_n})D_M$ (since $\tilde{\star}$ is a spectral semistar operation defined by the set $\mathscr{M}(\star_f)$). Hence J is a finite product of primes of D_M. Therefore D_M is a local Dedekind domain [24, Theorem 37.8 ((1)⇔(3))], that is, D_M is a DVR.

Now we show that each quasi-$\tilde{\star}$-prime ideal of D is quasi-$\tilde{\star}$-invertible. Let Q be a quasi-$\tilde{\star}$-prime of D and let $0 \neq x \in Q$. Then, by assumption, $(xD)^{\tilde{\star}} = (P_1^{e_1}P_2^{e_2} \cdot \ldots \cdot P_n^{e_n})^{\tilde{\star}}$, with $P_1, P_2, \ldots, P_n$ nonzero prime ideals of D and $e_i \geqslant 1$, $i = 1,2,\ldots,n$. Since xD is obviously invertible (and thus, clearly, quasi-$\tilde{\star}$-invertible), then each P_i is quasi-$\tilde{\star}$-invertible [21, Lemma 2.11]. Moreover, since Q is a quasi-$\tilde{\star}$-ideal of D, then $P_1^{e_1}P_2^{e_2} \cdot \ldots \cdot P_n^{e_n} \subseteq (P_1^{e_1}P_2^{e_2} \cdot \ldots \cdot P_n^{e_n})^{\tilde{\star}} \cap D \subseteq Q$. Therefore, $P_j \subseteq Q$ for some j, with $1 \leqslant j \leqslant n$, and since D_Q is a DVR, we have $Q = P_j$. Hence Q is a quasi-$\tilde{\star}$-invertible ideal of D. Therefore, by Theorem 4.9, we conclude that D is $\tilde{\star}$-Dedekind. □

Remark 4.26. It is clear that, if D is a $\star$-DD then, for each nonzero ideal I of D, such that $I^{\star_f} \neq D^{\star_f}$, we have a unique factorization $I^{\star_f} = (P_1^{e_1}P_2^{e_2} \cdot \ldots \cdot P_n^{e_n})^{\star_f}$, for some family of quasi-$\star_f$-prime ideals P_i, with $P_i^{\star_f} \neq D^{\star_f}$, and for some family of positive integers e_i, $i = 1,2,\ldots,n$, since $\tilde{\star} = \star_f$ (Proposition 1.6). The converse is not true. For instance, take D, T and $\star$ as in Remark 2.3(2). For each nonzero proper ideal I of D, we have $I^{\star_f} = IT = M^e = (M^e)^{\star_f}$, for some positive integer e, since T is a DVR. Note that this representation is unique, since D is local with maximal ideal M and $\dim(D) = 1$. But we have already observed that D is not a $\star$-DD.

Next result generalizes to the semistar setting [24, Theorem 38.5 ((1)⇔(3))].

Theorem 4.27. *Let D be an integral domain which is not a field and $\star$ a semistar operation on D. The following are equivalent*:

(1) *D is a $\star$-DD.*

(2) *For each nonzero ideal I and for each $a \in I$, $a \neq 0$, there exists $b \in I^{\tilde{\star}}$ such that $I^{\tilde{\star}} = ((a,b)D)^{\tilde{\star}}$.*

Proof. (1) ⇒ (2). We start by proving the following:

Claim. If D is a $\star$-DD, then the map $M \mapsto M^{\widetilde{\star}}$ establishes a bijection between the set $\mathscr{M}(\star_f)$ ($=\mathscr{M}(\widetilde{\star})$ by Proposition 1.4(5)) of the quasi-$\star_f$-maximal ideals of D and the set $\mathscr{M}(t_{D^{\widetilde{\star}}})$ of the $t_{D^{\widetilde{\star}}}$-maximal ideals of (the Krull domain) $D^{\widetilde{\star}}$.

For each $M \in \mathscr{M}(\star_f)$, since $D^{\widetilde{\star}} \subseteq D_M$, it is easy to see that $M^{\widetilde{\star}} = MD_M \cap D^{\widetilde{\star}}$. Therefore, $M^{\widetilde{\star}}$ is a $\dot{\widetilde{\star}}$-prime ideal of $D^{\widetilde{\star}}$ and $M^{\widetilde{\star}} \cap D = M$. Furthermore, by Corollary 4.20, we know that $D^{\widetilde{\star}}$ is a Krull domain and $\dot{\widetilde{\star}} = t_{D^{\widetilde{\star}}}$. On the other hand, for each $\dot{\widetilde{\star}}$-prime ideal N of $D^{\widetilde{\star}}$, we know that $N \cap D$ is a quasi-$\widetilde{\star}$-prime of D [20, Lemma 2.3 (4)]. Since D is a $\star$-DD (or, equivalently, a $\widetilde{\star}$-DD), we have that each quasi-$\widetilde{\star}$-prime is a quasi-$\widetilde{\star}$-maximal (Proposition 2.2(2)), thus we easily conclude.

Let $a \in I$, $a \neq 0$, and $\{M_1, M_2, \ldots, M_n\}$ the (finite) set of quasi-$\star_f$-maximal ideals such that $a \in M_i$. Since D_{M_i} is a DVR, then $ID_{M_i} = x_i D_{M_i}$, for some $x_i \in I$, for each $i = 1, 2, \ldots, n$. We use the fact that $D^{\widetilde{\star}}$ is a Krull domain and, by the Claim, that $\{D^{\widetilde{\star}}_{M^{\widetilde{\star}}} = D_M \mid M \in \mathscr{M}(\star_f)\}$ is the defining family of the rank-one discrete valuation overrings of $D^{\widetilde{\star}}$, in order to apply the approximation theorem to $D^{\widetilde{\star}}$. Let $v_1, v_2, \ldots, v_n$ be the valuations associated respectively to $D_{M_1}, D_{M_2}, \ldots, D_{M_n}$ and let $v_{M'}$ be the valuation associated to $D_{M'} = D^{\widetilde{\star}}_{M'^{\widetilde{\star}}}$, for $M' \in \mathscr{M}' := \mathscr{M}(\star_f) \setminus \{M_1, M_2, \ldots, M_n\}$. Set $k_1 := v_1(x_1), k_2 := v_2(x_2), \ldots, k_n := v_n(x_n)$. Then there exists $b \in K$ such that $v_i(b) = k_i$, for each $i = 1, 2, \ldots, n$, and $v_{M'}(b) \geqslant 0$, for each $M' \in \mathscr{M}'$ [24, Theorem 44.1]. We have $I^{\tilde{\star}} = ((a, b)D)^{\tilde{\star}}$. Indeed, let $M \in \mathscr{M}(\star_f)$. If $M = M_i$, for some i, then $ID_M = ID_{M_i} = x_i D_{M_i} = bD_{M_i} = (a, b)D_{M_i}$. If $M \neq M_i$ for each i, then $ID_M = D_M = (a, b)D_M$.

(2) $\Rightarrow$ (1). Let $M \in \mathscr{M}(\star_f)$ and J a nonzero ideal of D_M. Let $a \in J$, $a \neq 0$, there exists $s \in D$, $s \notin M$, such that $sa \in I := J \cap D$. Then, by assumption, there exists $b \in I^{\widetilde{\star}}$ such that $I^{\tilde{\star}} = ((sa, b)D)^{\tilde{\star}}$. Therefore, we have $J = ID_M = I^{\tilde{\star}} D_M = ((sa, b)D)^{\tilde{\star}} D_M = (sa, b)D_M = (a, b)D_M$. By [24, Theorem 38.5], D_M is a Dedekind domain, and hence a DVR. Thus, D is a $\star$-ADD, hence, in particular, is a P$\tilde{\star}$MD (Corollary 2.6 and Proposition 2.2(2)). In addition, from the assumption and from [21, Lemma 2.3], we deduce that D is $\tilde{\star}$-Noetherian (Lemma 3.3), hence D is a $\star$-DD (Corollary 4.3 and Proposition 4.1(1)). $\square$

Remark 4.28. Note that, if D is a $\star$-DD (and hence $\widetilde{\star} = \star_f$), then D satisfies also a statement concerning $\star_f$, analogous to the statement (2) in Theorem 4.27:

(2_f) for each nonzero ideal I of D and for each $0 \neq a \in I$, there exists $b \in I^{\star_f}$ such that $((a, b)D)^{\star_f} = I^{\star_f}$.

But (2_f) does not imply that D is a $\star$-DD. For instance, let D, T and $\star$ be as in Remark 2.3. Obviously, for each nonzero proper ideal I of D and for each nonzero $a \in I \subseteq D$ we have $I^{\star_f} = IT = X^n T = (a, X^n)T = ((a, X^n)D)^{\star_f}$, for some $n \geqslant 1$, (where $X^n \in I^{\star_f} \cap D$), but D is not a $\star$-DD.

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