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On Hilbert Functions and Cohomology

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INTRODUCTION

In recent years, (co)homological methods have been a very useful tool in the study of Hilbert functions. This line of research has its origin in Serre's computation of the multiplicity and in the expression of the difference between the function and polynomial due to Serre and Grothendieck. The last few years have witnessed many works in this direction, using either Koszul-like complexes or the local cohomology of the graded rings associated to the ideal.

The aim of this work is to study Hilbert functions in local rings by exploiting the interplay between these two points of view. Roughly speaking, the second method is the first taken to the limit, since local cohomology is an inductive limit of Koszul cohomologies. Therefore, we have chosen to work systematically with the cohomological Koszul complex instead of the homological one. With the cohomological approach we have functorial properties which are lost in the homological case, due to the nonnaturality of the duality between Koszul homology and cohomology. This allows us to give a quick computation of the first local cohomology module of the Rees algebra. Namely, in the I-adic case we get

$$\left[H^1_{R_+}(R)\right]_n \cong \begin{cases} \widetilde{I^n}/I^n & \text{if } n \geq 0, \\ A & \text{if } n < 0. \end{cases}$$

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The cohomological approach also clarifies the relationship between the various formulas for the difference between the Hilbert–Samuel function and polynomial present in the literature.

We will work in the wider framework of good filtrations, rather than I-adic. The study of good filtrations has an intrinsic interest in its own right, for example, if A is analytically unramified, the filtration of integral closures and the filtration of tight closures (provided that A contains a field) are good filtrations. But, moreover, many of the results in the literature concerning the I-adic case involve the use of nonadic filtrations, such as $\{\widetilde{I^n}\}, \{JI^{n-1}\}$ with J a reduction of I, hence it seems worthwhile to place ourselves in this more general setup. In this line, we extend the concept of Ratliff–Rush closure to good filtrations. This Ratliff–Rush closure is used to compute the first cohomology module of the Rees algebra associated with any Hilbert filtration. We will also be able to extend to Hilbert filtrations some results involving the Ratliff–Rush closure of ideals, e.g., Proposition 2.4 in [Sal93].

We show that, in the same way that the associated graded ring is used to study the Hilbert function $H^0_{\mathcal{F}}$, the natural object to consider in the study of the Hilbert–Samuel function $H^1_{\mathcal{F}}$ is the extended Rees algebra $R^*(\mathcal{F})$. Namely, Johnston and Verma [JV95] give a formula, which holds for $n \geq 0$, for the difference $H^1_{\mathcal{F}} - h^1_{\mathcal{F}}$ in terms of the local cohomology of the Rees algebra $R(\mathcal{F})$. Since this formula is analogous to the one involving $H^0_{\mathcal{F}}$ and the associated graded ring, one expects that it should hold for all $n \in \mathbb{Z}$. But for this end it is necessary to replace $R(\mathcal{F})$ by $R^*(\mathcal{F})$ (see Theorem 4.1). In fact, $R^*(\mathcal{F})$ appears naturally in the problem when one considers that $H^1_{\mathcal{F}}$ is extended by zero for negative integers, hence the natural convention to take in order to study Hilbert functions is $I_n = A$ if $n < 0$.

The paper begins with the definition and study of the Ratliff–Rush closure $\breve{\mathcal{F}}$ of a good filtration $\mathcal{F}$ in Sect. 2. By using $\breve{\mathcal{F}}$ we will be able to extend to Hilbert filtrations some results involving the Ratliff–Rush closure of ideals. Section 3 is devoted to the definition and study of the cohomological complexes $C^{\cdot}(s, k, n)$ which are strongly related to the homological complexes studied in [KM81] and [HM95]. These complexes will serve as a link between local and Koszul cohomology: we use $C^{\cdot}(d, k, n)$ to compute the local cohomology of Rees algebras. In Sect. 4 we prove our main result, Theorem 4.1, which expresses for all $n \in \mathbb{Z}$ the difference between the Hilbert function and polynomial in terms of the local cohomology of the extended Rees algebra. This result is somehow [KM81, Theorem 2], taken to the limit, and generalizes Johnston and Verma's formula to any Hilbert filtration and $n < 0$. We apply the preceding results to recover and generalize Proposition 2.4 in [Sal93] and answer

Question 3.2 of [JV95]. Finally, we give some applications in low dimensions.

1. NOTATIONS

Let A be a Noetherian ring. A sequence of ideals $\mathcal{F} = \{I_n\}_{n \geq 0}$ is called a filtration if it verifies $I_0 = A$, $I_1 \neq A$, $I_{n+1} \subseteq I_n$ for all $n \geq 0$, and $I_n I_m \subseteq I_{n+m}$ for all $n, m \geq 0$. It is called a good filtration if there exists $n_0 \geq 1$ such that $I_{n+1} = I_1 I_n$ for all $n \geq n_0$ (see [HZ94, Remark 2.2]). If $A = (A, \mathfrak{m})$ is a local ring, a good filtration $\mathcal{F}$ is called a Hilbert filtration if I_1 is an $\mathfrak{m}$-primary ideal.

We will use the notation $\mathcal{R} = R(\mathcal{F}) = \oplus_{n \geq 0} I_n t^n \subseteq A[t]$ for the Rees algebra associated with $\mathcal{F}$, $\mathcal{G} = \mathrm{gr}_{\mathcal{F}}(A) = \oplus_{n \geq 0} I_n / I_{n+1}$ for the associated graded ring of $\mathcal{F}$ and $\mathcal{R}^* = R^*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} I_n t^n \subseteq A[t, t^{-1}]$ for the extended Rees algebra associated with $\mathcal{F}$, where we are taking $I_n = A$ for $n < 0$. Hence, we will adopt the convention that $I_n = 0$ or $I_n = A$ if $n < 0$, depending on whether we want to study properties of the Rees algebra or the extended Rees algebra. In case that $\mathcal{F}$ is an I-adic filtration, i.e., $I_n = I^n$ for all $n \geq 1$, we will write $R(I)$, $\mathrm{gr}_I(A)$, and $R^*(I)$ instead of $R(\mathcal{F})$, $\mathrm{gr}_{\mathcal{F}}(A)$, and $R^*(\mathcal{F})$. For any ideal $J \subseteq I_1$ we will denote by $\mathcal{F}/J$ the filtration in A/J given by $\{(I_n + J)/J\}_{n \geq 0}$. Notice that if $\mathcal{F}$ is good, then $\mathcal{F}/J$ is also good.

If $\mathcal{F}$ is a good filtration and $x \in I_t$ we will say that x is a superficial element of $\mathcal{F}$ if there exists $c \geq 0$ such that $(I_{n+t} : x) \cap I_c = I_n$ for all $n \geq c$. If $\mathrm{grade}(I_1) \geq 1$ it is shown in [HM95] that this is equivalent to $x \notin z(A)$ and $(I_{n+t} : x) = I_n$ for $n \gg 0$. Unless otherwise specified, all our superficial elements will be superficial of degree 1 (which always exist if A is local with infinite residue field). If $\underline{x} = x_1, \ldots, x_s \in I_1$ we will say that $\underline{x}$ is a superficial sequence if, for all $1 \leq i < s$, x_{i+1} is superficial for $\mathcal{F}/(x_1, \ldots, x_i)$.

Let us recall the definition of reduction of a filtration (see [HZ94]): Let $\mathcal{F} = \{I_n\}_{n \geq 0}$, $\mathcal{H} = \{J_n\}_{n \geq 0}$ be good filtrations; $\mathcal{H}$ is called a good reduction of $\mathcal{F}$ if $\mathcal{H} \subseteq \mathcal{F}$ and there exists $m \geq 0$ such that $I_n = I_{n-1} J_1 + \cdots + I_{n-m} J_m$ for all $n \gg 0$. A good reduction is called minimal (good) reduction of $\mathcal{F}$ if it does not properly contain any good reduction of $\mathcal{F}$. It is shown in [HZ94] that $\mathcal{H}$ is a minimal reduction of $\mathcal{F}$ if and only if $\mathcal{H} = \{J^n\}_{n \geq 0}$, where J is a minimal reduction of I_1. Hence we can take $m = 1$ in the definition. We will say that J is a minimal reduction of $\mathcal{F}$. For such a reduction, the reduction number of $\mathcal{F}$ with respect to J is defined as $r_J(\mathcal{F}) = \min\{n \mid I_{k+1} = J I_k \text{ for all } k \geq n\}$. The reduction number of $\mathcal{F}$ is then $r(\mathcal{F}) = \min\{r_J(\mathcal{F}) \mid J \text{ a minimal reduction of } \mathcal{F}\}$.

By [HZ94, Lemma 2.11], any minimal reduction can be generated by a superficial sequence. Conversely, any superficial sequence generates a minimal reduction:

LEMMA 1.1. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen–Macaulay local ring and F a Hilbert filtration in A. Let $x_1, \dots, x_m \in I_1$ with $1 \le m \le d$. Then the following conditions are equivalent*:

(i) $x_1, \dots, x_m$ *is a superficial sequence for* F;

(ii) $x_1, \dots, x_m$ *is an A-sequence and there exists* $n_0 \in \mathbb{N}$ *such that for all* $1 \le i \le m$

$$(x_1, \dots, x_i) \cap I_n = (x_1, \dots, x_i) I_{n-1} \qquad \text{for all } n \ge n_0.$$

Moreover, if $m = d$ *the conditions above imply that* $(x_1, \dots, x_d)$ *is a minimal reduction of* F.

The proof is a standard argument using the claim in [HM95, Sect. 2] and [KM82, Proposition 2.1(b)]. See also [Swa94, Lemma 1 and 2].

If F is a Hilbert filtration, it makes sense to define the Hilbert–Samuel function of F by $H^1_{\mathsf{F}}(n) = \lambda_A(A/I_{n+1})$. It is well known that there exists a polynomial $h^1_{\mathsf{F}} \in \mathbb{Q}[X]$ such that $H^1_{\mathsf{F}}(n) = h^1_{\mathsf{F}}(n)$ for all $n \gg 0$; this is the Hilbert–Samuel polynomial of F and it can be written as

$$h^1_{\mathsf{F}}(n) = e_0 \binom{n+d}{d} - e_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

with $e_i = e_i(\mathsf{F}) \in \mathbb{Z}$, $e_0 > 0$. These are called the Hilbert coefficients of F. We also define the Hilbert function of F by $H^0_{\mathsf{F}}(n) = \lambda_A(I_n/I_{n+1})$ for all $n \ge 0$. Notice that $H^0_{\mathsf{F}}(n) = H^1_{\mathsf{F}}(n) - H^1_{\mathsf{F}}(n-1)$. In the case that the Hilbert filtration F is an I-adic filtration, we will write H^0_I, h^0_I, H^1_I, and $e_i(I)$ instead of H^0_{F}, h^0_{F}, H^1_{F}, h^1_{F}, and $e_i(\mathsf{F})$.

2. THE RATLIFF–RUSH CLOSURE OF A GOOD FILTRATION

For any ideal $I \subseteq A$, let

$$\tilde{I} = \bigcup_{k \ge 1} (I^{k+1} : I^k)$$

be the Ratliff–Rush closure of I (see [RR78] and [HJLS92]). This ideal has some nice properties; for instance, if $A = (A, \mathfrak{m})$ is local and I is $\mathfrak{m}$-primary it is the biggest ideal containing I and having the same Hilbert polynomial.

DEFINITION 2.1. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a good filtration. We define the Ratliff–Rush closure of $\mathcal{F}$ to be the sequence of ideals $\breve{\mathcal{F}} = \{\breve{I}_n\}_{n\geq 0}$ given by

$$\breve{I}_n = \bigcup_{k\geq 1} \left(I_{n+k} : I_1^k\right).$$

Remark 2.2. Notice that it makes no sense to consider $\breve{I}$ for an ideal I if I is not expressed as a piece of a good filtration. However, the usual Ratliff–Rush closure $\tilde{I}$ can be recovered from this definition (see Proposition 2.3(iv)).

PROPOSITION 2.3. *Assume* grade$(I_1) \geq 1$. *Then*

(i) $\breve{I}_n = \bigcup_{k\geq 1}(I_{nk+n} : I_n^k)$ *for all* $n \geq 0$;

(ii) *for all* $n \geq 0$ *it holds that* $I_n \subseteq \widetilde{I_n} \subseteq \breve{I}_n$, *with equality if* $n \gg 0$;

(iii) $\breve{\mathcal{F}}$ *is a good filtration*;

(iv) *if* $\mathcal{F}$ *is an I-adic filtration then* $\breve{\mathcal{F}} = \{\widetilde{I^n}\}_{n\geq 0}$;

(v) *if* $\mathcal{G} = \{J_n\}_{n\geq 0}$ *is a filtration such that* $J_n = I_n$ *for all* $n \gg 0$, *then* $\mathcal{G} \subseteq \breve{\mathcal{F}}$;

(vi) $\breve{\breve{\mathcal{F}}} = \breve{\mathcal{F}}$;

(vii) *let* $(x_1, \ldots, x_s)$ *be a minimal reduction of* $\mathcal{F}$; *then* $\breve{I}_n = \bigcup_{k\geq 1}(I_{n+k} : (x_1^k, \ldots, x_s^k))$ *for all* $n \geq 0$.

Proof. (i) First of all notice that both sequences of ideals $\{(I_{nk+n} : I_n^k)\}_{k\geq 1}$ and $\{(I_{n+k} : I_1^k)\}_{k\geq 1}$ are increasing chains; hence, for all $m \gg 0$,

$$\bigcup_{k\geq 1} \left(I_{nk+n} : I_n^k\right) = \left(I_{nm+n} : I_n^m\right) \quad \text{and} \quad \breve{I}_n = \left(I_{n+m} : I_1^m\right).$$

Let $x \in (I_{nm+n} : I_n^m)$. Since $I_1^{nm}x = (I_1^n)^m x \subseteq (I_n)^m x \subseteq I_{nm+n}$, we get $x \in \breve{I}_n$. Conversely, since $\mathcal{F}$ is good we can consider n_0 such that $I_{n+1} = I_1 I_n$ for all $n \geq n_0$. Then if $x \in \breve{I}_n$ we may assume that $x \in (I_{n+m} : I_1^m)$ with $m \geq n_0$. Hence $I_n^m x \subseteq I_{mn}x = I_{n_0} I_1^{mn-n_0} x \subseteq I_{n_0} I_{n+mn-n_0} \subseteq I_{mn+n}$, that is, $x \in \bigcup_{k\geq 1}(I_{nk+n} : I_n^k)$.

(ii) By definition of $\widetilde{I_n}$ we have $\widetilde{I_n} = \bigcup_{k\geq 1}(I_n^{k+1} : I_n^k)$. Since $I_n^{k+1} \subseteq I_{nk+n}$ for all $k \geq 1$ we get $(I_n^{k+1} : I_n^k) \subseteq (I_{nk+n} : I_n^k)$, hence $\widetilde{I_n} \subseteq \breve{I}_n$. Let $x \in I_1^s$ be a superficial element of degree $s \geq 1$ for $\mathcal{F}$; we know that such an element exists by [Rho71, Lemma 2.1]. Then for all $n \gg 0$ we have $(I_{n+s} : x) = I_n$, hence $I_n \subseteq (I_{n+1} : I_1) \subseteq (I_{n+2} : I_1^2) \subseteq \cdots \subseteq (I_{n+s} : I_1^s) \subseteq$

$(I_{n+s}:x)=I_n$, that is, $(I_{n+1}:I_1)=I_n$. So, for all $k\geq 1$ it holds that $(I_{n+k}:I_1^k)=((I_{n+k}:I_1):I_1^{k-1})=(I_{n+k-1}:I_1^{k-1})=\cdots=(I_{n+1}:I_1)=I_n$, hence $\breve{I}_n=I_n$ for $n\gg 0$.

(iii) Let us first show that $\breve{F}$ is a filtration. It is obvious that $\breve{I}_0=A$, and if $n>0$, for all $k\geq 1$ we have $(I_{n+k+1}:I_1^k)\subseteq(I_{n+k}:I_1^k)$, hence $\breve{I}_{n+1}\subseteq\breve{I}_n$. On the other side, $\breve{I}_n=(I_{n+k}:I_1^k)$ and $\breve{I}_m=(I_{m+k}:I_1^k)$ for $k\gg 0$, hence $\breve{I}_n\breve{I}_m=(I_{n+k}:I_1^k)(I_{m+k}:I_1^k)\subseteq(I_{n+k}I_{m+k}:I_1^{2k})\subseteq(I_{n+m+2k}:I_1^{2k})\subseteq\breve{I}_{n+m}$. Then by (ii) we have

$$\breve{I}_{n+1}=I_{n+1}=I_1I_n\subseteq\breve{I}_1\breve{I}_n\subseteq\breve{I}_{n+1}$$

for all $n\gg 0$, hence $\breve{I}_{n+1}=\breve{I}_1\breve{I}_n$, that is, $\breve{F}$ is good.

(iv) It is obvious from the definitions.

(v) Let $k\geq 0$ be such that $J_m=I_m$ for all $m\geq k$. Then for all $n\geq 0$ we have $I_1^kJ_n\subseteq I_kJ_n=J_kJ_n\subseteq J_{n+k}=I_{n+k}$, hence $J_n\subseteq\breve{I}_n$.

(vi) It is obvious from (ii) and (v).

(vii) Since $(I_{n+k}:I_1^k)\subseteq(I_{n+k}:(x_1^k,\ldots,x_s^k))$ for all $k,n\geq 0$, it is obvious that $\breve{I}_n\subseteq\bigcup_{k\geq 1}(I_{n+k}:(x_1^k,\ldots,x_s^k))$ for all $n\geq 0$. Conversely, recall that by [HZ94, Proposition 2.6], $(x_1,\ldots,x_s)$ is also a reduction of I_1, and let r be the corresponding reduction number. Then a computation analogous to that in [Sal82], Proposition 2(i)], shows that $I_1^{r+s(k-1)+1}=(x_1^k,\ldots,x_s^k)I_1^{r+s(k-1)+1-k}$ for all $k\geq 1$. So we get $(I_{n+k}:(x_1^k,\ldots,x_s^k))\subseteq(I_{n+r+s(k-1)+1}:I_1^{r+s(k-1)+1})\subseteq\breve{I}_n$ for all$k\geq 1$. ∎

Remark 2.4. When trying to generalize the concept of Ratliff–Rush closure to good filtrations, the first object that comes to mind is $\{\widetilde{I_n}\}_{n\geq 0}$. But, unlike $\breve{F}$, this sequence of ideals is not a filtration in general: for instance, let $A=k[\![t^3,t^4]\!]$ and $\mathfrak{m}=(t^3,t^4)$. Define $F=\{I_n\}_{n\geq 0}$ by $I_0=A$ and $I_n=t^3\mathfrak{m}^{n-1}$ for all $n\geq 1$. Since $t^3\mathfrak{m}^n=\mathfrak{m}^{n+1}$ for all $n\geq 2$, we get that F is a Hilbert filtration; but $\tilde{I}_2\not\subseteq\tilde{I}_1$ (see [HLS92, Example 1.11]).

The proposition below generalizes a result known for I-adic filtrations to any good filtration:

PROPOSITION 2.5. *Let F be a good filtration such that* grade$(I_1)\geq 1$, *and let G be the associated graded ring of F. Then*

$$H^0_{G_+}(G)=\bigoplus_{n\geq 0}\frac{\breve{I}_{n+1}\cap I_n}{I_{n+1}}.$$

In particular grade$(G_+)>0$ *if and only if* $F=\breve{F}$.

Proof. For any $x \in A$, let x^* denote its initial form in G. If $x^* \in [H^0_{G_+}(G)]_n$, then there exists $k \geq 1$ such that $G_+^k x^* = 0$ and in particular $[G_+^k]_k x^* = 0$. Since $[G_+^k]_k = (I_1^k + I_{k+1})/I_{k+1}$, we get $I_1^k x \subseteq (I_1^k + I_{k+1})x \subseteq I_{n+k+1}$, hence $x \in \breve{I}_{n+1} \cap I_n$.

Conversely, if $x^* \in (\breve{I}_{n+1} \cap I_n)/I_{n+1}$, we must show that there exists $k \gg 0$ such that $G_+^k x^* = 0$. Since $G_+^k \subseteq \oplus_{m \geq k} G_m$, it is enough to show that $G_m x^* = 0$ for all $m \gg 0$. From Proposition 2.3(ii) we can take n_0 such that $\breve{I}_m = I_m$ for all $m \geq n_0$; then $I_m x \subseteq \breve{I}_m x \subseteq \breve{I}_m \breve{I}_{n+1} \subseteq \breve{I}_{m+n+1} = I_{m+n+1}$ for all $m \geq n_0$. Therefore $G_m x^* = 0$ for all $m \geq n_0$.

Now it is obvious that if $F = \breve{F}$ then $\operatorname{grade}(G_+) > 0$. Conversely, if $\operatorname{grade}(G_+) > 0$ then $\breve{I}_{n+1} \cap I_n = I_{n+1}$ for all $n \geq 0$. For $n = 0$ we get $\breve{I}_1 = I_1$ and then by induction $I_{n+1} = \breve{I}_{n+1} \cap I_n = \breve{I}_{n+1} \cap \breve{I}_n = \breve{I}_{n+1}$. ∎

Another graded ring which can be naturally obtained from G and whose irrelevant ideal has positive grade is $G/H^0_{G_+}(G)$. Let us compare these two rings:

THEOREM 2.6. *Let F be a Hilbert filtration such that* $\operatorname{grade}(I_1) \geq 1$, *let* $G = \operatorname{gr}_A(F)$ (resp. $\breve{G} = \operatorname{gr}_A(\breve{F})$) *be the associated graded rings, and let* $\overline{G} = G/H^0_{G_+}(G)$. *Then $\overline{G}$ is isomorphic to a graded subalgebra of $\breve{G}$, and they coincide if and only if $\breve{F} = F$.*

Proof. By Proposition 2.5 we have

$$\overline{G} = \bigoplus_{n \geq 0} \frac{I_n}{\breve{I}_{n+1} \cap I_n} \cong \bigoplus_{n \geq 0} \frac{I_n + \breve{I}_{n+1}}{\breve{I}_{n+1}} \subseteq \bigoplus_{n \geq 0} \frac{\breve{I}_n}{\breve{I}_{n+1}} = \breve{G}.$$

Moreover $\overline{G} = \breve{G}$ if and only if $I_n + \breve{I}_{n+1} = \breve{I}_n$ for all $n \geq 0$, and by Proposition 2.3(ii) this is equivalent to $\breve{I}_n = I_n$ for all $n \geq 0$. ∎

As an immediate application of the Ratliff–Rush closure we can give a short proof of the following well-known result:

PROPOSITION 2.7. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring, F a Hilbert filtration in A. If $e_1(F) = \cdots = e_d(F) = 0$, then $r(F) = 0$, that is, F is I-adic with I a complete intersection.*

Proof. Consider the filtration $G = \{J^n\}_{n \geq 0}$, with $J = (x_1, \dots, x_d)$ a minimal reduction of F. Then the two filtrations have the same Hilbert polynomial and $G \subseteq F$; hence $J^n = I_n$ for $n \gg 0$. Then by Proposition 2.3 we get $F \subseteq \breve{G} = G$, hence $F = G$. ∎

3. KOSZUL COMPLEXES AND LOCAL COHOMOLOGY OF REES ALGEBRAS

Let $R = \oplus_{n \in \mathbb{Z}} R_n$ be a Noetherian graded ring, $x_1, \ldots, x_s \in R$ homogeneous elements of degrees $k_1, \ldots, k_s$, respectively. In the case that $R = (A, \mathfrak{m})$ is a local ring, we will consider it as a graded ring with the trivial graduation. Let us recall the definition of the (graded) cohomological Koszul complex $K^{\cdot}(x_1, \ldots, x_s; R)$; we will use [BH93] and [HIO88] as general references.

The complex $(K^{\cdot}(x_1, \ldots, x_s; R), d^{\cdot})$ is defined as the dual complex of the homological Koszul complex $(K_{\cdot}(x_1, \ldots, x_s; R), d_{\cdot})$, that is,

$$K^n(x_1, \ldots, x_s; R) = K_n(x_1, \ldots, x_s; R)^* = \operatorname{Hom}_R\bigl(K_n(x_1, \ldots, x_s; R), R\bigr)$$

and $d^{\cdot} = d_{\cdot}^*$; $K^n(x_1, \ldots, x_s; R)$ can be identified with the graded free R-module of rank $\binom{s}{n}$ given by

$$\bigoplus_{1 \le i_1 < \cdots < i_n \le s} R\bigl(k_{i_1} + \cdots + k_{i_n}\bigr) = \bigoplus_{1 \le i_1 < \cdots < i_n \le s} Re_{i_1, \ldots, i_n},$$

where $\deg(e_{i_1, \ldots, i_n}) = -(k_{i_1} + \cdots + k_{i_n})$. With this notation the differential $d^n \colon K^n(x_1, \ldots, x_s; R) \to K^{n+1}(x_1, \ldots, x_s; R)$ is given by

$$d^n(e_{i_1, \ldots, i_n}) = \sum_{k=1}^{n+1} \sum_{\substack{i_{k-1} < j < i_k \\ i_0 = 0,\ i_{n+1} = s+1}} (-1)^{k-1} x_j e_{i_1, \ldots, i_{k-1}, j, i_k, \ldots, i_n};$$

notice that d^n is a homogeneous morphism. We will denote by $H^i(x_1, \ldots, x_s; R) = H^i(K^{\cdot}(x_1, \ldots, x_s; R))$ the ith cohomology module of this complex. Since the differentials are homogeneous morphisms, it has a natural structure of graded R-module. It holds that

$$H^0(x_1, \ldots, x_s; R) = \bigl(0 : (x_1, \ldots, x_s)\bigr)$$

and

$$H^s(x_1, \ldots, x_s; R) = \frac{R}{(x_1, \ldots, x_s)}(k_1 + \cdots + k_s).$$

Let $H_i(x_1, \ldots, x_s; R) = H_i(K_{\cdot}(x_1, \ldots, x_s; R))$. If we forget about the graduation of R we have nonnatural isomorphisms of R-modules $H^i(x_1, \ldots, x_s; R) \cong H_{s-i}(x_1, \ldots, x_s; R)$ for all $0 \le i \le s$ (see [BH93, Proposition 1.6.10]). These morphisms are not homogeneous except in the special case $k_1 = \cdots = k_s = k$, where we get homogeneous isomorphisms

$H^i(x_1, \dots, x_s; R) \cong H_{s-i}(x_1, \dots, x_s; R)(ik)$. If R is either a local ring or a positively graded ring with $k_i > 0$ for all $1 \le i \le s$, then it holds that $x_1, \dots, x_s$ is an R-sequence if and only if $H^i(x_1, \dots, x_s; R) = 0$ for all $i \ne s$ (see, for example, [Mat86, Theorem 16.5]).

Now let $(A, \mathfrak{m})$ be a local ring, let $\mathsf{F} = \{I_n\}_{n \ge 0}$ be a Hilbert filtration, and let $x_1, \dots, x_m \in I_1$. Let us adopt the convention that $I_n = 0$ if $n < 0$. We will define, for $1 \le s \le m$, $k \ge 1$, and $n \in \mathbb{Z}$, cohomological complexes $C^{\cdot}(s, k, n)$,

$$0 \to C^0(s,k,n) \to C^1(s,k,n) \to \cdots \to C^{s-1}(s,k,n) \\ \to C^s(s,k,n) \to 0,$$

as follows: $C^{\cdot}(s, k, n)$ is the complex

$$0 \to A/I_n \to (A/I_{n+k})^{\binom{s}{1}} \to (A/I_{n+2k})^{\binom{s}{2}} \to \cdots \\ \to (A/I_{n+(s-1)k})^{\binom{s}{s-1}} \to A/I_{n+sk} \to 0,$$

where the differentials are induced by the differentials of the Koszul complex $K^{\cdot}(x_1^k, \dots, x_s^k; A)$. In other words, we have an epimorphism of complexes

$$K^{\cdot}(x_1^k, \dots, x_s^k; A) \to C^{\cdot}(s,k,n).$$

such that, with our convention about the negative pieces, it becomes an equality if $n < -sk$.

Remark 3.1. Several homological analogues of these complexes appear in the literature. For example, $C^{\cdot}(d, k, n)$ is a cohomological particular case of the complex $\overline{K}$ of the proof of Theorem 2 in [KM81]. Also, in the case $k = 1$ our $C^{\cdot}(s, 1, n - s)$ is the cohomological counterpart of the complex $C_{\cdot}(x_1, \dots, x_s, \mathsf{F}, n)$ defined in [HM95].

Part of the cohomology of $C^{\cdot}(s, k, n)$ can be computed:

PROPOSITION 3.2. *For all* $1 \le s \le m$, $k \ge 1$, $n \in \mathbb{Z}$, *the following hold*:

(i) $H^0(C^{\cdot}(s,k,n)) = (I_{n+k} : (x_1^k, \dots, x_s^k))/I_n$;

(ii) $H^s(C^{\cdot}(s,k,n)) = A/(I_{n+sk} + (x_1^k, \dots, x_s^k))$;

(iii) *if* $x_1, \dots, x_s$ *is an* A*-sequence*,

$$H^{s-1}(C^{\cdot}(s,k,n)) \cong \frac{(x_1^k, \dots, x_s^k) \cap I_{n+sk}}{(x_1^k, \dots, x_s^k) I_{n+(s-1)k}}.$$

Proof. Parts (i) and (ii) are straightforward. To prove (iii), consider the top part of the Koszul complex $K^{\cdot}(x_1^k, \dots, x_s^k; A)$,

$$A^{\binom{s}{s-2}} \xrightarrow{d_K^{s-2}} A^s \xrightarrow{d_K^{s-1}} \left(x_1^k, \dots, x_s^k\right) \to 0.$$

Since $x_1^k, \dots, x_s^k$ is an A-sequence, this is an exact sequence. Tensoring by $A/I_{n+(s-1)k}$ we get an exact sequence

$$\left(\frac{A}{I_{n+(s-1)k}}\right)^{\binom{s}{s-2}} \xrightarrow{\bar{d}_K^{s-2}} \left(\frac{A}{I_{n+(s-1)k}}\right)^s \xrightarrow{\bar{d}_K^{s-1}} \frac{\left(x_1^k, \dots, x_s^k\right)}{\left(x_1^k, \dots, x_s^k\right) I_{n+(s-1)k}} \to 0.$$

Let $d_C^{\cdot}$ denote the differential of $C^{\cdot}(s,k,n)$. Since $\operatorname{Im}(\bar{d}_K^{s-2}) = \operatorname{Im}(d_C^{s-2})$ it is clear that there is a commutative diagram of exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}(d_C^{s-2}) & \longrightarrow & \left(\frac{A}{I_{n+(s-1)k}}\right)^s & \xrightarrow{\bar{d}_K^{s-1}} & \frac{\left(x_1^k, \dots, x_s^k\right)}{\left(x_1^k, \dots, x_s^k\right) I_{n+(s-1)k}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & \operatorname{Ker}(d_C^{s-1}) & \longrightarrow & \left(\frac{A}{I_{n+(s-1)k}}\right)^s & \xrightarrow{d_C^{s-1}} & \frac{A}{I_{n+sk}}. & & \end{array}$$

Then by the snake lemma we get

$$H^{s-1}(C^{\cdot}(s,k,n)) = \operatorname{Coker}\alpha \cong \operatorname{Ker}\gamma = \frac{\left(x_1^k, \dots, x_s^k\right) \cap I_{n+sk}}{\left(x_1^k, \dots, x_s^k\right) I_{n+(s-1)k}}. \quad \blacksquare$$

Recall the definition of tensor product of complexes: if $K^{\cdot}, L^{\cdot}$ are complexes of graded R-modules, then $(K \otimes L)^{\cdot}$ is the complex given by $(K \otimes L)^n = \bigoplus_{i+j=n} K^i \otimes L^j$, with differential $d(x \otimes y) = dx \otimes y + (-1)^i x \otimes dy$ for $x \in K^i$, $y \in L^j$. Notice that $(K \otimes L)^n$ is a graded R-module with the graduation given by $\deg(x \otimes y) = \deg(x) + \deg(y)$. Hence $(K \otimes L)^{\cdot}$ is a complex of graded modules with homogeneous differential. If f, g are morphisms of complexes, then $f \otimes g$ is again a morphism of complexes, and if f^n and g^n are homogeneous morphisms for all n, so are $(f \otimes g)^n$. Then it is easily shown that $K^{\cdot}(x_1, \dots, x_s; R) = K^{\cdot}(x_1; R) \otimes \cdots \otimes K^{\cdot}(x_s; R)$.

Now let $a_1, \ldots, a_s \in R$ be homogeneous elements of degrees $\nu_1, \ldots, \nu_s$, respectively. Then the homogeneous morphisms of complexes $K^{\cdot}(a_j^k; R) \to K^{\cdot}(a_j^{k+1}; R)$ given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\cdot a_j^k} & R(k\nu_j) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot a_j} & & \\ 0 & \longrightarrow & R & \xrightarrow{\cdot a_j^{k+1}} & R((k+1)\nu_j) & \longrightarrow & 0 \end{array}$$

give us a homogeneous morphism $K^{\cdot}(a_1^k, \ldots, a_s^k; R) \to K^{\cdot}(a_1^{k+1}, \ldots, a_s^{k+1}; R)$ whose component of degree j is the multiplication by the $\binom{s}{j}$-tuple $(a_{i_1} \cdots a_{i_j} | 1 \le i_1 < \cdots < i_j \le s)$. Hence we have an inductive system of complexes which gives us the local cohomology of R; for all $i \ge 0$ it holds that

$$H^i_{(\underline{a})}(R) = \lim_{\vec{k}} H^i\left(a_1^k, \ldots, a_s^k; R\right)$$

(see [HIO88, Theorem 35.18]). When R is the Rees algebra $\mathsf{R} = R(\mathsf{F})$ and $(x_1, \ldots, x_d)$ is a reduction of F, then the ideals $(x_1 t, \ldots, x_d t)$ and R_+ have the same radical, so

$$H^i_{\mathsf{R}_+}(\mathsf{R}) = \lim_{\vec{k}} H^i\left((x_1 t)^k, \ldots, (x_d t)^k; \mathsf{R}\right).$$

THEOREM 3.3. *Let $(A, \mathfrak{m})$ be a d-dimensional local Cohen–Macaulay ring, let F be a Hilbert filtration and let $\underline{x} = x_1, \ldots, x_s \in I_1$ be an A-sequence. Then for all $n \in \mathbb{Z}$:*

(i) *For all $k \ge 0$ we have*

$$\left[H^i\left(\underline{(xt)^k}; \mathsf{R}\right)\right]_n \cong H^{i-1}(C^{\cdot}(s, k, n)) \qquad \textit{for all } 0 \le i \le s-1$$

and an exact sequence of A-modules

$$0 \to H^{s-1}(C^{\cdot}(s,k,n)) \to \left[H^s\left(\underline{(xt)^k}; \mathsf{R}\right)\right]_n \to H^s\left(\underline{x}^k; A\right) \to H^s(C^{\cdot}(s,k,n)) \to 0.$$

(ii) *If moreover $s = d$ and $\underline{x}$ is a superficial sequence for F then*

$$\left[H^i_{\mathsf{R}_+}(\mathsf{R})\right]_n \cong \lim_{\vec{k}} H^{i-1}(C^{\cdot}(d,k,n)) \qquad \textit{for all } 0 \le i \le d-1,$$

and we have an exact sequence of A-modules

$$0 \to \lim_{\vec{k}} \frac{(\underline{x}^k) \cap I_{dk+n}}{(\underline{x}^k) I_{(d-1)k+n}} \to \left[H^d_{\mathbf{R}_+}(\mathbf{R}) \right]_n$$

$$\to H^d_{\mathfrak{m}}(A) \to \lim_{\vec{k}} \frac{A}{I_{dk+n} + (\underline{x}^k)} \to 0.$$

In particular if $d \geq 2$ it holds that

$$\left[H^1_{\mathbf{R}_+}(\mathbf{R}) \right]_n \cong \begin{cases} \breve{I}_n / I_n & \text{if } n \geq 0, \\ A & \text{if } n < 0. \end{cases}$$

Proof. Let us look at the epimorphism of complexes

$$K^{\cdot}(x_1^k, \dots, x_s^k; A) \to C^{\cdot}(s, k, n).$$

It is easily seen that the kernel of this epimorphism is the nth-degree component $K_n^{\cdot}((x_1 t)^k, \dots, (x_s t)^k; \mathbf{R})$ of the (graded) Koszul complex $K^{\cdot}((x_1 t)^k, \dots, (x_s t)^k; \mathbf{R})$, where $\mathbf{R}$ is the Rees algebra associated with $\mathbf{F}$. Hence, we have an exact sequence of complexes of A-modules

$$0 \to K_n^{\cdot}((x_1 t)^k, \dots, (x_s t)^k; \mathbf{R}) \to K^{\cdot}(x_1^k, \dots, x_s^k; A) \to C^{\cdot}(s, k, n) \to 0.$$

The morphisms $K^{\cdot}(x_1^k, \dots, x_s^k; A) \to K^{\cdot}(x_1^{k+1}, \dots, x_s^{k+1}; A)$ obtained above can obviously be restricted to $K_n^{\cdot}((x_1 t)^k, \dots, (x_s t)^k; \mathbf{R})$, hence for all $k \geq 1$ we obtain morphisms of exact sequences

$$\begin{array}{ccccccc} 0 \to & K_n^{\cdot}((x_1 t)^k, \dots, (x_s t)^k; \mathbf{R}) & \to & K^{\cdot}(x_1^k, \dots, x_s^k; A) & \to & C^{\cdot}(s, k, n) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K_n^{\cdot}((x_1 t)^{k+1}, \dots, (x_s t)^{k+1}; \mathbf{R}) & \to & K^{\cdot}(x_1^{k+1}, \dots, x_s^{k+1}; A) & \to & C^{\cdot}(s, k+1, n) & \to 0 \end{array}$$

which provide us with an inductive system of exact sequences of complexes. So we obtain the theorem due to the naturality of the cohomology long exact sequence.

The only thing left to complete the proof is the last claim in (ii). For this, since

$$\left[H^1_{R_+}(R) \right]_n \cong \lim_{\vec{k}} H^0(C^{\cdot}(d, k, n)) = \lim_{\vec{k}} \frac{(I_{n+k} : (x_1^k, \dots, x_d^k))}{I_n},$$

where the morphisms in the inductive system are the natural inclusions, it is enough to show that $(I_{n+k} : (x_1^k, \dots, x_d^k)) = \breve{I}_n$ for large k and $n \geq 0$.

This we get from Proposition 2.3(vii), since $(x_1, \dots, x_d)$ is a reduction of $\mathcal{F}$ and the ideals $(I_{n+k} : (x_1^k, \dots, x_d^k))$ form an increasing chain. On the other side, if $n < 0$ we have $(I_{n+k} : (x_1^k, \dots, x_d^k)) = A$ for large k and $I_n = 0$. ∎

Remark 3.4. The combination of Proposition 2.4 in [Sal93] and Theorem 2.4 in [JV95] shows that if A is two-dimensional Cohen–Macaulay and $\mathcal{F}$ is I-adic then $\lambda(H^1_{\mathcal{R}_+}(\mathcal{R})_n) = \lambda(\widetilde{I^n}/I^n)$ for all $n \geq 0$. We have extended this result to any Hilbert filtration for all $n \in \mathbb{Z}$ and $d \geq 2$ in (ii). Moreover, it is obvious that the last claim holds true with the weaker assumption depth$(A) \geq 2$.

Now consider the extended Rees algebra $\mathcal{R}^*$. It has a natural structure of an $\mathcal{R}$-module given by the natural inclusion $\mathcal{R} \hookrightarrow \mathcal{R}^*$, so we will consider its local cohomology modules with respect to $\mathcal{R}_+$. In the same way as $C^{\cdot}(s, k, n)$ has been constructed, we can construct cohomological complexes $C^{\cdot}(s, k, n)^*$ for all $n \in \mathbb{Z}$, $k \geq 0$, taking the convention that $I_n = A$ for $n < 0$. All the results in this section hold also in this case by changing the convention about the negative pieces. In particular we get:

THEOREM 3.5. *Assume that* $(A, \mathfrak{m})$ *has infinite residue field and* depth$(A) \geq 2$. *Then it holds that*

$$\left[H^1_{\mathcal{R}_+}(\mathcal{R}^*)\right]_n \cong \begin{cases} \breve{I}_n/I_n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

In fact, the isomorphisms of Theorems 3.3 and 3.5 can be refined to isomorphisms of graded $\mathcal{R}$-modules:

THEOREM 3.6. *Let* $(A, \mathfrak{m})$ *be a local ring with infinite residue field and* depth$(A) \geq 2$, *let* $\mathcal{F}$ *be a Hilbert filtration, let* $\mathcal{R} = R(\mathcal{F})$, *and let* $\mathcal{R}^* = R^*(\mathcal{F})$. *Consider in* $\breve{\mathcal{R}}^* = R^*(\breve{\mathcal{F}})$ *the structure of* $\mathcal{R}$-*module given by the natural inclusion. Then there are isomorphisms of graded* $\mathcal{R}$-*modules*

$$H^1_{\mathcal{R}_+}(\mathcal{R}) \cong \breve{\mathcal{R}}^*/\mathcal{R} \quad \text{and} \quad H^1_{\mathcal{R}_+}(\mathcal{R}^*) \cong \breve{\mathcal{R}}^*/\mathcal{R}^*.$$

Proof. Consider the exact sequence of graded $\mathcal{R}$-modules

$$0 \to \mathcal{R} \to \breve{\mathcal{R}}^* \to \breve{\mathcal{R}}^*/\mathcal{R} \to 0.$$

Since $[\breve{\mathcal{R}}^*/\mathcal{R}]_n = 0$ for all $n \gg 0$ we get $H^0_{\mathcal{R}_+}(\breve{\mathcal{R}}^*/\mathcal{R}) = \breve{\mathcal{R}}^*/\mathcal{R}$. Moreover, $H^0_{\mathcal{R}_+}(\breve{\mathcal{R}}^*) = 0$, since I_1 contains a nonzero divisor, and

$$H^1_{\mathcal{R}_+}(\breve{\mathcal{R}}^*) = H^1_{\breve{\mathcal{R}}_+}(\breve{\mathcal{R}}^*) = 0$$

by Theorem 3.5. Hence the first isomorphism follows from the local cohomology long exact sequence. The second one follows similarly from the exact sequence

$$0 \to R^* \to \breve{R}^* \to \breve{R}^*/R^* \to 0. \quad \blacksquare$$

The following two results are the key tools we will use in our computations with the local cohomology of the extended Rees algebra. The first one is a well-known result, and a proof can be found in [Kor95, Proposition A.6]:

THEOREM 3.7. *Let $R = \oplus_{n \geq 0} R_n$ be a graded ring such that $A = R_0$ is Noetherian and R is an A-algebra finitely generated by elements of R_1. Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R-module. Then*:

(i) *Let $r = ara(R_+)$ the minimum integer such that there exists an ideal generated by r elements and having the same radical as R_+. Then $H^i_{R_+}(M) = 0$ for all $i > r$.*

(ii) *For all $i \geq 0$ we have $H^i_{R_+}(M)_n = 0$ for $n \gg 0$.*

(iii) *$H^i_{R_+}(M)_n$ is a finitely generated A-module.*

The next theorem relates the local cohomology groups of the Rees algebra and the extended Rees algebra:

THEOREM 3.8. *Let A be a Noetherian ring, F a good filtration in A. For all $i \geq 2$ there are isomorphisms of graded R-modules*

$$H^i_{R_+}(R) \cong H^i_{R_+}(R^*),$$

and there is an exact sequence of graded R-modules

$$0 \to H^0_{R_+}(R) \to H^0_{R_+}(R^*) \to R^*/R \to H^1_{R_+}(R) \to H^1_{R_+}(R^*) \to 0.$$

In particular

$$\left[H^0_{R_+}(R)\right]_n \cong \left[H^0_{R_+}(R^*)\right]_n \quad \text{and} \quad \left[H^1_{R_+}(R)\right]_n \cong \left[H^1_{R_+}(R^*)\right]_n$$

for all $n \geq 0$.

Proof. Let us consider the exact sequence of graded R-modules

$$0 \to R \to R^* \to R^*/R \to 0.$$

Since $R^*/R = \oplus_{n<0} At^n$ we get $H^0_{R_+}(R^*/R) = R^*/R$ and hence $H^i_{R_+}(R^*/R) = 0$ for all $i > 0$. Then the theorem follows from the local cohomology long exact sequence. $\blacksquare$

4. HILBERT FUNCTIONS

Our purpose in this section is to give a formula for the difference between the Hilbert−Samuel function and polynomial which extends Johnston and Verma's formula [JV95, Theorem 2.4] to $n < 0$ and holds for any Hilbert filtration, namely:

THEOREM 4.1. *Let* $(A, \mathfrak{m})$ *be a local ring, let* $\mathbf{F} = \{I_n\}_{n \geq 0}$ *be a Hilbert filtration, and let* $\mathbf{R}$ (resp. $\mathbf{R}^*$) *be the Rees algebra* (*resp. the extended Rees algebra*) *associated with* $\mathbf{F}$. *Then we have*:

(i) *For all* $i \geq 0$ *it holds that* $\lambda_A(H^i_{\mathbf{R}_+}(\mathbf{R}^*)_n) < +\infty$ *for all* $n \in \mathbb{Z}$.

(ii) *For all* $n \in \mathbb{Z}$ *it holds that*

$$h^1_{\mathbf{F}}(n) - H^1_{\mathbf{F}}(n) = \sum_{i=0}^{d} (-1)^i \lambda_A\Big(H^i_{\mathbf{R}_+}(\mathbf{R}^*)_{n+1}\Big).$$

Proof. Let $R = R(I_1) = \oplus_{n \geq 0} I_1^n t^n$ be the Rees algebra associated with I_1. Since $\mathbf{F}$ is a good filtration, $\mathbf{R} = R(\mathbf{F})$ is a finitely generated R-module, hence by Theorem 3.7 we have that $H^i_{\mathbf{R}_+}(\mathbf{R})_n$ are finitely generated A-modules and they vanish for $n \gg 0$. Furthermore $H^i_{R_+}(\mathbf{R}) = 0$ for all $i > d$. Consider the inclusion of graded rings $R \hookrightarrow \mathbf{R}$: we have $R_+\mathbf{R} = \oplus_{n \geq 1} I_1 I_{n-1} t^n$. Since $\mathbf{F}$ is a good filtration, it holds that $I_1 I_{n-1} = I_n$ for $n \gg 0$, hence $R_+\mathbf{R}$ and $\mathbf{R}_+$ have the same radical. So $H^i_{R_+}(\mathbf{R}) = H^i_{\mathbf{R}_+}(\mathbf{R})$ for all $i \geq 0$. Therefore, by Theorem 3.8, $H^i_{\mathbf{R}_+}(\mathbf{R}^*)_n$ vanishes for $n \gg 0$.

Now let $\mathbf{G} = \oplus_{n \geq 0} I_n / I_{n+1}$ be the associated graded ring of $\mathbf{F}$: we have an exact sequence of graded $\mathbf{R}$-modules

$$0 \to \mathbf{R}^*(1) \xrightarrow{\cdot t^{-1}} \mathbf{R}^* \to \mathbf{G} \to 0.$$

Since $H^i_{\mathbf{R}_+}(\mathbf{G}) = H^i_{\mathbf{G}_+}(\mathbf{G})$ for all $i \geq 0$, the local cohomology long exact sequence gives, for all $n \in \mathbb{Z}$,

$$0 \to H^0_{\mathbf{R}_+}(\mathbf{R}^*)_{n+1} \to H^0_{\mathbf{R}_+}(\mathbf{R}^*)_n \to H^0_{\mathbf{G}_+}(\mathbf{G})_n \to H^1_{\mathbf{R}_+}(\mathbf{R}^*)_{n+1}$$
$$\to \cdots \to H^d_{\mathbf{R}_+}(\mathbf{R}^*)_{n+1} \to H^d_{\mathbf{R}_+}(\mathbf{R}^*)_n \to H^d_{\mathbf{G}_+}(\mathbf{G})_n \to 0. \qquad (*)$$

Let us prove (i) by decreasing induction on n. For $n \gg 0$ it is obvious since $H^i_{\mathbf{R}_+}(\mathbf{R}^*)_n = 0$. Then we have, for all $n \in \mathbb{Z}$,

$$\cdots \to H^i_{\mathbf{R}_+}(\mathbf{R}^*)_{n+1} \to H^i_{\mathbf{R}_+}(\mathbf{R}^*)_n \to H^i_{\mathbf{G}_+}(\mathbf{G})_n \to \cdots,$$

$H^i_{R_+}(R^*)_{n+1}$ has finite length by induction hypothesis, and $H^i_{G_+}(G)_n$ has finite length since G_0 is Artinian. Hence $H^i_{R_+}(R^*)_n$ has finite length.

To prove (ii), define $f(n) = \sum_{i=0}^d (-1)^i \lambda_A(H^i_{R_+}(R^*)_{n+1})$ for all $n \in \mathbb{Z}$. From the exact sequence $(*)$ we get

$$f(n) - f(n-1) = -\sum_{i\geq 0} (-1)^i \lambda_A\big(H^i_{G_+}(G)_n\big).$$

On the other side, G is a Noetherian graded ring with G_0 Artinian. Hence, a proof analogous to that of Lemma 1.3 in [Mar93] shows that

$$H^0_F(n) - h^0_F(n) = \sum_{i\geq 0} (-1)^i \lambda_A\big(H^i_{G_+}(G)_n\big).$$

Let $g(n) = h^1_F(n) - H^1_F(n)$. We have obtained, for all $n \in \mathbb{Z}$,

$$f(n) - f(n-1) = g(n) - g(n-1).$$

Hence, since $f(n) = g(n) = 0$ for $n \gg 0$ we get $f(n) = g(n)$ for all $n \in \mathbb{Z}$. ∎

Remark 4.2. Kirby and Mehran [KM81] consider, for all $m \in \mathbb{Z}$, intermediate Rees modules given by $R^{[m]} = \oplus_{n\geq m} I_n t^n \subseteq A[t, t^{-1}]$, and they express the difference between the Hilbert–Samuel function and polynomial in terms of the Euler characteristic of the graded Koszul complex $K_{.}(x_1^{n_1}t^{n_1}, \dots, x_d^{n_d}t^{n_d}; R^{[m]})$. Namely, in Theorem 2 they show that if $N = n_1 + \cdots + n_d$ then

$$\sum_{i=0}^{d} (-1)^i \lambda_A\Big(H^i\big(K_{n+N}\big(x_1^{n_1}t^{n_1}, \dots, x_d^{n_d}t^{n_d}; R^{[m]}\big)\big)\Big)$$
$$= (-1)^d \big(h^1_F(n-1) - H^1_F(n-1)\big)$$

for all $n \geq m$ and $n_i \geq i_0 - n$, where i_0 is the regularity index of H^1_F. They also show that the individual homology modules are independent of $n_1, \dots, n_d$ provided they are big enough.

The formula proved for an I-adic filtration in [JV95, Theorem 2.4] is

$$h^1_F(n-1) - H^1_F(n-1) = \sum_{i=0}^{d} (-1)^i \lambda_A\big(H^i_{R_+}(R)_n\big)$$

for all $n \geq 0$. Notice that, by the duality between Koszul homology and cohomology, this is Kirby and Mehran's formula taken to the limit in the case $m = 0$. Since Johnston and Verma's formula is analogous to the one involving H^0_F and the associated graded ring, one expects that it should hold for all $n \in \mathbb{Z}$. Anyway, in the Cohen–Macaulay case they find the obstruction for this: it is the fact that $H^1_{R_+}(R)_{-1} \cong A$. By replacing $R(F)$ by $R^*(F)$, that is, making m tend to $-\infty$ in Kirby and Mehran's result, this obstruction has been avoided.

Our aim is now to give an affirmative answer to Question 3.2 of [JV95]. Namely, we want to show that if $\operatorname{depth}(\operatorname{gr}_A(F)) \geq d - 1$, then, for all $n \in \mathbb{Z}$,

$$h^1_F(n) - H^1_F(n) = (-1)^d \lambda_A\Big(H^d_{R_+}(R^*)_{n+1}\Big).$$

This gives a cohomological explanation to the fact that the Hilbert functions of filtrations with $\operatorname{depth}(\operatorname{gr}_A(F)) = d$ and $d - 1$ behave in the same way; see [Mar89], whose results hold also for Hilbert filtrations.

Let $(A, \mathfrak{m})$ be a local ring, let F be a Hilbert filtration in A, let $R = R(F)$, and let $R^* = R^*(F)$. Although R^* is not a finitely generated R-module, it is a Noetherian ring; therefore, by [Mat86, Remark before Theorem 16.6], all the maximal R^*-regular sequences contained in R_+ have the same (and finite) length. Hence we will denote this well-defined integer by $\operatorname{depth}_{R_+}(R^*)$. Again by the remark in [Mat86], the same argument as in the finite case shows that

$$\operatorname{depth}_{R_+}(R^*) = \min\Big\{i \mid H^i_{R_+}(R^*) \neq 0\Big\}.$$

The proposition below, together with Theorems 3.3 and 3.8, generalizes Proposition 2(iv) of [Sal82]:

PROPOSITION 4.3. *Assume that A is Cohen–Macaulay and let $G = \operatorname{gr}_A(F)$. Then it holds that*

(i) *if G is not Cohen–Macaulay then* $\operatorname{depth}_{R_+}(R^*) = \operatorname{depth}(G) + 1$;

(ii) *if G is Cohen–Macaulay then* $\operatorname{depth}_{R_+}(R^*) = d$.

Proof. (i) Let $s = \operatorname{depth}(G) \leq d - 1$ and let us first show that $\operatorname{depth}_{R_+}(R^*) \geq s + 1$ by induction on s.

In the case $s = 0$, since A is Cohen–Macaulay it is clear that $H^0_{R_+}(R^*) = 0$.

In the case $s = 1$ by Proposition 2.5 we get $F = \breve{F}$. Since G is not Cohen–Macaulay we have $d \geq 2$. Hence by Theorem 3.5 we get $H^1_{R_+}(R^*) = \bigoplus_{n \in \mathbb{Z}} \breve{I}_n / I_n = 0$.

Assume $s \geq 2$. Let $x \in I_1$ be such that $x^* \in G$ is a nonzero divisor. This is equivalent to $x \notin z(A)$ and $I_n \cap (x) = xI_{n-1}$ for all $n \in \mathbb{Z}$, or equivalently $(I_n : x) = I_{n-1}$ for all $n \in \mathbb{Z}$. Hence, if $\overline{A} = A/(x)$ and $\overline{F} = F/(x)$, we get that $\overline{G} = \operatorname{gr}_{\overline{A}}(\overline{F}) \cong G/x^*G$ and $\overline{R}^* = R^*(\overline{F}) \cong R^*/xt R^*$. Although $\overline{R} = R(\overline{F}) \ncong R/xt R$ (they differ in the degree-zero piece) we have a homogeneous epimorphism of graded rings $R \to \overline{R}$. Notice that we have a commutative diagram of morphisms of R-modules

$$\begin{array}{ccc} R & \longrightarrow & R^* \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow \overline{R^*} \cong & R^*/xt\,R^*, \end{array}$$

hence the two possible structures of R-module on $\overline{R^*}$ coincide. Since $R_+\overline{R} = \overline{R}_+$ we have $H^i_{R_+}(\overline{R^*}) = H^i_{\overline{R}_+}(\overline{R^*})$ for all $i \geq 0$.

Since depth$(\overline{G}) = s - 1$, by induction hypothesis we get depth$_{\overline{R}_+}(\overline{R^*}) \geq s$. Hence $H^i_{R_+}(\overline{R^*}) = 0$ for all $i \leq s - 1$. Consider the exact sequence of graded R-modules

$$0 \to R^*(-1) \xrightarrow{\cdot xt} R^* \to \overline{R^*} \to 0.$$

The local cohomology long exact sequence gives

$$H^{i-1}_{R_+}(\overline{R^*})_n = 0 \to H^i_{R_+}(R^*)_{n-1} \to H^i_{R_+}(R^*)_n$$

for all $i \leq s$ and all $n \in \mathbb{Z}$. Since $H^i_{R_+}(R^*)_n = 0$ for $n \gg 0$ we get $H^i_{R_+}(R^*) = 0$ for all $i \leq s$, that is, depth$_{R_+}(R^*) \geq s + 1$.

Let us now show that depth$(G) + 1 \geq p =$ depth$_{R_+}(R^*)$. Consider the exact sequence of graded R-modules

$$0 \to R^*(1) \xrightarrow{\cdot t^{-1}} R^* \to G \to 0.$$

Since $H^i_{R_+}(G) = H^i_{G_+}(G)$ for all i, the local cohomology long exact sequence gives, for all $i \leq p - 2$,

$$0 = H^i_{R_+}(R^*) \to H^i_{G_+}(G) \to H^{i+1}_{R_+}(R^*)(1) = 0.$$

Hence $H^i_{G_+}(G) = 0$ for all $i \leq p - 2$, that is, depth$(G) \geq p - 1$.

(ii) In the case where G is Cohen−Macaulay, the above exact sequence gives, for all $i < d$ and $n \in \mathbb{Z}$,

$$H^i_{R_+}(R^*)_{n+1} \to H^i_{R_+}(R^*)_n \to 0 = H^i_{G_+}(G)_n.$$

Since $H^i_{R_+}(R^*)_n = 0$ for all $n \gg 0$ we get $H^i_{R_+}(R^*) = 0$ for all $i < d$. ∎

We can now answer the question of Johnston and Verma:

PROPOSITION 4.4. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring, let $\mathbf{F}$ be a Hilbert filtration, let $\mathbf{R} = R(\mathbf{F})$, and let $s = \operatorname{depth}(\operatorname{gr}_A(\mathbf{F}))$. Then it holds that*

(i) *if* $\min\{s+1, d\} > 1$ *then, for all* $n \in \mathbb{Z}$,

$$h^1_{\mathbf{F}}(n) - H^1_{\mathbf{F}}(n) = \sum_{i=\min\{s+1,d\}}^{d} (-1)^i \lambda_A\left(H^i_{\mathbf{R}_+}(\mathbf{R})_{n+1}\right);$$

(ii) *if* $\min\{s+1, d\} = 1$ *then, for all* $n \geq -1$,

$$h^1_{\mathbf{F}}(n) - H^1_{\mathbf{F}}(n) = \sum_{i=1}^{d} (-1)^i \lambda_A\left(H^i_{\mathbf{R}_+}(\mathbf{R})_{n+1}\right)$$

and the formula cannot be extended to all $n \in \mathbb{Z}$ *since, when A is Cohen–Macaulay,* $\lambda_A(H^1_{\mathbf{R}_+}(\mathbf{R})_{n+1}) = \infty$ *for all* $n < -1$.

Remark 4.5. If $d = 1$ and $\operatorname{gr}_A(\mathbf{F})$ is Cohen–Macaulay we have $h^1_{\mathbf{F}}(n) - H^1_{\mathbf{F}}(n) = -\lambda_A(H^1_{\mathbf{R}_+}(\mathbf{R})_{n+1})$ for all $n \geq -1$ and $H^1_{\mathbf{R}_+}(\mathbf{R})_n$ does not have finite length for all $n \leq -1$: by the exact sequence of Theorem 3.8, since $H^0_{\mathbf{R}_+}(\mathbf{R}^*) = 0$ and $\lambda_A(H^1_{\mathbf{R}_+}(\mathbf{R}^*)_n) < \infty$ for all n, $H^1_{\mathbf{R}_+}(\mathbf{R})_n$ cannot have finite length if $n < 0$. This contradicts the $d = 1$ case of Proposition 3.1 of [JV95], where it should say "for all $n \geq 0$."

We can generalize Proposition 2.4 of [Sal93] to Hilbert filtrations and extend it to all $d \geq 2$ and all $n \in \mathbb{Z}$:

PROPOSITION 4.6. *Let $(A, \mathfrak{m})$ be a local ring such that* $\operatorname{depth}(A) \geq 2$, *and let $\mathbf{F}$ be a Hilbert filtration. Then for all $n \in \mathbb{Z}$ it holds that*

$$\sum_{i=2}^{d} (-1)^i \lambda\left(H^i_{\mathbf{R}_+}(\mathbf{R})_n\right) = h^1_{\mathbf{F}}(n-1) - \lambda\left(A/\breve{I}_n\right).$$

In particular if $d = 2$ and A is Cohen–Macaulay then $\lambda(H^2_{\mathbf{R}_+}(\mathbf{R})_0) = e_2(\mathbf{F})$ *and* $\lambda(H^2_{\mathbf{R}_+}(\mathbf{R})_1) = e_0(\mathbf{F}) - e_1(\mathbf{F}) + e_2(\mathbf{F}) - \lambda(A/\breve{I}_1)$.

Proof. Apply Theorem 4.1 to the filtration $\mathbf{F}$, and then use Theorems 3.5 and 3.8. ∎

LEMMA 4.7. *For all $d \geq 1$ and for all $n \in \mathbb{Z}$,*

$$\lambda\left(H^d_{\mathbf{R}_+}(\mathbf{R})_n\right) \leq \lambda\left(H^d_{\mathbf{R}_+}(\mathbf{R})_{n-1}\right).$$

Proof. It follows from the exact sequences:

$$0 \to \mathcal{R}(-1) \overset{\cdot xt}{\to} \mathcal{R} \to \mathcal{R}/xt\,\mathcal{R} \to 0,$$

$$0 \to K \to \mathcal{R}/xt\,\mathcal{R} \to \overline{\mathcal{R}} \to 0,$$

where x is a superficial element of $\mathcal{F}$. ∎

In particular we get the following facts in dimensions 1 and 2, which generalize results obtained by Sally for the maximal ideal [Sal82, Corollary to Proposition 4, (ii)]:

PROPOSITION 4.8. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $d = 2$, and let $\mathcal{F}$ be a Hilbert filtration. Then it holds that:*

(i) $e_0 - e_1 = \lambda(A/I_1)$ *if and only if* $e_2 = 0$ *and* $\breve{I}_1 = I_1$. *In particular, if I is an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay two-dimensional ring, such that $e_2 = 0$, then $\mathrm{gr}_{\breve{I}}(A)$ is Cohen–Macaulay.*

(ii) $h^0_{\mathcal{F}}(n) \le \lambda(\breve{I}_n/\breve{I}_{n+1})$ *for all* $n \in \mathbb{Z}$.

(iii) *Let* $d = \dim(A) \le 2$ *and* $\mathcal{R} = R(\mathcal{F})$. *Then* $r(\mathcal{F}) = 0$ *if and only if* $[H^d_{\mathcal{R}_+}(\mathcal{R})]_{1-d} = 0$.

Proof. (i) Assume $e_0 - e_1 = \lambda(A/I_1)$. Then by the Huneke–Ooishi theorem (see [HM95, Corollary 4.9]) we get that $\mathrm{gr}_{\mathcal{F}}(A)$ is Cohen–Macaulay and $r(\mathcal{F}) \le 1$. In particular, $\breve{I}_1 = I_1$, and also $e_2 = 0$. Conversely if $e_2 = 0$ by Proposition 4.6 we get $H^2_{\mathcal{R}_+}(\mathcal{R})_0 = 0$. Then, by Remark 4.7, $H^2_{\mathcal{R}_+}(\mathcal{R})_1 = 0$, hence, again by Proposition 4.6, $e_0 - e_1 = \lambda(A/\breve{I}_1) = \lambda(A/I_1)$.

(ii) By Lemma 4.7 we have, for all $n \in \mathbb{Z}$,

$$\begin{aligned} 0 &\le \lambda\left(H^2_{\mathcal{R}_+}(\mathcal{R})_n\right) - \lambda\left(H^2_{\mathcal{R}_+}(\mathcal{R})_{n+1}\right) \\ &= h^1_{\mathcal{F}}(n-1) - \lambda\left(A/\breve{I}_n\right) - h^1_{\mathcal{F}}(n) + \lambda\left(A/\breve{I}_{n+1}\right) \\ &= \lambda\left(\breve{I}_n/\breve{I}_{n+1}\right) - h^0_{\mathcal{F}}(n). \end{aligned}$$

(iii) By Proposition 2.7 it is enough to show that $e_1 = \cdots = e_d = 0$. Since $e_1 = \lambda_A(H^1_{\mathcal{R}_+}(\mathcal{R})_0)$ in the one-dimensional case and $e_2 = \lambda_A(H^2_{\mathcal{R}_+}(\mathcal{R})_0)$ and $e_1 + e_2 = \lambda_A(H^2_{\mathcal{R}_+}(\mathcal{R})_{-1})$ in the two-dimensional case, we are done. ∎

Note added in proof. After submitting this paper, the author learned about a recent preprint by L. T. Hoa, where he also introduces the Ratliff-Rush closure for good filtrations.

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