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Symmetries of Lagrangian fibrations

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Abstract

We construct fiber-preserving anti-symplectic involutions for a large class of symplectic manifolds with Lagrangian torus fibrations. In particular, we treat the K3 surface and the six-dimensional examples constructed by Castaño-Bernard and Matessi (2009) [8], which include a six-dimensional symplectic manifold homeomorphic to the quintic threefold. We interpret our results as corroboration of the view that in homological mirror symmetry, an anti-symplectic involution is the mirror of duality. In the same setting, we construct fiber-preserving symplectomorphisms that can be interpreted as the mirror to twisting by a holomorphic line bundle.

Keywords: Symplectic manifolds; Calabi–Yau manifolds; Lagrangian fibrations; Homological mirror symmetry

Contents

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1. Introduction

1.1. Statement of result

Let X be a symplectic manifold and let B be smooth manifold of half the dimension of X. We call a continuous map $f : X \to B$ a *Lagrangian fibration* if each fiber of f contains a relatively open dense set that is a smooth Lagrangian submanifold of X. Lagrangian fibrations arose classically in the context of integrable systems and toric geometry. More recently, Lagrangian fibrations have played a role in the conjectural interpretation of mirror symmetry introduced by Strominger, Yau and Zaslow [33]. We discuss this in greater detail in Section 1.2.

In [8], the first two authors introduced a general construction that produced a class $\mathcal{C}$ of Lagrangian fibrations. See Section 2 for the precise definition of $\mathcal{C}$. In short, $\mathcal{C}$ consists of fibrations on symplectic manifolds of dimensions 4 and 6. In those dimensions, $\mathcal{C}$ includes fibrations such that the total space is homeomorphic to any of the Calabi–Yau complete intersections in toric manifolds considered by Batyrev and Borisov [3] as candidates for mirror symmetry. Each fibration in $\mathcal{C}$ has a Lagrangian section.

An anti-symplectomorphism of a symplectic manifold X with symplectic form ω is a self-diffeomorphism ϕ of X such that

$$\phi^*\omega = -\omega.$$

An anti-symplectomorphism ϕ of X such that $\phi^2 = \mathrm{Id}_X$ is called an anti-symplectic involution.

In the present paper, we define a class of Lagrangian sections $\mathfrak{C}$ for fibrations $f : X \to B$ of class $\mathcal{C}$. For each fibration $f \in \mathcal{C}$, there exists at least one section $\sigma \in \mathfrak{C}$. Our main results are the following.

Theorem 1.1. *Let $f : X \to B$ be a Lagrangian fibration of class $\mathcal{C}$. Let σ be a Lagrangian section of f of class $\mathfrak{C}$. There exists a unique anti-symplectic involution $\phi_{f,\sigma}$ of X such that*

$$f \circ \phi_{f,\sigma} = f, \qquad \phi_{f,\sigma} \circ \sigma = \sigma. \tag{1}$$

That is, there exists a unique anti-symplectic involution $\phi_{f,\sigma}$ of X preserving the fibers of f and fixing the section σ. Assuming existence, uniqueness continues to hold for an arbitrary Lagrangian section σ.

Theorem 1.2. *Let $f : X \to B$ be a Lagrangian fibration of class $\mathcal{C}$ and let σ_0, σ_1, be two Lagrangian sections of class $\mathfrak{C}$. There exists a unique symplectomorphism $t : X \to X$ satisfying*

$$f \circ t = f, \qquad t \circ \sigma_0 = \sigma_1. \tag{2}$$

Assuming existence, uniqueness continues to hold for arbitrary Lagrangian sections of f.

There are several motivations for proving Theorem 1.1. Recently, there has been considerable research devoted to defining Gromov–Witten type invariants for symplectic manifolds equipped with an anti-symplectic involution [35,36,31]. Moreover, in the presence of an anti-symplectic involution, an open-string mirror correspondence was found [27]. Most known examples of anti-symplectic involutions come from real algebraic geometry. Theorem 1.1 constructs a vast number of examples of symplectic manifolds with anti-symplectic involutions in a purely symplectic way. In Section 1.2, we give a conjectural mirror symmetry interpretation of Theorem 1.1 that explains conditions (1). In [32, Theorem 1.1] is applied to show unobstructedness and calculate Lagrangian Floer cohomology for smooth fibers of f.

Theorem 1.2 is important in the proof that Theorem 1.1 holds for any section $\sigma \in \mathfrak{C}$. Moreover, as we explain in Section 1.2, Theorem 1.2 has a mirror symmetry interpretation of independent interest. In Section 1.4, Theorem 1.2 allows us to construct anti-symplectic involutions that do not fix a section of a fibration.

1.2. Mirror symmetry of symmetries

We briefly review some aspects of mirror symmetry necessary to put Theorems 1.1 and 1.2 in context. We discuss two conjectures and the evidence in their favor.

1.2.1. The Hodge diamond

A Calabi–Yau manifold is a Kähler manifold with trivial canonical bundle. Mirror symmetry predicts that there exist pairs of Calabi–Yau manifolds (X, Y) such that symplectic geometry on Y mirrors complex geometry on Y. Set $n = \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$. A concrete prediction of mirror symmetry is that

$$H^q\big(X, \Omega_X^p\big) \simeq H^q\big(Y, \Omega_Y^{n-p}\big). \tag{3}$$

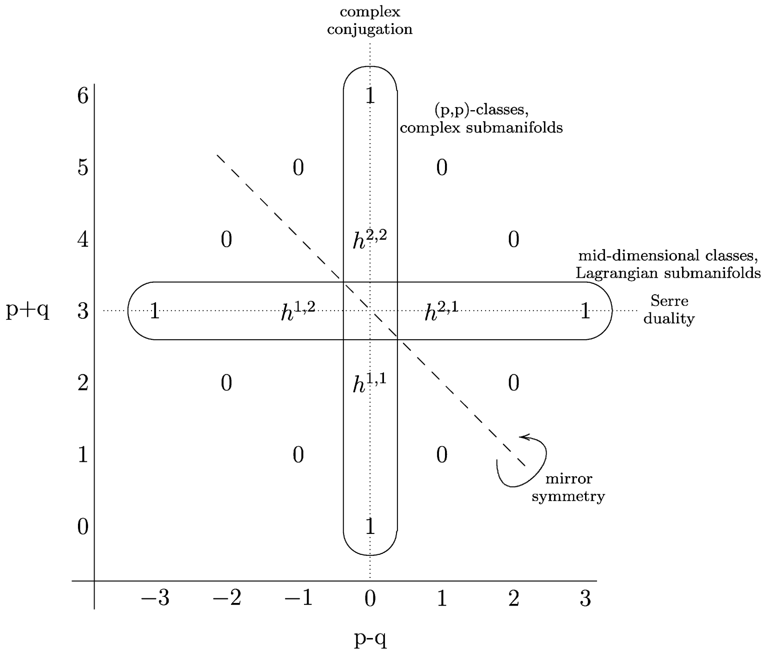

Fig. 1. The Hodge diamond.

Namely, the Hodge diamond of Y is the reflection of the Hodge diamond of X about a diagonal. See Fig. 1, which illustrates the case when $n = 3$ and $\pi_1(X) = \{1\}$. We use the notation $h^{p,q} = \dim H^q(X, \Omega_X^p)$.

It follows from the isomorphism (3) that the first order deformations of the Kähler class of X, which are classified by $H^1(X, \Omega_X^1)$, are isomorphic to the first order deformations of the complex structure on Y, which are classified by $H^1(Y, \Omega_Y^{n-1})$. The middle-dimensional cohomology of X, which contains the Poincaré duals of Lagrangian submanifolds, is isomorphic to the (p, p) classes of Y, which contain the Poincaré duals of complex submanifolds.

Recall that the Hodge diamond of any Kähler manifold has two symmetries: Serre duality and complex conjugation. Naturally, the mirror isomorphism (3) preserves these symmetries of the Hodge diamond. It is interesting to note, however, that mirror symmetry exchanges complex conjugation and Serre duality. In the remainder of this section, we trace the exchange of symmetries through successively more refined descriptions of mirror symmetry.

1.2.2. Homological mirror symmetry

Homological mirror symmetry [23] can be seen as a categorification of mirror symmetry on the level of Hodge diamonds. Mirror symmetry on the level of Hodge diamonds implies an isomorphism of vector spaces between the middle-dimensional cohomology of X and the (p, p)-classes of Y:

$$\bigoplus_{p+q=n} H^q\big(X, \Omega_X^p\big) \xrightarrow{\sim} \bigoplus_p H^p\big(Y, \Omega_Y^p\big).$$

Homological mirror symmetry replaces each vector space with a category, and asserts an equivalence of categories.

Let X be a symplectic manifold with $\dim_{\mathbb{R}}(X) = 2n$. We would like to replace $H^n(X)$ with a category. Since Lagrangian submanifolds of X have half the dimension of X, it is natural to look for a category with objects Lagrangian submanifolds. In fact, from X we can construct the A^∞ category $Fuk(X)$ [16,17,13,29]. An object of $Fuk(X)$, is a Lagrangian submanifold $L \subset X$ equipped with a unitary local system $E \to L$, a grading θ (see Section 7) and a *Pin* structure $\mathfrak{p}$. Depending on context, we may omit several of the data comprising an object of $Fuk(X)$ from our notation when it does not cause confusion. Morphisms between two objects (L_1, E_1) and (L_2, E_2) are given by the Floer cohomology groups with local coefficients $HF^*((L_1, E_1), (L_2, E_2))$. From $Fuk(X)$, one can construct a triangulated category $\mathcal{D}^b\,Fuk(X)$, as explained in [23]. In general, it seems necessary to enlarge $\mathcal{D}^b\,Fuk(X)$ further [23]. We denote the enlargement as well by $\mathcal{D}^b\,Fuk(X)$.

On the other hand, let Y be a Kähler manifold. To Y, we can associate the triangulated category $\mathcal{D}^b\,Coh(Y)$, the derived category of coherent sheaves on Y. Perhaps the simplest objects of $\mathcal{D}^b\,Coh(Y)$, are the structure sheaves of complex submanifolds. The Poincaré duals of complex submanifolds all belong to $\bigoplus_p H^p(Y, \Omega^p)$. According to the Hodge conjecture, the Poincaré duals of complex submanifolds should generate all rational (p, p)-classes. Thus, it makes sense to replace the vector space $\bigoplus_p H^p(Y, \Omega^p_Y)$ with the category $\mathcal{D}^b\,Coh(Y)$.

Suppose X and Y are Calabi–Yau manifolds, i.e. Kahler manifolds with trivial canonical bundle. Homological mirror symmetry [23] predicts that for certain pairs (X, Y) there exists an equivalence of triangulated categories

$$m : \mathcal{D}^b\,Fuk(X) \xrightarrow{\sim} \mathcal{D}^b\,Coh(Y).$$

Such pairs are called mirror pairs and Y is called a mirror of X.

In homological mirror symmetry, symmetries of a vector space should be replaced with auto-equivalences of a category. As for any smooth algebraic variety, the functor

$$\mathcal{D} := R\underline{Hom}(-, \mathcal{O}_Y) : \mathcal{D}^b\,Coh(Y) \xrightarrow{\sim} \mathcal{D}^b\,Coh(Y)^{op} \tag{4}$$

induces an equivalence of categories, and

$$\mathcal{D}^{op} \circ \mathcal{D} \simeq \mathrm{Id}. \tag{5}$$

The auto-equivalence $\mathcal{D}$ is closely related to the Serre duality symmetry of the Hodge diamond. It is natural to ask whether the functor mirror to $\mathcal{D}$,

$$\mathcal{D}^\vee := \left(m^{op}\right)^{-1} \circ \mathcal{D} \circ m : \mathcal{D}^b\,Fuk(X) \xrightarrow{\sim} \mathcal{D}^b\,Fuk(X)^{op},$$

is quasi-isomorphic to a geometrically defined functor. One goal of this paper is to construct a geometric functor

$$\mathcal{I} : \mathcal{D}^b\,Fuk(X) \xrightarrow{\sim} \mathcal{D}^b Fuk(X)^{op}$$

for which it is reasonable to conjecture that

$$\mathcal{I} \simeq \mathcal{D}.$$

However, before proceeding further, we pause to summarize what is conjectured about the geometry of m.

1.2.3. The SYZ conjecture

The Strominger–Yau–Zaslow (SYZ) conjecture [33] takes a first step toward giving a geometric interpretation of the homological mirror symmetry functor m. For each point $y \in Y$, let $\mathcal{O}_y$ denote the skyscraper sheaf at y. According to [33], the functor m^{-1} should carry $\mathcal{O}_y$ to a Lagrangian torus $L_y \subset X$ equipped with a flat unitary line bundle $E_y \to L_y$. One motivation for this conjecture was that $RHom(\mathcal{O}_y, \mathcal{O}_y) \simeq \Lambda^*(T_y Y)$, while it is reasonable to conjecture that

$$HF^*\big((L_y, E_y), (L_y, E_y)\big) \simeq H^*(L_y) \simeq \Lambda^*\big(H^1(L_y)\big).$$

Indeed, the first isomorphism would follow if the spectral sequence computing Floer cohomology [16] degenerates at the E_2 term.

Furthermore [33], the family of tori L_y should completely fill X and lead to a Lagrangian fibration $f : X \to B$, where B is a three-dimensional manifold. This fibration may have singular fibers. Conversely, given a Lagrangian fibration $f : X \to B$, it should be possible to construct a mirror Y_f, as the moduli space of pairs (L, E), where L is a fiber of f and $E \to L$ is a flat unitary line bundle. The Lagrangian fibrations constructed in [8] provide concrete examples in which the SYZ conjecture can be tested.

The SYZ conjecture is sharpened by Fukaya in [14]. From now on let $f : X \to B$ denote a Lagrangian fibration and let Y_f be the corresponding mirror. Let $\sigma : B \to X$ be a Lagrangian section of f. It is suggested in [14] that the choice of f should play an important role in an intrinsic construction of the equivalence of categories $\mathcal{D}^b\, Fuk(X) \xrightarrow{\sim} \mathcal{D}^b\, Coh(Y_f)$. Indeed, given a coherent sheaf $\mathcal{F}$ on Y_f and $y \in Y_f$, let $\mathcal{F}_y$ denote the fiber of $\mathcal{F}$ at y and let $\mathcal{O}_y$ denote the skyscraper sheaf at y. We have canonically,

$$\mathcal{F}_y^\vee \simeq Hom(\mathcal{F}, \mathcal{O}_y). \tag{6}$$

Assume the object (L, E) of $Fuk(X)$ is mirror to $\mathcal{F}$, and let (L_y, E_y) be the fiber of f mirror to $\mathcal{O}_y$. It follows from homological mirror symmetry and isomorphism (6) that we must have an isomorphism

$$\mathcal{F}_y^\vee \simeq HF^0\big((L, E), (L_y, E_y)\big).$$

That is, we may calculate the fibers of the sheaf $\mathcal{F}$ on Y_f only knowing its mirror L. A family version of Floer homology should piece the fibers of $\mathcal{F}$ together to give $\mathcal{F}$ itself. In [15], the choice of a Lagrangian section σ plays an important role in piecing together the fibers. The mirror functor sends the Lagrangian σ to the structure sheaf of Y_f.

In summary, the mirror functor depends on the choices of a Lagrangian fibration f and a Lagrangian section σ. From now on, we include f and σ in our notation for the mirror functor

$$m_{f,\sigma} : \mathcal{D}^b\, Fuk(X) \xrightarrow{\sim} \mathcal{D}^b\, Coh(Y_f).$$

Similarly, we write

$$\mathcal{D}^{\vee}_{f,\sigma} = \left(m^{op}_{f,\sigma}\right)^{-1} \circ \mathcal{D} \circ m_{f,\sigma}.$$

1.2.4. Duality conjecture

Suppose $f : X \to B$ is a Lagrangian fibration of class $\mathcal{C}$. Let $\phi_{f,\sigma}$ be the involution of X given by Theorem 1.1. Let $(L, E, \theta, \mathfrak{p})$ be an object of $\mathcal{D}^b\, Fuk(X)$. We define

$$\mathcal{I}_{f,\sigma}((L, E, \theta, \mathfrak{p})) = \left(\phi_{f,\sigma}(L), \phi_{f,\sigma *}E^{\vee}, -\theta \circ \phi^{-1}_{f,\sigma}, \phi_{f,\sigma *}\mathfrak{p}\right). \tag{7}$$

A short calculation shows that formula (7) defines a functor

$$\mathcal{I}_{f,\sigma} : \mathcal{D}^b\, Fuk(X) \to \mathcal{D}^b\, Fuk(X)^{op}.$$

See [25]. The reversal of morphisms results from the fact that $\phi_{f,\sigma}$ changes the sign of the symplectic form. For signs, see [32]. The following conjecture has appeared in various forms throughout the mirror-symmetry literature. See for example Arinkin and Polishchuk [1] in the case of the elliptic curve and Nadler [25] in the case of the cotangent bundle. In both cases, the conjecture is a theorem.

Conjecture 1.3. *The geometrically defined functor $\mathcal{I}_{f,\sigma}$ is quasi-isomorphic to $\mathcal{D}^{\vee}_{f,\sigma}$.*

We briefly present some evidence for Conjecture 1.3. First of all, it is clear from the definition that $\mathcal{I}^{op}_{f,\sigma} \circ \mathcal{I}_{f,\sigma} \simeq \mathrm{Id}$ by analogy with Eq. (5). Furthermore, just as $\mathcal{D}$ preserves the structure sheaf of Y_f, so too $\mathcal{I}_{f,\sigma}$ preserves the Lagrangian section σ equipped with the trivial local system. In Section 7, we explain the appropriate choice of grading for σ.

A distinctive property of $\mathcal{D}$ is that it preserves $\mathcal{O}_y$ up to a shift in grading by $\dim_{\mathbb{C}} Y$. It follows that a geometric functor $\mathcal{I}_{f,\sigma}$ isomorphic to $D^{\vee}_{f,\sigma}$ should preserve (L_y, E_y) up to a shift in grading by $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$. Indeed, Theorem 1.1 guarantees that for a fiber L_y of f, we have $\phi_{f,\sigma}(L_y) = L_y$. Suppose L_y is smooth and hence a torus. By Corollary 1.8, $\phi_{f,\sigma}$ acts on L_y by the inverse map of the torus group. So, if E_y is a flat unitary line bundle then $\phi_{f,\sigma *}E_y = E^{\vee}_y$. It follows that $\phi_{f,\sigma *}E^{\vee}_y = E_y$. In Section 7 we verify under some reasonable assumptions that for the natural choice of grading θ_y on a torus fiber L_y, $\mathcal{I}_{f,\sigma}$ shifts θ_y by $\dim_{\mathbb{C}} X$. Thus $\mathcal{I}_{f,\sigma}$ preserves the mirror of $\mathcal{O}_y$ up to a shift by $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$.

Finally, we state a theorem concerning the derived category of coherent sheaves that mirrors the uniqueness claim of Theorem 1.1. The proof appears in Section 8. In the following, $\mathcal{D}$ is the functor defined in (4).

Theorem 1.4. *Let Y be a smooth projective variety of dimension n. Let $\mathcal{D}' : \mathcal{D}^b\, Coh(Y) \to \mathcal{D}^b\, Coh(Y)^{op}$ denote an equivalence of categories such that*

$$\mathcal{D}'(\mathcal{O}_y) \simeq \mathcal{O}_y[n],\ \forall y \in Y, \qquad \mathcal{D}'(\mathcal{O}_Y) = \mathcal{O}_Y.$$

Then $\mathcal{D}' \simeq \mathcal{D}$.

1.2.5. Twist conjecture

Let $\mathcal{L} \to Y_f$ denote a holomorphic line bundle over Y_f. Tensoring with $\mathcal{L}$ defines an auto-equivalence

$$\mathcal{T} : \mathcal{D}^b\, Coh(Y) \xrightarrow{\sim} \mathcal{D}^b\, Coh(Y).$$

As before, we define the mirror auto-equivalence $\mathcal{T}^\vee$ by

$$\mathcal{T}^\vee = (m_{f,\sigma})^{-1} \circ \mathcal{T} \circ m_{f,\sigma} : \mathcal{D}^b\, Fuk(X) \xrightarrow{\sim} \mathcal{D}^b\, Fuk(X).$$

Let $(\sigma_\mathcal{L}, E_\mathcal{L})$ denote the Lagrangian submanifold with unitary local system mirror to $\mathcal{L}$ by the mirror isomorphism $m_{f,\sigma}$.

We assume that $\sigma_\mathcal{L}$ is a Lagrangian section of $f : X \to B$ and $E_\mathcal{L}$ has rank one. Furthermore, we assume that $\sigma_\mathcal{L} \in \mathfrak{C}$. By Theorem 1.2, there exists a unique symplectomorphism $t : X \to X$ such that $f \circ t = f$ and $t \circ \sigma = \sigma_\mathcal{L}$. Let $\widehat{E}_\mathcal{L} = f^* f_* E_\mathcal{L}$, which is a flat unitary line bundle on X since $\sigma_\mathcal{L}$ is a section. Define an auto-equivalence $\mathfrak{t}$ of $\mathcal{D}^b\, Fuk(X)$ by

$$\mathfrak{t}((L, E, \theta, \mathfrak{p})) = \big(t(L), (t_* E) \otimes \widehat{E}_\mathcal{L}, \theta \circ t^{-1}, t_* \mathfrak{p}\big).$$

We make the following conjecture.

Conjecture 1.5. *The geometrically defined functor* $\mathfrak{t}$ *is quasi-isomorphic to* $\mathcal{T}^\vee$.

Previously, Kontsevich described the functor $\mathfrak{t}$ in terms of monodromy transformations arising from complex structure moduli of X. See [22,30]. Given the right Lagrangian section $\sigma_\mathcal{L}$, the functor $\mathfrak{t}$ can be used to construct the homogeneous coordinate ring of Y as described in [37].

We briefly present some evidence in favor of Conjecture 1.5. Let $E_0 \to \sigma$ denote the trivial rank-1 unitary local system. Just as

$$\mathcal{T}(\mathcal{O}_Y) = \mathcal{L},$$

so too, since $t(\sigma) = \sigma_\mathcal{L}$ and $\widehat{E}_\mathcal{L}|_{\sigma_\mathcal{L}} = E_\mathcal{L}$, we have

$$\mathfrak{t}((\sigma, E_0)) = (\sigma_\mathcal{L}, E_\mathcal{L}).$$

Also, just as

$$\mathcal{T}(\mathcal{O}_y) \simeq \mathcal{O}_y, \quad \forall y \in Y,$$

so too, since $t(L_y) = L_y$ and $\widehat{E}_\mathcal{L}|_{L_y}$ is trivial, we have

$$\mathfrak{t}((L_y, E_y)) \simeq (L_y, E_y).$$

Finally, we state a theorem for the derived category of coherent sheaves that mirrors the uniqueness claim of Theorem 1.2. The proof appears in Section 8.

Theorem 1.6. *Let Y be a smooth projective variety. Let $\mathcal{T}'$ be an auto-equivalence of $\mathcal{D}^b\, Coh(Y)$ such that*

$$\mathcal{T}'(\mathcal{O}_y) \simeq \mathcal{O}_y, \quad \forall y \in Y, \qquad \mathcal{T}'(\mathcal{O}_Y) \simeq \mathcal{L}.$$

Then $\mathcal{T}' \simeq \mathcal{T}$.

Finally, here is a justification of our assumption that $\sigma_{\mathcal{L}}$ is a section and $E_{\mathcal{L}}$ has rank 1. By homological mirror symmetry, we have

$$HF^*\big((\sigma_{\mathcal{L}}, E_{\mathcal{L}}), (L_y, E_y)\big) \simeq RHom(\mathcal{L}, \mathcal{O}_y) \simeq \mathcal{L}_y^{\vee}.$$

Since the fiber $\mathcal{L}_y$ is one-dimensional, we conclude that $HF^*((\sigma_{\mathcal{L}}, E_{\mathcal{L}}), (L_y, E_y))$ is one-dimensional. The Floer complex is a direct sum of tensor products of the fibers of the local coefficient systems at intersection points of the Lagrangian submanifolds. So, it is natural to assume that σ intersects each fiber L_y at one point and $\operatorname{rk} E_{\mathcal{L}} = 1$.

1.3. Idea of proof

We briefly explain the idea of our proofs of Theorems 1.1 and 1.2. First suppose $f_0 : X_0 \to B_0$ is a Lagrangian fibration that is a smooth submersion. Let σ_0 be a smooth Lagrangian section.

Recall that the cotangent bundle T^*B_0 has a canonical symplectic form. Let Z denote the zero section of T^*B_0. Let $\pi : T^*B_0 \to B_0$ denote the canonical projection. We also use π to denote the induced projection of quotients of T^*B_0.

Proposition 1.7. *(See [12].) There exists a unique lattice bundle $\Lambda_0 \subset T^*B_0$ and a unique symplectomorphism*

$$\Theta : T^*B_0/\Lambda_0 \to X_0$$

such that

$$\Theta \circ Z = \sigma_0, \qquad f_0 \circ \Theta = \pi. \tag{8}$$

Sketch of proof. Let $b \in B_0$ and let ξ be a cotangent vector to B_0 at b. Choose a function h on B_0 such that $dh|_b = \xi$. Define $H = h \circ f_0$. Let Φ_H be the time-one map of the Hamiltonian flow of H. Define $\widetilde{\Theta} : T^*B_0 \to X_0$ by

$$\widetilde{\Theta}(b, \xi) = \Phi_H \circ \sigma_0(b).$$

Define $\Lambda_0 = \widetilde{\Theta}^{-1}(\sigma_0)$. It is not hard to check that $\widetilde{\Theta}$ descends to define a map Θ on the quotient T^*B_0/Λ_0 with the required properties.

To check uniqueness, assume $\Lambda_0' \subset T^*B_0$ is a lattice bundle and

$$\Theta' : T^*B_0/\Lambda_0' \to X_0$$

is a symplectomorphism such that $\Theta' \circ Z = \sigma_0$ and $f_0 \circ \Theta' = \pi$. Let $\widetilde{\Theta}'$ denote the composition of Θ' with the quotient map $T^*B_0 \to T^*B_0/\Lambda_0'$. Let b, ξ and h, be as above. Define a function K on T^*B_0 by $K = h \circ \pi$. Let Φ_K be the time-one map of the Hamiltonian flow of K. Since

$$H \circ \widetilde{\Theta}' = h \circ f_0 \circ \widetilde{\Theta}' = h \circ \pi = K,$$

and $\widetilde{\Theta}'$ is a symplectomorphism, we conclude that

$$\widetilde{\Theta}' \circ \Phi_K = \Phi_H \circ \widetilde{\Theta}'. \tag{9}$$

An explicit calculation shows that

$$\Phi_K(b, 0) = (b, \xi). \tag{10}$$

Using Eqs. (9) and (10), we conclude

$$\widetilde{\Theta}'(b, \xi) = \widetilde{\Theta}' \circ \Phi_K \circ Z(b) = \Phi_H \circ \widetilde{\Theta}' \circ Z(b) = \Phi_H \circ \sigma_0(b) = \widetilde{\Theta}(b, \xi).$$

Thus $\widetilde{\Theta}' = \widetilde{\Theta}$. It follows immediately that $\Lambda_0' = \Lambda_0$. $\square$

Corollary 1.8. *There exists a unique anti-symplectic involution ϕ_0 of X_0 such that*

$$f_0 \circ \phi_0 = f_0, \qquad \phi_0 \circ \sigma_0 = \sigma_0. \tag{11}$$

Proof. Let $-\mathrm{Id}$ denote the anti-symplectic involution of T^*B_0 given by negative the identity transformation on fibers. We also use $-\mathrm{Id}$ to denote the induced involution of quotients of T^*B_0. Let Θ denote the symplectomorphism constructed in Proposition 1.7. The diffeomorphism ϕ_0 of X_0 defined by

$$\phi_0 = \Theta \circ (-\mathrm{Id}) \circ \Theta^{-1} \tag{12}$$

is an anti-symplectic involution satisfying conditions (11). We have proved existence.

To prove uniqueness, let ϕ_0' denote any anti-symplectic involution satisfying conditions (11). It follows that

$$\phi_0' \circ \Theta \circ (-\mathrm{Id}) : T^*B_0/\Lambda_0 \to X_0$$

is a symplectomorphism satisfying conditions (8). By the uniqueness claim of Proposition 1.7, we conclude $\phi_0' \circ \Theta \circ (-\mathrm{Id}) = \Theta$, and consequently $\phi_0' = \phi_0$. $\square$

Corollary 1.9. *Let σ_0 and σ_0' be two Lagrangian sections of f_0. There exists a unique symplectomorphism $t_0 : X_0 \to X_0$ such that*

$$f_0 \circ t_0 = f_0, \qquad t_0 \circ \sigma_0 = \sigma_0'. \tag{13}$$

Proof. Let Θ (resp. Θ') denote the symplectomorphism given by applying Proposition 1.7 to σ_0 (resp. σ_0'). Clearly,

$$t_0 = \Theta' \circ \Theta^{-1} : X_0 \to X_0$$

is a symplectomorphism satisfying conditions (13).

To prove uniqueness, let $t_0' : X_0 \to X_0$ be any symplectomorphism satisfying conditions (13). Then

$$t_0' \circ \Theta : T^*B_0/\Lambda_0 \to X_0$$

satisfies conditions (8) for section σ'. It follows that $t_0' \circ \Theta = \Theta'$, and consequently $t_0' = t_0$. $\square$

With this background, we outline the proofs of Theorems 1.1 and 1.2. Suppose $f : X \to B$ is a Lagrangian fibration of class $\mathcal{C}$. The construction of f given in [8] realizes X as the compactification of an open dense submanifold $X_0 \subset X$ such that $f_0 := f|_{X_0}$ is a Lagrangian fibration that is a smooth submersion with a Lagrangian section σ_0. To obtain X from X_0, local models of singular fibrations are glued onto X_0 matching up σ_0 to sections of the local models. The main technical part of this paper is devoted to constructing a fiber preserving anti-symplectic involution fixing a section on each local model. By the denseness of X_0 in X, Corollary 1.8 guarantees that all local involutions piece together to form the global involution ϕ of Theorem 1.1. Uniqueness of ϕ follows similarly. The same approach together with Corollary 1.9 is used to construct and prove the uniqueness of t as in Theorem 1.2.

1.4. Involutions that do not fix a section

A simple example of Calabi–Yau manifold with an anti-symplectic involution is the Fermat quintic,

$$Q = \left\{ (z_0, \dots, z_4) \in \mathbb{C}P^4 \;\middle|\; \sum_{i=0}^{4} z_i^5 = 0 \right\},$$

with the anti-symplectic involution ϕ_Q induced by complex conjugation. It is easy to see that the fixed locus of ϕ_Q is connected. In fact, it is homeomorphic to $\mathbb{R}P^3$.

On the other hand, the fixed locus of any involution constructed by Theorem 1.1 cannot be connected. In particular, any anti-symplectic involution fixing a section cannot have connected fixed locus for the following reason. By the proof of Corollary 1.8, if $\dim_{\mathbb{R}} X = 2n$, the intersection of the fixed locus with any smooth fiber is 2^n points. The section is clearly a component of the fixed locus, but it intersects each fiber in only one point, thus it cannot be the whole fixed locus.

In Remark 1.12 below, we show how to construct anti-symplectic involutions that do not fix a Lagrangian section.

The proof of the following proposition appears in Section 6.

Proposition 1.10. *Let $f : X \to B$ be of class $\mathcal{C}$. If $\phi_f : X \to X$ is an anti-symplectomorphism satisfying $f \circ \phi_f = f$ then*

$$\phi_f^2 = \mathrm{Id}_X. \tag{14}$$

In other words, all fiber-preserving anti-symplectomorphisms are involutions. Furthermore, if $t : X \to X$ is a symplectomorphism such that $f \circ t = f$ then the following equation holds

$$\phi_f \circ t^{-1} = t \circ \phi_f. \tag{15}$$

Remark 1.11. Let $f : X \to B$ be of class $\mathcal{C}$ and $\sigma_0 \in \mathfrak{C}$. Let σ_1 be an arbitrary Lagrangian section such that there exists a symplectomorphism $t : X \to X$ satisfying conditions (2). Then

$$\phi_{f,\sigma_1} := t \circ \phi_{f,\sigma_0} \circ t^{-1}$$

is an anti-symplectic involution satisfying conditions (1). Moreover, by Proposition 1.10, we have

$$\phi_{f,\sigma_1} = t^2 \circ \phi_{f,\sigma_0}.$$

Remark 1.12. Let $f : X \to B$ be of class $\mathcal{C}$ and $\sigma_0, \sigma_1 \in \mathfrak{C}$. Let t be as in Theorem 1.2. It follows from Proposition 1.10 that $\phi' := t \circ \phi_{f,\sigma_0}$ is an anti-symplectic involution. It follows from our definition of $\mathfrak{C}$ that if ϕ' fixes a Lagrangian section σ_2 then $\sigma_2 \in \mathfrak{C}$. In this case, Remark 1.11 applied to σ_0 and σ_2 implies that t has a square root. Conversely, if for some reason t does not have a square root, we conclude that ϕ' does not fix a Lagrangian section.

The following corollary is an immediate consequence of Proposition 1.10 and Theorems 1.1 and 1.2.

Corollary 1.13. *Let $f : X \to B$ be a Lagrangian fibration of class $\mathcal{C}$ and let σ_0, σ_1, be two Lagrangian sections of class $\mathfrak{C}$. There exists a unique anti-symplectic involution $\phi : X \to X$ satisfying*

$$f \circ \phi = f, \qquad \phi \circ \sigma_0 = \sigma_1.$$

Assuming existence, uniqueness continues to hold for arbitrary Lagrangian sections of f.

Remark 1.14. The proof in Section 6 implies that Proposition 1.10 continues to hold for any Lagrangian fibration that is a smooth submersion with a smooth Lagrangian section. Corollary 1.13 holds for any Lagrangian fibration that is a smooth submersion and any smooth Lagrangian sections.

2. Lagrangian fibrations

In [8], the first two authors provide a method to construct Lagrangian torus fibrations of 6-dimensional symplectic manifolds homeomorphic to known Calabi–Yau manifolds. Recall that an integral affine structure $\mathscr{A}$ on a topological manifold is an atlas of charts whose change of coordinate maps are affine maps with integral linear part, i.e. elements of $\mathbb{R}^n \rtimes \mathrm{SL}(\mathbb{Z}, n)$. The basic idea is to start with an integral affine manifold with singularities, $(B, \Delta, \mathscr{A})$, where the B is a topological n-manifold such that $B_0 = B - \Delta$ has an integral affine structure $\mathscr{A}$. Here the discriminant Δ has codimension 2 and the affine structure is assumed to be *simple*. Roughly speaking, simplicity means that around points of Δ, B_0 is locally affine isomorphic to given models of integral affine manifolds, satisfying certain natural properties, such as having unipotent monodromy, cf. [8, Definition 3.14] for details.

The affine structure on $B_0 = B - \Delta$ induces a family of maximal lattices $\Lambda \subseteq T^*B_0$, together with a symplectic manifold X_0 and an exact sequence

$$0 \to \Lambda \to T^*B_0 \to X_0 \to 0.$$

This gives us a Lagrangian T^n bundle $f_0 : X_0 \to B_0$. The manifold X_0 can be compactified to a topological n-manifold X by gluing on Gross' local models of topological T^n fibrations [18]. To define a symplectic structure on X, in other words, to achieve a *symplectic* compactification of X_0, one needs local models of *Lagrangian* fibrations with singular fibers. These models were studied in [5,7,9]. In dimension $n = 2$, Δ consists of a finite collection of points and the symplectic compactification of X_0 is achieved by gluing a standard model of a Lagrangian fibration over a disc with a nodal central fiber; this model is known in symplectic geometry as a simple focus–focus fibration. This construction gives compact symplectic 4-manifolds with Lagrangian 2-torus fibrations. For a specific choice of integral affine S^2 with 24 singularities, one obtains a symplectic 4-manifold diffeomorphic to a K3 surface (cf. [24] or [8, Theorem 3.22]). In dimension $n = 3$, Δ is typically a graph with trivalent vertices, labeled either positive or negative. In this case the affine structure around edges or positive and negative vertices is isomorphic to the one induced on the base of models of three different kinds of local Lagrangian fibrations: respectively the so-called *generic*, *positive* and *negative* fibrations (see Section 3). The models for generic and positive fibrations can be regarded as 3-dimensional analogues of focus–focus fibrations; in particular, the fibrations are T^2-invariant, have codimension 2 discriminant and are given by smooth fibration maps. On the other hand, the model for a negative fibration is S^1-invariant, the fibration map is piecewise smooth and its discriminant locus has mixed codimension 1 and 2. This model can be regarded as a perturbation Gross' topological version of the negative fibration used in [18]. This perturbation forces the discriminant locus to drop codimension in a small neighborhood of a negative vertex. As a consequence, the compactification of $f : X_0 \to B_0$ achieved in [8], required a modification of Δ, more precisely, a fattening near the negative vertices.

2.1. The class $\mathcal{C}$

Given a simple integral affine 3-manifold with singularities $(B, \Delta, \mathscr{A})$ a *localized thickening* of Δ is given by the data $(\Delta^{\blacklozenge}, \{D_{p^-}\}_{p^- \in \mathcal{N}})$ where:

(i) $\Delta^{\blacklozenge}$ is the closed subset obtained from Δ after replacing a neighborhood of each negative vertex with a localized amoeba, i.e. an amoeba, as in Fig. 2 below, after the end of each leg is pinched down to dimension one.
(ii) $\mathcal{N}$ is the set of negative vertices and for each $p^- \in \mathcal{N}$, D_{p^-} is a 2-disk containing the codimension 1 component of $\Delta^{\blacklozenge}$ around p^-.

Given a localized thickening, define

$$B_{\blacklozenge} = B - \left(\Delta \cup \bigcup_{p^- \in \mathcal{N}} D_{p^-}\right),$$

and denote by $\mathscr{A}_\blacklozenge$ the restriction of the affine structure on $B_\blacklozenge$. Let $X_\blacklozenge = T^*B_\blacklozenge/\Lambda$ with standard symplectic form and $f_\blacklozenge : X_\blacklozenge \to B_\blacklozenge$ be the projection.

The main result of [8] is the following:

Theorem 2.1. *Given a compact simple integral affine* 3*-manifold with singularities* $(B, \Delta, \mathscr{A})$*, satisfying some additional mild hypothesis, there is a localized thickening* $(\Delta_\blacklozenge, \{D_{p^-}\}_{p^-\in\mathcal{N}})$ *and a smooth, compact symplectic* 6*-manifold* X *together with a piecewise smooth Lagrangian fibration* $f : X \to B$ *such that*

(i) f *is smooth except along* $\bigcup_{p^-\in\mathcal{N}} f^{-1}(D_{p^-})$*;*
(ii) *the discriminant locus of* f *is* $\Delta_\blacklozenge$*;*
(iii) *there is a commuting diagram*

$$\begin{array}{ccc} X_\blacklozenge & \xrightarrow{\psi} & X \\ {\scriptstyle f_\blacklozenge}\downarrow & & \downarrow{\scriptstyle f} \\ B_\blacklozenge & \xrightarrow{\iota} & B \end{array}$$

where ψ *is a symplectomorphism and* ι *the inclusion;*
(iv) *over a neighborhood of a positive vertex of* $\Delta_\blacklozenge$ *the fibration is positive, over a neighborhood of a point on an edge the fibration is generic-singular, over a neighborhood of* D_{p^-} *the fibration is Lagrangian negative;*
(v) f *has a section,* σ*, such that* $\sigma(B) \subset X$ *is a smooth Lagrangian submanifold such that* $\sigma(B) \cap \operatorname{Crit} f = \emptyset$*.*

Compact symplectic manifolds X and Lagrangian fibrations $f : X \to B$ as in Theorem 2.1 define a class, $\mathcal{C}$. The class $\mathcal{C}$ includes a large number of symplectic models of mirror pairs of Calabi–Yau manifolds with SYZ dual Lagrangian fibrations. Examples of these include the quintic 3-fold and its mirror and Batyrev–Borisov pairs of Calabi–Yau manifolds (cf. [8] for details).

In the next section, we review the local models used for the compactification in Theorem 2.1, and provide a case-by-case proof of existence of anti-symplectic involutions for each model.

3. Local existence of involutions

3.1. Focus–focus fibrations

We show how to construct fiber-preserving anti-symplectic involutions in dimension $n = 2$. Consider the case of proper fibrations with focus–focus type singularities. Here is a simple example:

Example 3.1. Let $X = \mathbb{C}^2 - \{z_1z_2 + 1 = 0\}$ and let ω be the restriction to X of the standard symplectic form on $\mathbb{C}^2$. The following map $f : X \to \mathbb{R}^2$ is a Lagrangian fibration:

$$f(z_1, z_2) = \left(\frac{|z_1|^2 - |z_2|^2}{2}, \log|z_1z_2 + 1|\right). \tag{16}$$

The only singular fiber is $f^{-1}(0)$, which is nodal with one node at the point $(0,0)$. Clearly conjugation $\phi : (z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ on $\mathbb{C}^2$ is a fiber-preserving anti-symplectic involution. The fixed locus of ϕ is $\mathbb{R}^2 - \{x_1x_2 + 1 = 0\}$ which has 3 connected components. Two of them (i.e. the connected components of $\{x_1x_2 + 1 < 0\}$) are sections of f not containing the singular point. The other one (i.e. the set $\{x_1x_2 + 1 > 0\}$) is mapped 2 to 1 by f except at $(0,0)$ which is a branched point.

We now describe the construction of a general nodal fibration. Details can be found in [4]. First, let us discuss a local model for the singularity. The standard focus–focus singularity is the (non-proper) map $q : \mathbb{C}^2 \to \mathbb{C}$ given by

$$q(z_1, z_2) = z_1\bar{z}_2. \tag{17}$$

Here $z_1 = y_1 + iy_2$, $z_2 = x_1 + ix_2$ and the symplectic form is $\omega = \sum dx_j \wedge dy_j$. The real and imaginary parts of the map q are $q_1 = x_1y_1 + x_2y_2$ and $q_2 = x_1y_2 - x_2y_1$ respectively. If v_{q_j} denotes the Hamiltonian vector field corresponding to q_j and g_j^t its flow, we have that

$$\begin{aligned} g_1^t(z_1, z_2) &= \left(e^{-t}z_1, e^t z_2\right), \\ g_2^t(z_1, z_2) &= \left(e^{it}z_1, e^{it}z_2\right). \end{aligned}$$

Notice that v_{q_2} induces an S^1 action. In fact, if $\tau = e^{-t_1+it_2}$ the we have

$$\left(g_1^{t_1} \circ g_2^{t_2}\right)(z_1, z_2) = \left(\tau z_1, \bar{\tau}^{-1}z_2\right) \tag{18}$$

which gives a $\mathbb{C}^*$ action.

If $B = \{b = b_1 + ib_2,\ |b| < 1\}$, we restrict the above map q to $Y = q^{-1}(B)$. We have two Lagrangian sections of q, $\Sigma_j : B \to Y$, $j = 1, 2$, given by $\Sigma_1(b) = (1, \bar{b})$ and $\Sigma_2(b) = (b, 1)$. Define the maps

$$\begin{aligned} \phi_j : \mathbb{C}^* \times B &\to Y, \\ (\tau, b) &\mapsto \tau \cdot \Sigma_j(b). \end{aligned}$$

Clearly ϕ_j describes the orbit of the section Σ_j via the Hamiltonian flow. Now let

$$V_1 = \left\{(\tau, b) \in \mathbb{C}^* \times B \;\middle|\; |b| < |\tau| < 1\right\}, \tag{19}$$

$$V_2 = \left\{(\tau, b) \in \mathbb{C}^* \times B \;\middle|\; 1 < |\tau| < |b|^{-1}\right\}, \tag{20}$$

and let $U_j = \phi_j(V_j)$. Denote $U = (U_1 \cup U_2)$. Clearly $U \cup \{(0,0)\}$ is an open neighborhood of the singular point $(0,0)$.

Now, suppose we are given a proper nodal fibration $f : X \to B$ with singular point $p \in X$, and a Lagrangian section σ of f with $\sigma(B) \subset X^\# = X - p$. We can describe $X^\#$ in terms of action angle-coordinates using a non-proper version of Proposition 1.7. Consider $B \subset \mathbb{C}$ the unit disk and T^*B with its standard symplectic form. For any smooth function $H : B \to \mathbb{R}$ define the following closed 1-forms on B:

$$\lambda_1 = -\log|b|\,db_1 + \operatorname{Arg} b\,db_2 + dH,$$
$$\lambda_2 = 2\pi\,db_2,$$

and consider the integral lattice $\Lambda \subset T^*B$ spanned by λ_1 and λ_2 and let $J_H = T^*B/\Lambda$. Then one can prove (cf. [5, Theorems 2.5 and 3.1]) that with a suitable choice of H, there is a unique fiber-preserving symplectomorphism

$$\Theta : J_H \to X^\# \tag{21}$$

which maps the zero section to σ. Uniqueness of Θ follows from Proposition 1.7 and by continuity.

Conversely, we now show how to use the above descriptions to construct a proper nodal fibration. Any such fibration can be obtained by gluing the neighborhood $U \cup \{(0,0)\}$ of the focus–focus singularity to the space J_H defined above. First of all, notice that J_H also has a $\mathbb{C}^*$ action, namely for every $\alpha \in T_b^*B$,

$$\tau \cdot (b,\alpha) = \big(b, -\log|\tau|\,db_1 + \operatorname{Arg}\tau\,db_2 + \alpha\big) \tag{22}$$

which are just translations along the fibers. Now let L_1 be the Lagrangian section in J_H given by the graph of dH and let L_2 be the one given by the zero section. We can define maps

$$\psi_j : \mathbb{C}^* \times B \to J_H,$$
$$(\tau, b) \mapsto \tau \cdot L_j(b).$$

Notice that $\psi_1(1,b) = L_1(b)$ and $\psi_1(b,b) = L_2(b)$. Take $V_j \subseteq \mathbb{C}^* \times B$, $j = 1,2$ as in (19) and (20) and denote $U_j' = \psi_j(V_j)$ and $U' = U_1' \cup U_2'$. Now define $g : U' \to U$ by $g|_{U_j'} = \phi_j \circ \psi_j^{-1}$. We can use g to glue the focus–focus singularity to J_H, to form a symplectic manifold with a nodal fibration.

All nodal fibrations can be realized this way for any given function H on B. Furthermore, given two nodal fibrations of X and X' determined by functions H and H' there is a fiber preserving symplectomorphism in a neighborhood of the singular fibers if an only if, up to a constant term, H and H' have the same germ at the origin. In other words, the invariant of a germ of nodal fibration is a germ of a function on B at the origin, a formal power series in two variables, uniquely defined up to a constant term. Moreover, this power series is independent of the choice of Lagrangian section of f (cf. [5]).

We have the following useful result:

Lemma 3.2. *Let σ_1 and σ_2 be two Lagrangian sections of a nodal fibration $f : X \to B$ not intersecting the singular point $p \in X$. There exists a unique symplectomorphism $t : X \to X$ such that*

$$f \circ t = f, \qquad t \circ \sigma_1 = \sigma_2. \tag{23}$$

Proof. For the section σ_1 there is a function H on B and a map $\Theta : J_H \to X^\#$ which maps the zero section to σ_1 (see above). The section σ_2 corresponds to a section σ' in J_H. Now we can define a symplectomorphism t' on J_H which is given by translation by σ' on the fibers (clearly t' maps the zero section to σ'). With this, one can show that $t^{\#} = \Theta \circ t' \circ \Theta^{-1}$ extends to a symplectomorphism t of X satisfying the required conditions. In fact, if $\sigma'(b) = s_1(b)\,db_1 + s_2(b)\,db_2$, then one can describe t' in terms of the $\mathbb{C}^*$ action by

$$t'(b, \alpha) = \tau(b) \cdot (b, \alpha),$$

where $\tau(b) = e^{-s_1(b)+is_2(b)}$ (cf. (22)). Since the map g, which glues the singularity to J_H, also matches the $\mathbb{C}^*$ actions, we have that t' corresponds to the following map on the local model for the focus–focus singularity (see (18)):

$$t'(z_1, z_2) = \big(\tau(b)z_1, \bar{\tau}(b)^{-1}z_2\big)$$

where $b = q(z_1, z_2) = z_1\bar{z}_2$. Clearly this map extends smoothly to the singularity. To prove uniqueness of t, one restricts t to X_0 and applies Corollary 1.9. □

Finally we have:

Proposition 3.3. *Let $f : X \to B$ be a nodal fibration with a Lagrangian section σ not intersecting the singular point $p \in X$. There exists a unique anti-symplectic involution $\iota_{f,\sigma}$ of X such that*

$$f \circ \iota_{f,\sigma} = f, \qquad \iota_{f,\sigma} \circ \sigma = \sigma. \tag{24}$$

Proof. Consider first the local model $q : \mathbb{C}^2 \to \mathbb{C}$ given by (17). Clearly the map $\iota : (z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$ is a fiber-preserving anti-symplectic involution, which exchanges the Lagrangian sections Σ_1 and Σ_2. Similarly, J_H has the anti-symplectic involution

$$\iota : (b, \alpha) \mapsto \big(b, dH(b) - \alpha\big),$$

which exchanges L_1 and L_2 and fixes $\frac{1}{2}L_1$. Since the gluing map g satisfies $g \circ \iota = \iota \circ g$, ι extends to a fiber-preserving anti-symplectic involution on (X, ω, f) fixing the section $\sigma' := \frac{1}{2}L_1$.

Now suppose that σ is another section of f not intersecting the singular point. From Lemma 3.2 we know that there is a fiber-preserving symplectomorphism t sending σ to σ'. Then the map $\iota_{f,\sigma} = t^{-1} \circ \iota \circ t$ is a fiber preserving anti-symplectomorphism with the required properties. To prove uniqueness one simply restricts f and σ to X_0 and B_0, respectively, and applies Corollary 1.8. □

3.2. Generic-singular fibration

An almost identical construction can be carried out for generic-singular fibrations in dimension 6. The (non-proper) local model for the singularity is the map $q : \mathbb{C}^2 \times S^1 \times (0, 1) \to \mathbb{C} \times S^1$ given by

$$q : \big(z_1, z_2, e^{i\theta}, r\big) \mapsto (z_1\bar{z}_2, r),$$

which is singular along $\mathrm{Crit}\, q = \{0\} \times S^1 \times (0, 1)$. The Hamiltonian flow induces a $\mathbb{C}^* \times S^1$ action on $\mathbb{C}^2 \times S^1$ given by

$$\left(\tau, e^{is}\right) \cdot \left(z_1, z_2, e^{i\theta}, r\right) = \left(\tau z_1, \bar{\tau}^{-1} z_2, e^{i(\theta+s)}, r\right).$$

The space $X^\#$ is constructed as follows. Let $D \subset \mathbb{C}$ be the open unit disk and $B = D \times (0, 1)$. Given a smooth function H on B, we form the lattice $\Lambda \subseteq T^*B$ generated by the periods

$$\begin{aligned} \lambda_1 &= -\log|b|\, db_1 + \arg b\, db_2 + dH, \\ \lambda_2 &= 2\pi\, db_2, \\ \lambda_2 &= dr. \end{aligned}$$

Then $X^\# = T^*B/\Lambda$ also has a $\mathbb{C}^* \times S^1$ action

$$\left(\tau, e^{is}\right) \cdot (b, \alpha) = \left(b, -\log|\tau|\, db_1 + \operatorname{Arg} \tau\, db_2 + s\, dr + \alpha\right).$$

We can now glue a neighborhood of $\mathrm{Crit}(q)$ to $X^\#$ using this action just like in the 4-dimensional case. This gives a symplectic manifolds X and a proper Lagrangian fibration over a cylinder $B = D \times (0, 1)$ with discriminant $\Delta = \{0\} \times (0, 1)$ and singular fibers being a product of S^1 and a nodal fiber. This construction provides all generic-singular fibrations. All generic-singular fibrations can be realized using the above method and, as in the nodal case, germs of generic-singular fibrations are classified [5]. The proof of the following two statements is the same as the proof of Lemma 3.2 and Proposition 3.3:

Lemma 3.4. *Let σ_1 and σ_2 be two Lagrangian sections of a generic-singular fibration $f : X \to B$ not intersecting the singular set* $\mathrm{Crit}\, f \subset X$. *There exists a unique symplectomorphism $t : X \to X$ such that*

$$f \circ t = f, \qquad t \circ \sigma_1 = \sigma_2. \tag{25}$$

Proposition 3.5. *Let $f : X \to B$ be a generic-singular fibration with a Lagrangian section σ not intersecting the critical locus* $\mathrm{Crit}\, f \subset X$. *There exists a unique anti-symplectic involution $\iota_{f,\sigma}$ of X such that*

$$f \circ \iota_{f,\sigma} = f, \qquad \iota_{f,\sigma} \circ \sigma = \sigma. \tag{26}$$

In this case the fixed point locus consists of seven components, six of which are sections. We leave the details of the proof to the reader.

3.3. Positive fibration

The situation is completely analogous to the case of fibrations of nodal type discussed above. First we give an explicit example.

Example 3.6. Let $X = \mathbb{C}^3 - \{1 + z_1z_2z_3 = 0\}$ with standard symplectic structure. Let $f : X \to \mathbb{R}^3$ be given by

$$f(z_1, z_2, z_3) = \bigl(\log|1 + z_1z_2z_3|, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2\bigr).$$

The map f defines a proper Lagrangian fibration having the topology of a positive fibration. The singular fibers lie over a trivalent vertex:

$$\Delta = \{b_1 = 0, b_2 = b_3 \geqslant 0\} \cup \{b_1 = b_2 = 0, b_3 \leqslant 0\} \cup \{b_1 = b_3 = 0, b_2 \leqslant 0\}.$$

The fibers over the edges have generic-singular type discussed above while the central fiber is homeomorphic to $S^1 \times T^2$ after a 2 cycle, $\{x\} \times T^2$, is collapsed to $x \in S^1$. It is easy to see that f is invariant under the anti-symplectic involution of X given by conjugation $\iota : (z_1, z_2, z_3) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3)$. The fixed set of ι is the set $\mathbb{R}^3 - \{1 + x_1x_2x_3\}$, which has 5 connected components. The one containing $(0, 0, 0)$ is mapped by f generically 4 to 1, while the other four components are sections not intersecting the singular locus.

A neighborhood of the singular locus in the above example is modeled on the following example due to Harvey and Lawson [20].

Example 3.7. On $\mathbb{C}^3$ define

$$F(z_1, z_2, z_3) = \bigl(\operatorname{Im} z_1z_2z_3, |z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2\bigr). \tag{27}$$

Here the last two components define the moment map μ of a Hamiltonian T^2-action. The critical locus of F is $\operatorname{Crit}(F) = \bigcup_{ij}\{z_i = z_j = 0\}$ and its discriminant locus is Δ as in Example 3.6. The regular fibers are homeomorphic to $\mathbb{R} \times T^2$. The singular fiber over $0 \in \Delta$ is homeomorphic to $\mathbb{R} \times T^2$ after $\{p\} \times T^2$ is collapsed to $p \in \mathbb{R}$. All the other singular fibers are homeomorphic to $\mathbb{R} \times T^2$ after a two cycle $\{p\} \times T^2 \subset \mathbb{R} \times T^2$ is collapsed to a circle. The map $\iota : (z_1, z_2, z_3) \mapsto (-\bar{z}_1, \bar{z}_2, \bar{z}_3)$ defines a fiber preserving anti-symplectic involution. Notice that the smooth part of every singular fiber has two connected components and ι sends one to the other.

In the above example, the Hamiltonian flow associated to the components of F induces an $\mathbb{R} \times T^2$ action on $\mathbb{C}^3$ which is free and transitive on the smooth fibers. Let us denote by $t = (t_1, t_2, t_3)$ the coordinates on $\mathbb{R} \times T^2$, where t_2 and t_3 are periodic (of period 1) and by $(t, z) \mapsto t \cdot z$ the action on $\mathbb{C}^3$. Consider a neighborhood B of $(0, 0, 0) \in \mathbb{R}^3$ and let $Y = F^{-1}(B)$. We can find sections of F, $\Sigma_j : B \to Y$, $j = 1, 2$, chosen so that $\iota \circ \Sigma_1 = \Sigma_2$. There is a function $\tau : B \to \mathbb{R} \times T^2$ such that $\tau(b) \cdot \Sigma_1(b) = \Sigma_2(b)$. Actually, τ is defined only on $B - \Delta$, and it is shown in [5] that the first component τ_1 of τ tends to $+\infty$ as b approaches the discriminant locus Δ. Define the maps

$$\begin{aligned} \phi_j : \bigl(\mathbb{R} \times T^2\bigr) \times B &\to Y, \\ (t, b) &\mapsto t \cdot \Sigma_j(b). \end{aligned}$$

Notice that $\phi_1(0, b) = \Sigma_1(b)$ and $\phi_1(\tau(b), b) = \Sigma_2(b)$ if $b \notin \Delta$. Since τ_1 is big near Δ, we may assume $\tau_1(b) > 0$ for all $b \in B$. For $j = 1, 2$ define subsets

$$V_1 = \{(t,b) \in (\mathbb{R} \times T^2) \times B \mid t_1 \in (0, \tau_1(b))\}, \tag{28}$$
$$V_2 = \{(t,b) \in (\mathbb{R} \times T^2) \times B \mid t_1 \in (-\tau_1(b), 0)\} \tag{29}$$

and let $U_j = \phi_j(V_j)$. We have that $U_1 \cup U_2$ is a T^2 invariant open set, whose closure $\bar{U}$ is a (closed) neighborhood of $\mathrm{Crit}(F)$ having as boundary the T^2 orbit of $\Sigma_1(B) \cup \Sigma_2(B)$.

Now, for any smooth function H on B, define one forms on B as follows

$$\lambda_1 = \sum_{j=1}^{3} \tau_j \, db_j + dH, \qquad \lambda_2 = db_2, \qquad \lambda_3 = db_3. \tag{30}$$

It can be shown that these are all closed one forms. Let $X^\# = T^*B/\Lambda$, where Λ is the integral lattice generated by the λ_j's. Also on $X^\#$ there is an $\mathbb{R} \times T^2$ action given by

$$t \cdot (b, \alpha) = \left(b, \sum_{j=1}^{3} t_j \, db_j + \alpha\right).$$

Now let L_1 be the Lagrangian section in $X^\#$ given by the graph of dH and let L_2 be the zero section. We can define maps

$$\psi_j : (\mathbb{R} \times T^2) \times B \to X^\#,$$
$$(t,b) \mapsto t \cdot L_j(b).$$

Notice that $\psi_1(0,b) = L_1(b)$ and $\psi_1(\tau(b), b) = L_2(b)$. Take V_j, $j = 1,2$, as in (28) and (29) and denote $U'_j = \psi_j(V_j)$ and $U' = U'_1 \cup U'_2$. We can now define a map $g : U' \to U$ such that $g|_{U'_j} = \phi_j \circ \psi_j^{-1}$ and use it to glue the singularity to $X^\#$, to form X with the positive fibration $f : X \to B$. Also in this case we have:

Lemma 3.8. *Let σ_1 and σ_2 be two Lagrangian sections of a positive fibration $f : X \to B$ not intersecting the singular set* $\mathrm{Crit}\, f \subset X$. *There exists a unique symplectomorphism $t : X \to X$ such that*

$$f \circ t = f, \qquad t \circ \sigma_1 = \sigma_2. \tag{31}$$

Proposition 3.9. *Let $f : X \to B$ be a positive fibration with a Lagrangian section σ not intersecting the singular set* $\mathrm{Crit}\, f \subset X$. *There exists a unique anti-symplectic involution $\iota_{f,\sigma}$ of X such that*

$$f \circ \iota_{f,\sigma} = f, \qquad \iota_{f,\sigma} \circ \sigma = \sigma. \tag{32}$$

The proofs are almost word by word like in Lemma 3.2 and Proposition 3.3. As in Example 3.7, the fixed point locus of the involution consists of 5 connected components, four of which are sections.

3.4. A piecewise smooth fibration

Here we prove the existence of a fiber-preserving anti-symplectic involution on interesting examples of piecewise smooth fibrations used in [8]. We now recall the construction of this fibration. Consider the following S^1 action on $\mathbb{C}^3$:

$$e^{i\theta}(z_1, z_2, z_3) = \left(e^{i\theta} z_1, e^{-i\theta} z_2, z_3\right). \tag{33}$$

It is Hamiltonian with moment map:

$$\mu(z_1, z_2, z_3) = \frac{|z_1|^2 - |z_2|^2}{2}. \tag{34}$$

The only critical value of μ is $t = 0$. Now let $\gamma : \mathbb{C}^2 \to \mathbb{C}$ be the following S^1-invariant, piecewise smooth map

$$\gamma(z_1, z_2) = \begin{cases} \frac{z_1 z_2}{|z_1|}, & \text{when } \mu(z_1, z_2) \geqslant 0, \\ \frac{z_1 z_2}{|z_2|}, & \text{when } \mu(z_1, z_2) < 0. \end{cases} \tag{35}$$

Define $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ to be

$$\pi(z_1, z_2, z_3) = \left(\gamma(z_1, z_2), z_3\right)$$

and $\mathrm{Log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ to be

$$\mathrm{Log}(u_1, u_2) = \left(\log|u_1|, \log|u_2|\right).$$

It was shown in [7] that given a symplectomorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ the map

$$f = (\mu, \mathrm{Log} \circ \Phi \circ \pi) \tag{36}$$

defines a piecewise smooth Lagrangian fibration on the open subset of $\mathbb{C}^3$ given by

$$X = (\Phi \circ \pi)^{-1}\left(\left(\mathbb{C}^*\right)^2\right).$$

Now consider the anti-symplectic involution $\iota : \mathbb{C}^3 \to \mathbb{C}^3$ given by conjugation. It is easy to see that if Φ commutes with conjugation on $\mathbb{C}^2$, then $\iota(X) = X$ and $f \circ \iota = f$, and therefore ι is fiber-preserving.

Example 3.10. If in the above construction we take

$$\Phi(u_1, u_2) = \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2 - \sqrt{2}) \tag{37}$$

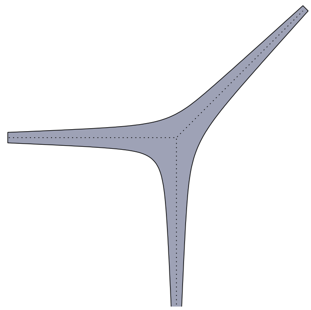

Fig. 2. Amoeba of $v_1 + v_2 + 1 = 0$.

then the map f becomes

$$f(z_1, z_2, z_3) = \left(\frac{1}{2}\left(|z_1|^2 - |z_2|^2\right), \log\frac{1}{\sqrt{2}}|\gamma - z_3|, \log\frac{1}{\sqrt{2}}|\gamma + z_3 - \sqrt{2}|\right). \tag{38}$$

The discriminant locus Δ is described as follows. Consider, inside $(\mathbb{C}^*)^2$, the surface

$$\Sigma = \{v_1 + v_2 + 1 = 0\},$$

which is, topologically, a pair of pants. Then

$$\Delta = \{0\} \times \operatorname{Log}(\Sigma).$$

Clearly Δ has the shape of an amoeba as shown in Fig 2.

For a discussion of the topology of the fibers in this example we refer to [7]. Observe that Φ commutes with conjugation, and therefore ι is fiber-preserving. The fixed locus S of ι is the complement in $\mathbb{R}^3$ of the set

$$K = \left\{\gamma(x_1, x_2) - x_3 = 0\right\} \cup \left\{\gamma(x_1, x_2) + x_3 - \sqrt{2} = 0\right\}.$$

The reader may verify that $S = \mathbb{R}^3 - K$ has five connected components, $S_1, S_2, \ldots, S_5$ containing respectively $(0, 0, 2)$, $(0, 0, -1)$, $(-1, -1, 1/2)$, $(1, 1, 1/2)$, $(0, 0, 1/2)$. Then S_1, S_2 and S_5 are mapped generically 2 to 1 while S_3 and S_4 are sections.

Now we verify that the same involution ι as above is fiber-preserving also with respect to the version of the above example where the legs are pinched down to codimension 2 toward the ends. Here is how we construct it.

Example 3.11. Consider the smooth function:

$$H_0 = \frac{\pi}{4}\operatorname{Im}(u_1\bar{u}_2)$$

and let Φ_{H_0} be the Hamiltonian symplectomorphism associated to H_0, i.e.

$$\Phi_{H_0} : (u_1, u_2) \mapsto \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2).$$

We now want a symplectomorphism which acts like Φ_{H_0} in a small ball centered at the origin and like the identity outside a slightly bigger ball. So choose a cut-off function $k : \mathbb{R}_{\geqslant 0} \to [0, 1]$ such that, for some $\epsilon > 0$,

$$k(t) = \begin{cases} 1, & \text{when } 0 < t \leqslant \epsilon; \\ 0, & \text{when } t \geqslant 2\epsilon \end{cases} \tag{39}$$

and define the Hamiltonian

$$H = k\big(|u_1|^2 + |u_2|^2\big) H_0.$$

The Hamiltonian symplectomorphism Φ_H associated to H satisfies

$$\Phi_H(u_1, u_2) = \begin{cases} \mathrm{Id}_{\mathbb{C}^2}, & \text{when } |u_1|^2 + |u_2|^2 \geqslant 2\epsilon; \\ \frac{1}{\sqrt{2}}(u_1 - u_2, u_1 + u_2), & \text{when } |u_1|^2 + |u_2|^2 \leqslant \epsilon. \end{cases}$$

Now let Ψ be the affine symplectomorphism

$$\Psi : (v_1, v_2) \mapsto \frac{1}{\sqrt{2}}(v_1 - v_2, v_1 + v_2 - \sqrt{2}),$$

and finally, define $\Phi = \Psi \circ \Phi_H$. It is clear that

$$\Phi(u_1, u_2) = \begin{cases} \Psi, & \text{when } |u_1|^2 + |u_2|^2 \geqslant 2\epsilon; \\ (-u_2, u_1 - 1), & \text{when } |u_1|^2 + |u_2|^2 \leqslant \epsilon. \end{cases} \tag{40}$$

The fibration f defined by (36) with this choice of Φ has as discriminant locus a 3-legged amoeba with the horizontal leg pinched down to a line towards its ends.

Observe that H satisfies $H(\bar{u}_1, \bar{u}_2) = -H(u_1, u_2)$. The reader may check that this property implies that Φ_H commutes with conjugation. It follows that also Φ commutes with conjugation (since also Ψ does). Similarly one shows that the symplectomorphism which pinches down all three legs at once is:

$$\Phi(u_1, u_2) = \begin{cases} (-u_2, u_1 - 1), & \text{when } |u_1|^2 + |u_2|^2 \leqslant \epsilon; \\ (u_1 - 1, u_2 - \sqrt{2}), & \text{when } |u_1|^2 + |u_2 - \sqrt{2}|^2 \leqslant \epsilon; \\ \frac{1}{\sqrt{2}}(u_1 - u_2,\ u_1 + u_2), & \text{when } |u_2|^2 \geqslant M; \\ \Psi, & \text{everywhere else} \end{cases} \tag{41}$$

which also commutes with conjugation. So ι is fiber-preserving also with respect to f constructed with this Φ.

The fibration above has the right topology to be a good candidate for a Lagrangian negative fibration. However, it fails to be smooth over a large hyperplane containing the discriminant locus, and therefore it is not a suitable model for the compactification as in Theorem 2.1. A suitable model should be smooth away from a disc containing the codimension 1 part of the discriminant. The smoothing process is a delicate issue, and involves studying a class of piecewise smooth Lagrangian fibrations, called stitched fibrations [9].

Remark 3.12. One may use lifts of sections of the Log fibration to obtain many Lagrangian sections of the fibration in Example 3.10 not intersecting its singular locus (details cf. [8, Proposition 5.9]). From all these, only the sections S_3 and S_4 are fixed by the involution given by complex conjugation. The same situation holds for the thin-legged Example 3.11. In what follows, we shall assume that our choice of a Lagrangian section σ of Example 3.11 is either given by S_3 or S_4.

3.5. *Involutions of stitched fibrations*

This section is rather technical, closely related to [9], and may be skipped on a first reading. The main (new) results are Theorems 3.24 and 3.26 where we provide conditions for the existence of a smooth, fiber-preserving, anti-symplectic involution of stitched fibrations, which will be used to prove the existence of involutions of the negative fibration.

Definition 3.13. Let X be a smooth $2n$-dimensional symplectic manifold. Suppose there is a free Hamiltonian S^1 action on X with moment map $\mu : X \to \mathbb{R}$. Let $X^+ = \{\mu \geqslant 0\}$ and $X^- = \{\mu \leqslant 0\}$. Given a smooth $(n-1)$-dimensional manifold M, a map $f : X \to \mathbb{R} \times M$ is said to be a *stitched Lagrangian fibration* if there is a continuous S^1 invariant function $G : X \to M$, such that the following holds:

(i) Let $G^\pm = G|_{X^\pm}$. Then G^+ and G^- are restrictions of C^∞ maps on X;
(ii) f can be written as $f = (\mu, G)$ and f restricted to $X^\pm$ is a proper submersion with connected Lagrangian fibers.

We call $Z = \mu^{-1}(0)$ the *seam* and $\Gamma = f(Z) \subseteq \{0\} \times M$ the *wall*. We denote $f^\pm = f|_{X^\pm}$.

Denote $B = f(X)$ and $B^\pm = f(X^\pm)$. In general, a stitched fibration will only be piecewise C^∞, however all its fibers are smooth Lagrangian tori. Throughout this section we will always assume (unless otherwise stated) that the pair (B, Γ) is diffeomorphic to the pair (D^n, D^{n-1}), where $D^k \subset \mathbb{R}^k$ is an open unit ball centered at the origin and $\mathbb{R}^{n-1}$ is embedded in $\mathbb{R}^n$.

Observe that the fibration in Example 3.10, when restricted to $X - f^{-1}(\Delta)$, defines a stitched Lagrangian fibration. In fact this is the main example. The seam is $Z = \mu^{-1}(0) - f^{-1}(\Delta)$, notice that in this case Z has three connected components.

The seam of a stitched fibration is an S^1-bundle $p : Z \to \bar{Z} := Z/S^1$ such that f factors through p, i.e. we have the diagram:

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "f|_Z"'] & & \bar{Z} \arrow[dl, "\bar{f}"] \\ & \Gamma & \end{tikzcd}$$

where $\bar{Z}$ has the reduced symplectic form and $\bar{f}$ is the reduced Lagrangian fibration over the wall Γ. We also have the vertical $(n-1)$-plane distribution:

$$\mathfrak{L} = \ker \bar{f}_* \subset T\bar{Z}$$

tangent to the fibers of $\bar{f}$. In what follows we will define certain invariants of the stitched fibration consisting of sections of $\mathfrak{L}^*$ which are *fiberwise closed*, in the sense that they restrict to closed 1-forms on the fibers of $\bar{f}$.

On the base of a stitched fibration we allow a more general set of coordinates than just the smooth ones, which we define bellow.

Definition 3.14. A set of coordinates on $B \subseteq \mathbb{R} \times M$, given by a map $\phi : B \to \mathbb{R}^n$, is said to be *admissible* if the components of $\phi = (\phi_1, \ldots, \phi_n)$ satisfy the following properties:

(i) ϕ_1 is the restriction to B of the projection map $\mathbb{R} \times M \to \mathbb{R}$;
(ii) for $j = 2, \ldots, n$ the restrictions of ϕ_j to B^+ and B^- are locally restrictions of smooth functions on B.

Essentially, admissible coordinates are those such that $\phi \circ f$ is again stitched. Let $f : X \to B$ be a stitched Lagrangian fibration and let ϕ be a set of admissible coordinates. For $j = 2, \ldots, n$, $f_j^{\pm} = \phi_j \circ f|_{X^{\pm}}$ is the restriction of a C^∞ function on X to $X^{\pm}$ and we can write $f = (\mu, f_2^{\pm}, \ldots, f_n^{\pm})$.

Now we want to put stitched fibrations in a normal form. In the smooth case, a proper Lagrangian submersion locally always admits action-angle coordinates, defined up to the choice of a basis of $H_1(F_b, \mathbb{Z})$, where F_b is the fiber over $b \in B$. In the case of stitched fibrations we can generalize this idea as follows. Assuming B contractible, we choose a pair of bases $\gamma^{\pm} = (\gamma_1, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm})$ of $H_1(X, \mathbb{Z})$ such that

(a) γ_1 is represented by an orbit of the S^1 action;
(b) $\gamma_j^+ = \gamma_j^- + m_j \gamma_1$, for some $m_2, \ldots, m_n \in \mathbb{Z}$.

Condition (b) simply means that $p_*\gamma^+ = p_*\gamma^-$ under the map $p_* : H_1(X, \mathbb{Z}) \to H_1(X/S^1, \mathbb{Z})$. The following proposition generalizes the notion of action-angle coordinates on the base.

Proposition 3.15. *Let $f : X \to B$ be a stitched fibration and let $\gamma^{\pm}$ be bases of $H_1(X, \mathbb{Z})$ satisfying the above conditions. Then the maps f^+ and f^- induce lattices Λ^+ and Λ^- together with embeddings,*

$$\Lambda^{\pm} \hookrightarrow T^*_{B^{\pm}}.$$

Let $\alpha^{\pm} : B^{\pm} \to \mathbb{R}^n$ be the action coordinates associated with the choice of bases $\gamma^{\pm}$ and satisfying $\alpha^{\pm}(b) = 0$ for some $b \in \Gamma$. Then the map

$$\alpha = \begin{cases} \alpha^+ & \text{on } B^+, \\ \alpha^- & \text{on } B^- \end{cases}$$

is an admissible change of coordinates. If $b_1, \dots, b_n$ *denote the action coordinates on* B *given by* α*, then* $\{db_1, \dots, db_n\}$ *is a basis of* Λ^+ *and* Λ^-.

Recall that to establish the existence of action-angle coordinates, in the classical case, one chooses a smooth Lagrangian section. In the stitched case we choose a continuous section $\sigma : B \to X$ such that $\sigma|_{B^\pm}$ are the restrictions of smooth maps and $\sigma(B)$ is a smooth Lagrangian submanifold. Such sections always exist locally, for instance in Example 3.10 a component of the fixed locus of the anti-symplectic involution is a section of this type. We denote a stitched fibration $f : X \to B$ together with a choice of basis γ of $H_1(X, \mathbb{Z})$ and a section σ as above by $\mathcal{F} = (X, B, f, \gamma, \sigma)$. When $W \subseteq B$ is an open set we usually denote by $\mathcal{F}|_W$ the fibration $(f^{-1}(W), W, f, \gamma, \sigma|_W)$.

Definition 3.16. Two stitched fibrations $(X, B, f, \gamma, \sigma)$ and $(X', B', f', \gamma', \sigma')$, with seams Z and Z' respectively are *symplectically conjugate* if there are neighborhoods $W \subseteq B$ of $\Gamma := f(Z)$ and $W' \subseteq B'$ of $\Gamma' := f'(Z')$, an S^1 equivariant C^∞ symplectomorphism $\psi : f^{-1}(W) \to f'^{-1}(W')$ sending Z to Z' and a C^∞ diffeomorphism $\phi : W \to W'$ such that: $f' \circ \psi = \phi \circ f$, $\psi \circ \sigma = \sigma' \circ \phi$ and $\psi_* \gamma = \gamma'$. The set of equivalence classes under this relation will be called *germs of stitched fibrations*.

Notice that in the above definition we are allowed to shrink to a smaller neighborhood of Γ but not to a smaller Γ. So germs are meant to be defined around Γ and not around a point.

The following is a basic construction of stitched fibrations.

Example 3.17 *(Normal forms).* Let $(b_1, \dots, b_n)$ be the standard coordinates on $\mathbb{R}^n$. Let (U, Γ) be a pair of subsets of $\mathbb{R}^n$ diffeomorphic to (D^n, D^{n-1}) and $\Gamma = U \cap \{b_1 = 0\}$. Define $U^+ = U \cap \{b_1 \geqslant 0\}$ and $U^- = U \cap \{b_1 \leqslant 0\}$. Consider the lattice $\Lambda = \operatorname{span}\langle db_1, \dots, db_n\rangle_{\mathbb{Z}}$ and form the symplectic manifold T^*U/Λ. Denote by π the standard projection onto U and let $Z = \pi^{-1}(\Gamma)$. We may consider the S^1 action on T^*U/Λ given by translations by multiples of db_1 in the fibers of π, whose moment map is $\mu = b_1$. Suppose there is an open neighborhood $V \subseteq T^*U/\Lambda$ of Z and a map $u : V \to \mathbb{R}^n$ which is a proper, smooth, S^1-invariant Lagrangian submersion with components $(u_1, \dots, u_n)$ such that $u|_Z = \pi$ and $u_1 = b_1$. Now define the following subsets of T^*U/Λ,

$$Y^+ := \pi^{-1}\big(U^+\big), \qquad Y := Y^+ \cup V, \qquad Y^- := Y \cap \pi^{-1}\big(U^-\big)$$

and define the map $f_u : Y \to \mathbb{R}^n$ by

$$f_u = \begin{cases} u & \text{on } Y^-, \\ \pi & \text{on } Y^+. \end{cases} \tag{42}$$

Clearly $f_u : Y \to \mathbb{R}^n$ is a stitched fibration. Denote $B_u := f_u(Y)$. The zero section σ_0 of π is, perhaps after a change of coordinates in the base, a section of f_u. Let γ_0 be the basis of $H_1(Y, \mathbb{Z})$ induced by Λ. We call the stitched fibration $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$ a *normal form*.

Now consider a normal form $\mathcal{F}_u$ and let $(b, y) = (b_1, \dots, b_n, y_1, \dots, y_n)$ be canonical coordinates on T^*B_u so that y gives coordinates on the fiber $T^*_b B_u$. Let W be a neighborhood of Γ inside $u(V)$. If $r \in \mathbb{R}$ is a parameter, for any $b = (0, b_2, \dots, b_n) \in \Gamma$, let (r, b) denote the point $(r, b_2, \dots, b_n) \in \mathbb{R}^n$. Given $(r, b) \in W$, denote by $L_{r,b}$ the fiber $u^{-1}((r, b))$. For every fiber $F_b \subset Z$ of π, consider the symplectomorphism

$$\left(y_1, \dots, y_n, \sum_{k=1}^{n} x_k\, dy_k\right) \mapsto (x_1, b_2 + x_2, \dots, b_n + x_n, y_1, \dots, y_n), \tag{43}$$

between a neighborhood of the zero section of T^*F_b and a neighborhood of F_b in V. If W is sufficiently small, for every $(r, b) \in W$, the Lagrangian submanifold $L_{r,b}$ will be the image of the graph of a closed 1-form on F_b. Due to the S^1 invariance of u and the fact that $u_1 = b_1$, this 1-form has to be of the type

$$r\, dy_1 + \ell(r, b),$$

where $\ell(r, b)$ is S^1 invariant, i.e. it may be considered as a 1-form on $\bar{F}_b := F_b/S^1$. Denote by $\ell(r)$ the smooth one parameter family of sections of $\mathfrak{L}^*$ such that $\ell(r)|_{\bar{F}_b} = \ell(r, b)$. The condition $u|_Z = \pi$ implies that $\ell(0, b) = 0$. Furthermore, the N-th order Taylor series expansion of $\ell(r)$ in the parameter r can be written as

$$\ell(r) = \sum_{k=1}^{N} \ell_k r^k + o\big(r^N\big), \tag{44}$$

where the ℓ_k's are fiberwise closed sections of $\mathfrak{L}^*$.

Definition 3.18. With the above notation, we define

(i) $\mathscr{L}_Z$ the set of sequences $\ell = \{\ell_k\}_{k\in\mathbb{N}}$ such that ℓ_k is a fiberwise closed section of $\mathfrak{L}^*$;
(ii) $\mathscr{U}_Z$ the set of pairs (V, u) where $V \subseteq T^*U/\Lambda$ is a neighborhood of Z and $u : V \to \mathbb{R}^n$ is a proper, smooth, S^1-invariant Lagrangian submersion with components $(u_1, \dots, u_n)$ such that $u|_Z = \pi$ and $u_1 = b_1$.

As above, to a given $(V, u) \in \mathscr{U}_Z$ we can associate a unique sequence $\ell \in \mathscr{L}_Z$. Conversely, in [9, Section 5] it is shown that for any given sequence $\ell \in \mathscr{L}_Z$ there is some $(V, u) \in \mathscr{U}_Z$, therefore a normal form, associated to it. Clearly, this (V, u) is not unique.

In [9] the following result is proved:

Proposition 3.19. *Every stitched fibration $\mathcal{F} = (X, B, f, \sigma, \gamma)$, such that the pair (B, Γ) is diffeomorphic to the pair (D^n, D^{n-1}), is symplectically conjugate to a normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$.*

When $\mathcal{F}$ is smooth, its normal form is $\mathcal{F}_\pi$, this is Arnold–Liouville theorem. Given a stitched Lagrangian fibration $\mathcal{F} = (X, B, f, \sigma, \gamma)$ with normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$, we respectively denote by Z_{nor} and Γ_{nor} the seam and the wall of $\mathcal{F}_u$ and by $\bar{Z}_{\mathrm{nor}}$ the S^1 reduction of Z_{nor}.

Definition 3.20. Let $\mathcal{F} = (X, B, f, \sigma, \gamma)$ be a stitched fibration with normal form $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$. Let $\ell \in \mathscr{L}_{\bar{Z}_{\mathrm{nor}}}$ be the unique sequence determined by $(V, u) \in \mathscr{U}_{Z_{\mathrm{nor}}}$ defining $\mathcal{F}_u$. We call $\mathrm{inv}(\mathcal{F}) := (\bar{Z}_{\mathrm{nor}}, \ell)$ the *invariants* of $\mathcal{F}$. We say that the invariants of $\mathcal{F}$ vanish if for all $k \in \mathbb{N}$, $\ell_k \equiv 0$ when restricted to the reduced fibers of $\mathcal{F}_u$. We say that the invariants of $\mathcal{F}$ are fiberwise constant if all the ℓ_k's are fiberwise constant.

One can prove that $\mathrm{inv}(\mathcal{F})$ is independent on the choice of normal form. Moreover, we also have the following classification results from [9]:

Theorem 3.21. *Given any pair $(U, \Gamma_{\mathrm{nor}})$ of subsets of $\mathbb{R}^n$, diffeomorphic to (D^n, D^{n-1}) and with $\Gamma_{\mathrm{nor}} = U \cap \{b_1 = 0\}$, a sequence $\ell = \{\ell_k\}_{k\in\mathbb{N}} \in \mathscr{L}_{\bar{Z}_{\mathrm{nor}}}$ and integers $m_2, \dots, m_n$ such that*

$$\int_{[db_j]} \ell_1 = m_j, \quad \text{for all } j = 2, \dots, n, \tag{45}$$

there exists a smooth symplectic manifold (X, ω) and a stitched Lagrangian fibration $f : X \to U$ satisfying the following properties:

(i) *the coordinates $(b_1, \dots, b_n)$ on U are action coordinates of f with $\mu = f^*b_1$ the moment map of the S^1 action;*
(ii) *the periods $\{db_1, \dots, db_n\}$, restricted to $U^\pm$ correspond to bases $\gamma^\pm = \{\gamma_1, \gamma_2^\pm, \dots, \gamma_n^\pm\}$ of $H_1(X, \mathbb{Z})$ satisfying conditions* (a) *and* (b) *prior to Proposition* 3.15*;*
(iii) *there is a Lagrangian section σ of f, such that $(\bar{Z}_{\mathrm{nor}}, \ell)$ are the invariants of $(X, f, U, \sigma, \gamma^+)$.*

Theorem 3.22. *Let $\mathcal{F} = (X, B, f, \sigma, \gamma)$ and $\mathcal{F}' = (X', B', f', \sigma', \gamma')$ be stitched fibrations, such that the pairs (B, Γ) and (B', Γ') are diffeomorphic to the pair (D^n, D^{n-1}). Then,*

(i) *$\mathcal{F}$ and $\mathcal{F}'$ are symplectically conjugate if and only if* $\mathrm{inv}(\mathcal{F}) = \mathrm{inv}(\mathcal{F}')$*;*
(ii) *$\mathcal{F}$ is smooth if and only if* $\mathrm{inv}(\mathcal{F})$ *vanish;*
(iii) *$\mathcal{F}$ becomes smooth after an admissible change of coordinates on the base if and only if* $\mathrm{inv}(\mathcal{F})$ *are fiberwise constant.*

In the above, fiberwise constant means that in the normal form, the forms ℓ_k are independent of the y coordinates. The set of germs of stitched fibrations is therefore classified by the pairs $(\bar{Z}_{\mathrm{nor}}, \ell)$. We say that a fibration is *fake stitched* if it becomes smooth after an admissible change of coordinates on the base. The important consequence of Theorem 3.21, which was exploited in [8], is that from a given set of invariants we can form another one for example by summing to the sequence ℓ another sequence or by multiplying elements ℓ_k by pull backs of smooth functions on the base. The new invariants give rise to new stitched fibrations.

Remark 3.23. Observe that Corollary 1.9 does not hold for stitched fibrations in general, it only holds for stitched fibrations that are fake. Indeed, let f be a stitched fibration with a section σ and invariants ℓ_k. Suppose σ' is another Lagrangian section of f and assume there exists a symplectomorphism t of X such that $f \circ t = f$ and $t \circ \sigma = \sigma'$. This implies that $\bar{t}^*\ell_k = \ell_k$ where $\bar{t}$ is the translation induced on the reduced fibration. Therefore each ℓ_k is fiberwise constant, hence f is smooth after a suitable change of coordinates in the base.

On a smooth Lagrangian fibration $f : X \to B$, with B diffeomorphic to D^n, with a Lagrangian section $\sigma : B \to X$, there always exists a unique fiber-preserving anti-symplectic involution

$\iota : X \to X$ fixing σ. In fact, if (b, y) are action-angle coordinates on X then we must have

$$\iota(b, y) = (b, -y). \tag{46}$$

How about stitched fibrations? Do they admit smooth fiber-preserving anti-symplectic involutions? First observe that given $(X, B, f, \sigma, \gamma)$ with seam Z and wall Γ, then the smooth Lagrangian fibration $\bar{f} : \bar{Z} \to \Gamma$ has a unique smooth fiber-preserving anti-symplectic involution $\bar{\iota} : \bar{Z} \to \bar{Z}$ fixing $\sigma|_\Gamma$. Can $\bar{\iota}$ be extended to X? We have the following result:

Theorem 3.24. *A stitched fibration $\mathcal{F} = (X, B, f, \sigma, \gamma)$ with invariants $(\bar{Z}, \ell)$ has a unique smooth fiber-preserving anti-symplectic involution $\iota : X \to X$ fixing σ if and only if*

$$\bar{\iota}^* \ell_k = -\ell_k, \tag{47}$$

for every $\ell_k \in \ell$.

Proof. Observe that if $\tilde{f}^+$ and $\tilde{f}^-$ are smooth, proper, Lagrangian extensions of f^+ and f^- defined on open subsets $\tilde{X}^+$ and $\tilde{X}^-$ of X such that $X^\pm \subseteq \tilde{X}^\pm$, then there are unique fiber-preserving anti-symplectic involutions $\iota^+ : \tilde{X}^+ \to \tilde{X}^+$ and $\iota^- : \tilde{X}^- \to \tilde{X}^-$ fixing σ. Therefore we may define $\iota : X \to X$ to be such that $\iota|_{X^\pm} = \iota^\pm$. The question is if ι is smooth.

Let $\mathcal{F}_u = (Y, B_u, f_u, \sigma_0, \gamma_0)$ be a normal form for $\mathcal{F}$ (as described in Example 3.17), and let (b, y) be the canonical coordinates on $T^* B_u$, then we have the smooth anti-symplectic involution ι_0 such that $\iota_0(b, y) = (b, -y)$. We now show that if (47) holds, we can construct $\mathcal{F}_u$ so that ι_0 is fiber-preserving with respect to f_u.

We use the same notation as in the construction after Example 3.17. Observe that given the sequence $\ell = \{\ell_k\}_{k \in \mathbb{N}}$ satisfying (47), we can construct a one parameter family of fiberwise closed sections $\ell(r)$ of $\mathfrak{L}^*$ such that (44) holds and such that

$$\bar{\iota}^* \ell(r) = -\ell(r) \tag{48}$$

for every small r. This can be done by refining the methods used in [9] to construct $\ell(r)$ satisfying (44). From $\ell(r)$ we construct $(V, u) \in \mathcal{U}_Z$ so that the fibers of u are the images $L_{r,b}$ of the graph of the one form $r\,dy_1 + \ell(r, b)$ via the symplectomorphism (43). It can be easily verified that condition (48) implies that $\iota_0(L_{r,b}) = L_{r,b}$, i.e. that ι_0 is fiber-preserving with respect to u. By uniqueness, ι_0 must coincide with the map ι constructed above under the identification of $\mathcal{F}_u$ with $\mathcal{F}$ given by Theorem 3.22, part (i). Therefore ι is smooth.

Vice versa, suppose now that ι is smooth, we show that (47) must hold. Let $\mathcal{F}_u$ be a normal form for $\mathcal{F}$ and let $(V, u) \in \mathcal{U}_Z$ be the pair defining $\mathcal{F}_u$. Then we have that

$$\iota|_{Y^+} : (b, y) \mapsto (b, -y). \tag{49}$$

Moreover u satisfies

$$(u \circ \iota)|_{Y^-} = u|_{Y^-} \tag{50}$$

and so also the Taylor expansions with respect to b_1 evaluated at $b_1 = 0$ of the two sides of the above identity must coincide. This provides a certain relation which must be satisfied by the coefficients of the Taylor expansion of u. Notice that the coefficients of the Taylor expansion of $u \circ \iota$ only depend on the coefficients of the Taylor expansions of u and ι, but the Taylor coefficients of ι are the same as those of the map $(b, y) \mapsto (b, -y)$ since (49) holds, therefore the relation among the Taylor coefficients of u implied by Eq. (50) is the same as the one obtained assuming (50) holds with ι satisfying $\iota(b, y) = (b, -y)$ for all $(b, y) \in Y$. We now compute this relation in terms of the sequence $\ell = \{\ell_k\}_{k\in\mathbb{N}}$. Given that $\iota : (b, y) \mapsto (b, -y)$, it is easy to see $\iota(L_{r,b}) = L_{r,b}$ if and only if the one parameter family $\ell(r)$ obtained from u satisfies (48). Therefore the coefficients ℓ_k of the Taylor series (44) must satisfy (47). □

3.5.1. Non-proper stitched fibrations

Let X be a smooth symplectic 6-manifold together with a smooth Hamiltonian S^1 action with moment map $\mu : X \to \mathbb{R}$. Assume μ has exactly one critical value $0 \in \mathbb{R}$ and a codimension four submanifold $\Sigma = \operatorname{Crit}\mu$. Let M be a smooth 2-dimensional manifold and let $B \subseteq \mathbb{R} \times M$ be a contractible open neighborhood of a point $(0, m) \in \mathbb{R} \times M$. Let $\Gamma = B \cap (\{0\} \times M)$. As usual we define $Z = \mu^{-1}(0)$ and $\bar{Z}$ the S^1 quotient of Z and $X^+ = \{\mu \geqslant 0\}$, $X^- = \{\mu \leqslant 0\}$.

We consider fibrations satisfying the following:

Assumption 3.25. The map $f : X \to B$ is a topological T^3 fibration with discriminant locus $\Delta \subset \Gamma$ such that $f(\Sigma) = \Delta$ satisfying

(a) (X, ω, f, B) is topologically conjugate to a generic-singular fibration.
(b) There is a continuous S^1 invariant map $G : X \to M$ such that
  (i) if $G^{\pm} = G|_{X^{\pm}}$ then G^+ and G^- are restrictions of C^∞ maps on X;
  (ii) f can be written as $f = (\mu, G)$ and f restricted to $X^{\pm}$ is a proper map with connected Lagrangian fibers.
(c) There is a connected, S^1 invariant, open neighborhood $\mathfrak{U} \subseteq X$ of Σ such that $f(\mathfrak{U}) = B$ and such that $f_{\mathfrak{U}} = f|_{\mathfrak{U}}$ is a C^∞ map with non-degenerate singular points.

This kind of fibrations are studied in [8]. Examples of fibrations satisfying the above properties can be obtained from the fibration as in Example 3.10, after a suitable perturbation of f near the portion of Σ projecting onto the codimension two part of Δ (we will recall this smoothing in the next section). Clearly, the piecewise smoothness occurs along cylindrical portions of fibers contained in $\mu^{-1}(0)$. Now we recall some of the basic facts of fibrations satisfying Assumption 3.25.

One can construct fibrations of this type as follows. Over $B = D \times (0, 1)$ consider periods given by

$$\begin{aligned} \lambda_1 &= 2\pi\, db_1, \\ \lambda_2 &= dH + \lambda_0, \\ \lambda_3 &= db_3, \end{aligned}$$

where H is a smooth function and $\lambda_0 = \arg(b_1 + ib_2)\, db_1 + \log|b_1 + ib_2|\, db_2$. If Λ_H denotes the lattice generated by these periods, let $X^\# = T^*B/\Lambda_H$ and denote by $\pi^\#$ the projection. Now, in Section 3.2 we argued that the map $\pi^\#$ can be extended to a proper map π to give a smooth proper Lagrangian fibration of generic-singular type. This can be achieved by gluing $\mathfrak{U}$ to $X^\#$. The moment map of the S^1 action is, as usual, b_1. Let $X^{\pm}$, $B^{\pm}$, Z and $\bar{Z}$ be defined as usual.

Now let $\mathfrak{W}$ and $\mathfrak{U}$ be open S^1 invariant neighborhoods of the critical set, such that $\bar{\mathfrak{W}} \subseteq \mathfrak{U}$. Then, $X^\circ = X - \bar{\mathfrak{W}}$ can be viewed as an open neighborhood of the zero section of $X^\#$ over which the restriction π° of $\pi^\#$ is a (topologically trivial) Lagrangian open cylinder fibration (the fibers are homeomorphic to $T^2 \times \mathbb{R}$). The set $\mathfrak{U} - \bar{\mathfrak{W}}$ covers the two ends of each fiber. Suppose $u : X^\circ \to B$ is another Lagrangian open cylinder fibration, whose fibers coincide with the fibers of π° over $\mathfrak{U} - \bar{\mathfrak{W}}$, thus the fibers of u are compactly supported perturbations of the fibers of π°. If we also assume that $u|_Z = \pi^\circ$ then we can define:

$$f_u^\circ = \begin{cases} \pi^\circ & \text{on } X^+, \\ u & \text{on } X^-. \end{cases} \tag{51}$$

The map f_u° defines a piecewise smooth Lagrangian open cylinder fibration whose fibers coincide with those of π° on $\mathfrak{U} - \bar{\mathfrak{W}}$. We can therefore glue back the critical set and define the following proper piecewise smooth Lagrangian fibration:

$$f_{u,H} = \begin{cases} \pi & \text{on } \mathfrak{U}, \\ f_u^\circ & \text{on } X^\circ. \end{cases} \tag{52}$$

Clearly $f_{u,H} : X \to B$ is well defined and satisfies Assumption 3.25. In [8] it is proved that any fibration satisfying Assumption 3.25 is fiberwise symplectomorphic to $f_{u,H}$ for a certain choice of u and H and therefore $f_{u,H}$ defines a normal form. Moreover the invariants that classify such fibrations are given by triples $(Z_H^\#, \ell, H_\Delta)$, where $Z_H^\#$ is the zero level set of the S^1 moment map (restricted to $X^\#$), H_Δ is the germ of H along the discriminant Δ, and ℓ a sequence of fiberwise closed sections of $\mathcal{L}^*$, where $\mathcal{L} = \ker \bar{\pi}_*^\#$. In this case, each ℓ_k is a form with compact support inside cylindrical portions of the fibers. These invariants classify fibrations as in Assumption 3.25. For the details we refer the reader to [8, Section 6].

Theorem 3.26. *A piecewise smooth fibration $f : X \to B$ satisfying Assumption* 3.25 *with invariants $(Z_H^\#, \{\ell_k\}, H_\Delta)$ and a section σ has a unique smooth fiber-preserving anti-symplectic involution $\iota : X \to X$ fixing σ if and only if the invariants satisfy $\bar{\iota}^* \ell_k = -\ell_k$.*

Proof. Given a normal form $(X, f_{u,H})$ for a fibration satisfying Assumption 3.25, consider the anti-symplectic involution $\iota : X \to X$ constructed in Section 3.2 preserving the fibers of the smooth fibration π. Using the same arguments as in the proof of Proposition 3.3, one can prove that ι also preserves the fibers of $f_{u,H}$ if the invariants satisfy $\bar{\iota}^* \ell_k = -\ell_k$. Vice versa given invariants satisfying this identity one can construct a fibration $f_{u,H}$ having these invariants and whose fibers are preserved by ι. □

3.6. Negative fibration

Let $f : X \to B$ be the piecewise smooth fibration in Example 3.11. Recall that $X \subset \mathbb{C}^3$ and the construction of f makes use of a choice of symplectomorphism Φ as in (41) giving rise to a fibration whose discriminant locus Δ is the amoeba of Fig. 2 after its legs are pinched down to a line. We proved this fibration is invariant under the standard conjugation on $\mathbb{C}^3$. The fixed locus consists of 5 connected components, two of which are sections. The section σ fixed by the involution is given by the choice of any of such sections (cf. Remark 3.12).

In [8, Theorem 7.3], the first two authors propose a method to make the aforementioned f smoother, obtaining examples of fibrations of negative type.

Definition 3.27. Let X be a 6-dimensional symplectic manifold and $B \subseteq \mathbb{R}^3$ an open subset. A piecewise smooth Lagrangian fibration $f : X \to B$ is called a *Lagrangian negative fibration* if it satisfies the following properties:

(i) $f : X \to B$ is topologically conjugate to the fibration of Example 3.11, i.e. they define the same germ;
(ii) there exists a submanifold with boundary $D \subset B$, homeomorphic to a closed disc in $\mathbb{R}^2$, such that $\Delta \cap (B - D)$ consists of three one-dimensional disjoint segments (the legs of Δ) and f is smooth when restricted to $X - f^{-1}(D)$;
(iii) on $B - (D \cup \Delta)$, the affine structure induced by the fibration map is simple;
(iv) f has a section σ such that $\sigma(B)$ is a smooth Lagrangian submanifold disjoint from the singular locus $\Sigma \subset X$ of f.

We now present an abbreviated description of the smoothing process that leads to the proof of existence of negative fibrations (details cf. [8, Section 7]) and show that the existence of an anti-symplectic involution survives this process.

Let $f : X \to B$ be the fibration in Example 3.11 and a section as in Remark 3.12. Recall that the anti-symplectic involution preserving f is just conjugation. Let b_1, b_2, b_3 standard coordinates in the base $B \subseteq \mathbb{R}^3$. Then Δ is contained in the plane $b_1 = 0$. Let $\Sigma \subset X$ be the critical surface — i.e. the locus where vanishing cycles collapse, a pair of pants projecting onto Δ under f. For positive $M \in \mathbb{R}$, let $\Delta_{h,M} = \Delta \cap \{b_2 \leqslant -M\}$. For M large enough, $\Delta_{h,M}$ is one-dimensional — i.e. the thin part of the horizontal leg — and let $\Sigma_{h,M}$ be the portion of Σ projecting onto $\Delta_{h,M}$. For the following analysis, it is convenient to use S^1-invariant coordinates, $t = \mu$, $u_1 = z_1 z_2$ and $u_2 = z_3$. Then u_1 and u_2 can be thought of as coordinates on each reduced space $\mu^{-1}(t)/S^1$. On a suitable small neighborhood $N_{h,M}$ of $\Sigma_{h,M}$ the restriction of f to $N_{h,M}$ can be explicitly written as:

$$f = (\mu, G_t)$$

where

$$G_t(u_1, u_2) = \left(\log|u_2|, \log\left| \frac{u_1}{\sqrt{|t| + \sqrt{t^2 + |u_1|^2}}} - 1 \right| \right). \tag{53}$$

Clearly, f fails to be smooth at $t = 0$ since G_t does. In [8] it is shown that one can perturb f on $N_{h,M}$ by replacing G_t with a map of type:

$$\tilde{G}_t = \left(\log|u_2|, \log\left| \frac{u_1}{\rho(|u_1|, t, |u_2|^2)} - 1 \right| \right).$$

Here ρ is chosen so that $\tilde{G}_t$ coincides with G_t away from $N_{h,M}$ and it is smooth on $N_{h,M}$ (details cf. [8, Lemma 7.4]). It is clear that $\tilde{G}_t$ is invariant under the involution on $\mathbb{C}^2$, i.e. under conjugation. The perturbation $\tilde{f} = (\mu, \tilde{G}_t)$ of f is therefore invariant under the standard involution on $\mathbb{C}^3$.

Similarly, one perturbs f along small neighborhoods of $N_{v,M}$, and $N_{d,M}$ of $\Sigma_{v,M}$ and $\Sigma_{d,M}$ projecting onto $B_{v,M}$ and $B_{d,M}$ open neighborhoods of the vertical and diagonal legs, respectively. This produces an involution-invariant fibration $\tilde{f}$.

Now the smoothing needs to be extended to $X_{h,M} := \tilde{f}^{-1}(B_{h,M})$. First observe that the restriction of $\tilde{f}$ to $X_{h,M}$ is a piecewise smooth fibration satisfying the hypothesis of Theorem 3.26. Since $X_{h,M}$ has a fiber-preserving involution ι fixing a section, if $\{\ell_k\}$ are the invariants of $\tilde{f}$, then $\bar{\iota}^*\ell_k = -\ell_k$.

Now in [8, Lemma 7.6] it is shown that for some positive $m > M$ there is a neighborhood $B_{h,m} \subset B_{h,M} \cap \{b_2 \leqslant -m\}$ and a perturbation of $\tilde{f}$, making it smooth on $X_{h,m} := \tilde{f}^{-1}(B_{h,m})$. This is achieved by perturbing the invariants ℓ_k of $\tilde{f}$ in such a way that ℓ_k vanish identically on $B_{h,m} \cap \{b_1 = 0\}$. The perturbed invariants are (with a slight abuse of notation) of the form $\nu\ell_k$, where ν is a bump function on $B_{h,M} \cap \{b_1 = 0\}$ vanishing identically on $B_{h,m} \cap \{b_1 = 0\}$. Since ν is a function depending only on coordinates of the base, it is clear that $\bar{\iota}^*(\nu\ell_k) = -\nu\bar{\ell}_k$. Therefore the resulting fibration after this perturbation is invariant under ι and the section resulting from this perturbation is fixed by ι.

One may proceed in an analogous way with the other two legs. This gives a piecewise smooth fibration which is smooth over large open neighborhoods $B_{h,m}$, $B_{v,m}$, $B_{d,m}$, of $\Delta_{h,m}$, $\Delta_{v,m}$, $\Delta_{d,m}$, respectively, and a smooth fiber-preserving anti-symplectic involution defined on the total space of the fibration.

Finally, to produce a fibration satisfying the properties of Definition 3.27, one needs to perturb the fibration away from a (planar) tubular neighborhood N of Δ. Observe that the complement of Δ in the plane $\{b_1 = 0\}$ consists of three connected components, Γ_c, Γ_d, Γ_e which are the walls of three stitched fibrations f_c, f_d, f_e, each fibration being the restriction of the fibration obtained in the previous paragraph. If ℓ^c, ℓ^d and ℓ^e are the corresponding invariants, then Theorem 3.24 implies that $\bar{\iota}^*\ell_k^c = -\ell_k^c$, $\bar{\iota}^*\ell_k^d = -\ell_k^d$ and $\bar{\iota}^*\ell_k^e = -\ell_k^e$. Now, in [8, Lemma 7.12], it is shown that f_c, can be made smooth away from $N \cap \Gamma_c$. As before, this is accomplished after deforming ℓ_k^c to $\tilde{\ell}_k^c = \rho\ell_k^c$ for a suitably chosen bump function ρ on Γ_c. Again, being ρ dependent on coordinates on the base, implies that the resulting fibration is still ι-invariant and the resulting section fixed by ι. One proceeds in a similar way with f_d and f_e. This completes the required smoothing of f.

Observe that if D is the region over which f fails to be smooth, there are regions $D' \subset D$ and $B' \subset B$ such that $B' \cap \{b_1 = 0\} \subset D'$ where the section σ obtained after the smoothing of f remains unchanged, i.e. $\sigma(B')$ coincides with the section in Remark 3.12. It also follows that $\sigma(B)$ is smooth. This completes the proof the following:

Theorem 3.28. *Let $f : X \to B$ be a Lagrangian negative fibration. Then there is a Lagrangian section σ not intersecting the singular locus $\Sigma \subset X$ of f and unique smooth fiber preserving anti-symplectic involution $\iota_{f,\sigma}$ of X preserving the fibers of f and fixing σ.*

Remark 3.29. Let σ_1 and σ_2 Lagrangian sections of a negative fibration $f : X \to B$ and $B' \subset B$ and $D' \subset D$ as above. Since f is stitched along $f^{-1}(D) \subset X$, it follows from Remark 3.23, that, in general, there is no symplectomorphism t of X such that $f \circ t$ and $t \circ \sigma_1 = \sigma_2$. This contrasts with the nodal, generic-singular and positive models for which t always exists.

Lemma 3.30. *Let σ_1 and σ_2 be sections of a negative fibration $f : X \to B$ and $D \subset B \cap \{b_1 = 0\}$ the locus over which f fails to be smooth. If there exists an open neighborhood $B' \subset B$ of D such that $\sigma_1|_{B'} = \sigma_2|_{B'}$, then there is a unique symplectomorphism t of X such that $f \circ t = f$ and $t \circ \sigma_1 = \sigma_2$.*

Proof. On $X_\circ = f^{-1}(B - B')$, the fibration is smooth. Corollary 1.9 and Lemma 3.4 give a unique symplectomorphism $t_\circ$ of $X_\circ$ sending $\sigma_1|_{B-B'}$ to $\sigma_2|_{B-B'}$. Extending $t_\circ$ to X as the identity map on $X - X_\circ$ gives a smooth symplectomorphism t with the required properties. □

4. Global existence

Let $(B, \Delta, \mathscr{A})$ be a compact simple integral affine manifold with singularities. Let $\mathcal{N}$ be the set of negative vertices of Δ and let $(\Delta_\blacklozenge, \{D_p\}_{p\in\mathcal{N}})$ be a localized thickening and let $(B_\blacklozenge, \Delta_\blacklozenge, \mathscr{A}_\blacklozenge)$ be the integral affine manifold as in Section 2.1. Then there is a smooth symplectic manifold

$$X_\blacklozenge = T^*B_\blacklozenge/\Lambda_\blacklozenge$$

where $\Lambda_\blacklozenge$ is the period lattice induced by $\mathscr{A}|_{B_\blacklozenge}$, and a Lagrangian submersion:

$$f_\blacklozenge : X_\blacklozenge \to B_\blacklozenge.$$

Notice that if $B_0 = B - \Delta$, Λ_0 is the lattice induced by $\mathscr{A}$, $X_0 = T^*B_0/\Lambda_0$, and $f_0 : X_0 \to B_0$ the standard projection, then $X_\blacklozenge \subset X_0$ and $f_\blacklozenge = f_0|_{X_\blacklozenge}$.

Let σ_0 be a section of f_0 which can be taken to be induced by the zero section on T^*B_0. Then, Corollary 1.8 implies there is a unique fiber preserving anti-symplectic involution ϕ_0 of X_0 also preserving σ_0. With abuse of notation, denote by σ_0 and ϕ_0 their restrictions to $X_\blacklozenge$ and $B_\blacklozenge$ respectively. Theorem 2.1 gives a class $\mathcal{C}$ of fibrations $f : X \to B$ where X is the compact symplectic manifold obtained from $X_\blacklozenge$ after gluing models of generic, positive and negative fibrations as in Section 3 over Δ and matching local sections of each local model with σ_0. The latter provides the fibration with a section σ.

4.1. The class $\mathfrak{C}$

We now impose extra conditions on the sections of fibrations of class $\mathcal{C}$.

Definition 4.1. Let $f : X \to B$ be a fibration of class $\mathcal{C}$ with a section σ such that $\sigma(B) \cap \operatorname{Crit} f = \emptyset$ where $\operatorname{Crit} f \subset X$ is the singular set of f. Assume that identifications of neighborhoods of singular fibers with the local models of Section 3 are fixed. For each negative vertex $p \in \mathcal{N}$, let $B_p \subset B$ be a small open neighborhood of p such that $D_p \subset B_p$, where D_p is the locus over which f is piecewise smooth. Let $f^- : X^- \to B^-$ be the model for the negative fibration and let σ^- be a choice of section of f^- fixed by the local anti-symplectic involution as in Theorem 3.28. We say that σ is of class $\mathfrak{C}$ if for each $p \in \mathcal{N}$, the restriction of σ to B_p coincides with σ^-.

Notice that the definition of $\mathfrak{C}$ clearly depends on the choice of σ^- and $\{B_p\}_{p\in\mathcal{N}}$. Notice also that another section σ' is of class $\mathfrak{C}$ if and only if σ' coincides with σ when restricted to each B_p.

4.2. Proof of Theorem 1.2

Let $f : X \to B$ a fibration of class $\mathcal{C}$ with two sections $\sigma_1, \sigma_2 \in \mathfrak{C}$. Let $X_\blacklozenge \subset X$ and $f_\blacklozenge : X_\blacklozenge \to B_\blacklozenge$ be the Lagrangian submersion as above. By Corollary 1.9 there is a unique fiber-preserving symplectomorphism $t_0 : X_\blacklozenge \to X_\blacklozenge$ sending σ_1 to σ_2. We will show that t_0 extends to X.

Since $\sigma_1, \sigma_2 \in \mathfrak{C}$, it follows from Definition 4.1 that for each negative vertex $p \in \mathcal{N}$, $\sigma_1|_{B_p} = \sigma_2|_{B_p}$. Trivially, there is a unique local fiber-preserving symplectomorphism sending the restriction $\sigma_1|_{B_p}$ to the restriction $\sigma_2|_{B_p}$. Similarly, Lemma 3.8 guarantees that for each positive vertex v of Δ, there is an open neighborhood $B_v \subset B$ of v and a unique local fiber-preserving symplectomorphism sending the restriction $\sigma_1|_{B_v}$ to the restriction $\sigma_2|_{B_v}$. For the edges of Δ one applies Lemma 3.4 analogously. Each of these local symplectomorphisms provide a local extension of t_0 to X. By uniqueness, these extensions glue together along common intersection, giving a unique extension t of t_0 to X. Details are left to the reader.

Remark 4.2. Notice that due to the piecewise smoothness of $f : X \to B$ of class $\mathcal{C}$, if σ_1 is of class $\mathfrak{C}$ but σ_2 is not, it cannot follow that there is a symplectomorphism t of X such that $f \circ t = f$ and $t \circ \sigma_1 = \sigma_2$ (cf. Remark 3.29).

4.3. Proof of Theorem 1.1

It is enough to find a fiber-preserving anti-symplectic involution ϕ fixing one section $\sigma' \in \mathfrak{C}$. In fact, if σ is any other section in $\mathfrak{C}$ and t is the symplectomorphism taking σ to σ' constructed in Theorem 1.2, then $\phi_{f,\sigma} = t^{-1} \circ \phi \circ t$ is the anti-symplectic involution fixing σ.

Consider as above, the anti-symplectic involution ϕ_0 of $X_\blacklozenge$ fixing the section σ_0. We need to show that the section σ_0 extends to a section $\sigma' \in \mathfrak{C}$ and the involution ϕ_0 of X_0 extends to a smooth fiber-preserving anti-symplectic involution ϕ of X fixing σ'. The proof follows immediately from Theorem 2.1 and the results of Section 3. Let us denote by $f^\nu : X^\nu \to B^\nu$ a fibration of either generic-singular, positive or negative type, used in the compactification as in Theorem 2.1. This presumes that each affine base, $B_0^\nu = B^\nu - \Delta^\nu$, is locally affine isomorphic to $U_0 = U - \Delta \cap U$, where U is a suitable neighborhood of $x \in \Delta$, and x is either an edge point, a positive or a negative vertex. If ν is either generic or positive, we let σ^ν be any choice of a section of f^ν fixed by the local anti-symplectic involution, not intersecting the critical locus of f^ν. If ν is a negative vertex, the choice of σ^ν is σ^- as in Definition 4.1.

Then, the affine isomorphism induces a symplectomorphism of bundles Φ^ν and a commuting diagram

$$\begin{tikzcd} X_0^\nu \arrow[r, "\Phi^\nu"] \arrow[d, "f^\nu"'] & f_\blacklozenge^{-1}(U_0) \arrow[d, "f_\blacklozenge"] \\ B_0^\nu \arrow[r, "A^\nu"] & U_0 \end{tikzcd} \tag{54}$$

where $\Phi^\nu(\sigma^\nu|_{B_0^\nu}) = \sigma_0 \circ A^\nu$ and A^ν extends continuously to B^ν. Then X^ν is glued to $X_\blacklozenge$ over U using Φ^ν. Moreover, this gluing extends σ_0 to a smooth section on U. The gluing of two generic-singular fibrations along common edges in Δ requires taking care of further technicalities, as it involves gluing along singular fibers. A smooth symplectic deformation of the fibrations along a common intersection may be required but, in any case, two generic-singular fibrations can be glued matching its corresponding prescribed Lagrangian sections (cf. [8, Proposition 4.18]).

Now $f_\blacklozenge$ and each f_ν carry a unique fiber preserving smooth anti-symplectic involution ϕ_0 and ϕ^ν fixing σ_0 and σ^ν, respectively. Since σ_0 and σ^ν coincide over U_0, it follows that ϕ_0 and ϕ^ν coincide along $f_\blacklozenge^{-1}(U_0)$. Then ϕ_0 extends smoothly to $f_\blacklozenge^{-1}(U)$. Repeating this process for a suitable open cover $\{U\}$ of Δ produces the required section σ' of f, and the extension ϕ of ϕ_0. This completes the proof of Theorem 1.1. By construction $\sigma \in \mathfrak{C}$.

5. Examples

The same arguments discussed in Section 4 apply in dimension $n = 2$:

Theorem 5.1. *Let $(B, \Delta, \mathscr{A})$ be a 2-dimensional simple affine manifold with singularities and let $f : X_0 \to B_0 = B - \Delta$ be the Lagrangian submersion of $X_0 = T^*B_0/\Lambda$ onto B_0. Then*

- (i) *There is a symplectic manifold X and a Lagrangian fibration $f : X \to B$ such that $f|_{X_0} = f_0$.*
- (ii) *If a Lagrangian section σ is specified which avoids the critical points, then there is a unique fiber preserving anti-symplectic involution $\phi_{f,\sigma}$ fixing σ.*
- (iii) *If two Lagrangian sections σ_1 and σ_2 are specified (both avoiding the critical points), there is a unique symplectomorphism $t : X \to X$ such that $f \circ t = f$ and $t \circ \sigma_1 = \sigma_2$.*

The first claim is the content of [8, Theorem 3.22], while the second and third claims are new. The proof of (ii) is a verbatim of the one in dimension 3, where one can use the model for a focus–focus fibration of Section 3.1 together with the given anti-symplectic involution. The proof of (iii) is the same as the proof of Theorem 1.2.

5.1. The K3

Starting with explicit examples of integral affine base $B \cong S^2$ with 24 singularities Leung and Symington [24] illustrate how part (i) of Theorem 5.1 can be used to build well known Lagrangian fibrations on a symplectic 4-manifold $X \cong$ K3 with a section (see also [8, Example 3.16]). The construction involves making several choices, which produce different germs of Lagrangian fibrations. So even though the compactification X is the same (modulo symplectomorphism) regardless of the choices made, there are actually infinitely many germs of Lagrangian fibrations with the same topology (cf. [8, Corollary 3.24]). Given a choice of such fibration germ f, part (ii) gives a unique fiber preserving anti-symplectic involution.

In this case, the fixed locus of ϕ_f is a Lagrangian submanifold with 2 connected components: one of them is a sphere (i.e. the section) and the other is a genus $g = 10$ surface Σ which is a $3:1$ branch cover of S^2, with 24 branch points.

5.2. Almost toric 4-manifolds

Symington and Leung [24] propose a class of symplectic 4-manifolds with Lagrangian fibrations having focus–focus and toric singular fibers, called *almost toric*. Within this class, the integral affine bases which are simple (*simple* in the sense of Theorem 5.1) are the disc D^2, the cylinder $S^1 \times I$, the Klein bottle, the sphere S^2 and $\mathbb{RP}^2$. Namely, these are the only cases that have singularities of nodal type. Theorem 5.1 equips each of the corresponding fibrations with fiber-preserving anti-symplectic involutions. For instance, the Enriques surface is equipped with a Lagrangian fibration over $\mathbb{RP}^2$ with 12 focus–focus singularities and a fiber preserving anti-symplectic involution. The base S^2 gives a K3 surface discussed above.

5.3. The quintic

Starting with an explicit example of affine 3-manifold with singularities proposed by Gross [19, Example 4.3], the first two authors use Theorem 2.1 to produce a symplectic 6-manifold X homeomorphic to a smooth quintic 3-fold. Now Theorem 1.1 shows that such manifold has a fiber preserving anti-symplectic involution.

5.4. Mirror pairs

The example above generalizes to a much wider class. When B is an integral affine 3-manifold arising from toric degeneration in the sense of Gross and Siebert, Theorem 2.1 produces pairs of SYZ dual Lagrangian fibrations, with total spaces homeomorphic to mirror pairs of Calabi–Yau manifolds (cf. [8] for details). Now Theorem 1.1 equips these pairs of symplectic 6-manifolds with fiber-preserving anti-symplectic involutions fixing a section.

In the examples discussed above, the fixed locus set Σ appears to have nice topological properties. For instance, for X Calabi–Yau, there is an intriguing relation between the mod 2 cohomology of Σ and the Hodge numbers of X. These properties are being further investigated in [6].

6. Fiber-preserving anti-symplectomorphisms

In this section, we prove Proposition 1.10.

In the following lemmas, $f : X \to B$ is a Lagrangian fibration that is a smooth submersion, and σ is a smooth Lagrangian section of f. We denote by Λ the lattice bundle and by $\Theta : T^*B/\Lambda \to X$ the symplectomorphism of Proposition 1.7 applied to σ. We denote by Z the zero sections of T^*B and T^*B/Λ, and we denote by π the canonical projections to B. We denote by $-\mathrm{Id}$ the anti-symplectomorphisms of T^*B and T^*B/Λ given by negative the identity map on each fiber.

Let η be a 1-form on B. We define a symplectomorphism $T_\eta : T^*B \to T^*B$ by

$$T_\eta(p,\xi) = \big(p, \xi + \eta(p)\big), \quad \forall p \in B,\ \xi \in T_p^*B.$$

We also denote by T_η the symplectomorphism that T_η induces on T^*B/Λ.

Lemma 6.1. *Assume that $\pi_1(B) = \{1\}$. Let ϕ be an anti-symplectomorphism of X such that $f \circ \phi = \phi$. Then $\phi^2 = \mathrm{Id}_X$. In particular,*

$$\phi = \Theta \circ T_\eta \circ (-\mathrm{Id}) \circ \Theta^{-1}. \tag{55}$$

Proof. Define

$$Z' = \Theta^{-1} \circ \phi \circ \Theta \circ Z.$$

It is easy to see that Z' is a Lagrangian section of T^*B/Λ. Since $\pi_1(B) = \{1\}$, we may lift Z' to a Lagrangian section $\widetilde{Z}'$ of T^*B. Let η be the one form on B such that $\widetilde{Z}'$ is its graph. Clearly, $\pi \circ T_\eta = \pi$ and $T_\eta \circ Z = Z'$. By the uniqueness claim of Corollary 1.9 applied to the Lagrangian fibration $\pi : T^*B/\Lambda \to B$, we conclude that

$$T_\eta = \Theta^{-1} \circ \phi \circ \Theta \circ (-\mathrm{Id}).$$

Formula (55) follows. Observe that

$$T_\eta \circ (-\mathrm{Id}) = (-\mathrm{Id}) \circ T_{-\eta}, \quad T_{-\eta} = T_\eta^{-1}.$$

Consequently, $(T_\eta \circ (-\mathrm{Id}))^2 = \mathrm{Id}_{T^*B/\Lambda}$. The lemma follows. □

We omit the proof of the following lemma since it is similar and we do not use it.

Lemma 6.2. *Assume that $\pi_1(B) = \{1\}$. Let t be a symplectomorphism of X such that $f \circ t = f$. There exists a 1-form on B such that*

$$t = \Theta \circ T_\eta \circ \Theta^{-1}.$$

Proof of Proposition 1.10. Since smooth fibers are dense and the claim is a closed condition, we may assume without loss of generality that $f : X \to B$ is a smooth submersion. Since ϕ_f preserves fibers of f, the claim is local on the base B. So, without loss of generality we focus on the special case when B is the n-disk. Eq. (14) follows from Lemma 6.1. Eq. (15) follows formally from Eq. (14). Indeed, $\phi_f \circ t$ is an anti-symplectomorphism such that $f \circ \phi_f \circ t = f$. So, we conclude $\phi_f \circ t \circ \phi_f \circ t = \mathrm{Id}_X$, which implies Eq. (15). □

7. Gradings

In this section we will explain the definition of the grading of a Lagrangian submanifold $L \subset X$ in the special case where X is a symplectic Calabi–Yau manifold. Then we will assume that $f : X \to B$ is a special Lagrangian fibration and $\phi_{f,\sigma}$ is anti-holomorphic as well as anti-symplectic. In this case, we conclude that $\mathcal{I}_{f,\sigma}$ shifts the natural grading on the fibers of f by $\dim_{\mathbb{C}} X$.

The notion of a grading for a Lagrangian submanifold was introduced by Kontsevich [23]. Here we follow a slightly modified version of the exposition of [34]. We use the generalized definition of special Lagrangian submanifolds due to Salur [28] that applies to symplectic Calabi–Yau manifolds that may not have an integrable complex structure.

Let X be a symplectic manifold with symplectic form ω. An almost complex structure on J on X is said to be ω-tame if

$$\omega(\xi, J\xi) > 0$$

for all $\xi \neq 0$. We define the first Chern class $c_1(TX)$ by choosing an ω-tame almost complex structure on X. The definition of c_1 only depends on ω because the space of ω-tame almost complex structures is contractible.

From now on we assume that (X, ω) is a symplectic $2n$-real-dimensional Calabi–Yau manifold, i.e. that $c_1(TX) = 0$. We fix an ω-tame almost complex structure J on X. Use J to decompose complex valued differential forms on X by type. Fix a nowhere vanishing $(n,0)$-form Ω on X. The existence of Ω is guaranteed by the Calabi–Yau condition. We emphasize that we do not require Ω to be closed. Finally, we define the metric g by

$$g(\xi, \eta) = \frac{\omega(\xi, J\eta) + \omega(\eta, J\xi)}{2}.$$

Let $L \subset X$ be a Lagrangian submanifold. A small generalization of arguments of [20] shows that

$$\Omega|_L = \psi e^{\pi i \theta} vol_g, \tag{56}$$

where ψ is a strictly positive real-valued function, θ is an S^1-valued function, and vol_g is the volume form of L induced by g. If the Maslov class of L vanishes, then θ can be lifted to a real valued function. The choice of a real-valued lift of θ, which we also denote by θ, is a grading of L. A graded Lagrangian submanifold is called special Lagrangian if the grading θ is constant.

For the rest of this section, we assume that $f : X \to B$ is a special Lagrangian fibration. That is, each fiber of f contains a relatively open dense subset that is a smooth special Lagrangian submanifold of X. We assume that f has a section σ, and we assume that X has an anti-symplectic involution ϕ satisfying conditions (11). We assume also that ϕ is anti-J-holomorphic and

$$\phi^* \Omega = \bar{\Omega}. \tag{57}$$

In Lemma 7.5 below, we show that assumption (57) is not hard to satisfy given the previous assumptions. Moreover, we have the following lemma.

Lemma 7.1. *(See [10].) Let the Lagrangian fibration $f : X \to B$ be a smooth submersion. For each J, there exists a choice of Ω such that f is special Lagrangian.*

Let θ_σ denote a grading on the Lagrangian submanifold given by the section σ and let θ_y denote a grading on the fiber L_y of f.

Lemma 7.2. *The gradings θ_σ and θ_y must satisfy*

$$\theta_\sigma \in \mathbb{Z}, \quad \theta_y \in n/2 + \mathbb{Z}.$$

Proof. According to Corollary 1.8, ϕ acts on each smooth fiber L_y of f by a diffeomorphism of sign $(-1)^n$. So, equating the phase on each side of (57) and using the fact that θ_y is constant on L_y, we have

$$e^{\pi i \theta_y} = (-1)^n e^{-\pi i \theta_y}.$$

We conclude that $\theta_y \in n/2 + \mathbb{Z}$. On the other hand, ϕ acts on σ by the identity map. So, the same argument implies that $\theta_\sigma \in \mathbb{Z}$. $\square$

As noted previously, the mirror correspondence maps σ along with the appropriate local system, spin structure and grading to the structure sheaf $\mathcal{O}_Y$. We would like to identify the choice of grading θ_σ that corresponds to $\mathcal{O}_Y$. Since $\mathcal{O}_Y$ is fixed under $\mathcal{D}$, for consistency of Conjecture 1.3, we must assume that $\mathcal{I}_{f,\sigma}(\sigma, \theta_\sigma) = (\sigma, \theta_\sigma)$. It follows that

$$\theta_\sigma = -\theta_\sigma = 0.$$

To fully determine the choice of θ_y that makes L_y into the mirror of $\mathcal{O}_y$, we employ the mirror correspondence once again. Since σ is sent by the mirror correspondence to the structure sheaf $\mathcal{O}_Y$, we should have an isomorphism of graded vector spaces

$$m_{f,\sigma} : HF^*(\sigma, L_y) \xrightarrow{\sim} R\underline{Hom}(\mathcal{O}_Y, \mathcal{O}_y) \simeq \mathbb{C},$$

where the grading of $\mathbb{C}$ is 0. We will deduce θ_y from the definition of the grading on $HF^*(\sigma, L_y)$.

We recall the definition of the grading on HF^*. Let $L_1, L_2 \subset X$, be two transversely intersecting graded Lagrangian submanifolds with gradings θ_1, θ_2. By definition, $HF^*(L_1, L_2)$ is the cohomology of the complex $CF^*(L_1, L_2)$, which is generated by the intersection points of L_1 and L_2. The grading of a point $p \in L_1 \cap L_2$ is defined as follows. Identify T_pX with $\mathbb{C}^n$ by a complex linear transformation t taking L_1 to $\mathbb{R}^n \subset \mathbb{C}^n$ and L_2 to $M \cdot \mathbb{R}^n$. Take M to be unitary. So, it is conjugate to a diagonal matrix of the form

$$M = \begin{pmatrix} e^{i\pi\alpha_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{i\pi\alpha_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{i\pi\alpha_3} & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{i\pi\alpha_n} \end{pmatrix}$$

where $\alpha_i \in (0, 1)$. Set $\alpha = \sum_i \alpha_i$. Define the grading of p to be

$$ind_p(L_1, L_2) = \alpha - \theta_2(p) + \theta_1(p). \tag{58}$$

Lemma 7.3. *Let L_y be a smooth fiber. Assuming $\theta_\sigma \equiv 0$, and $HF^*(\sigma, L_y)$ is a one-dimensional vector space of grading* 0*, it follows that $\theta_y = n/2$.*

Proof. By definition of a section, there is a unique intersection point $p \in \sigma \cap L_y$. Let $t : T_pX \to \mathbb{C}^n$ such that $t(T_p\sigma) = \mathbb{R}^n$. It follows from Corollary 1.8 that we can choose t so that $t(L_y) = i\mathbb{R}^n$. Then for $i = 1, \ldots, n$, we have $\alpha_i = 1/2$. So, $\alpha = n/2$. Rearranging Eq. (58) we obtain

$$\theta_y(p) = \alpha - ind_p(\sigma, L_y) + \theta_\sigma(p) = \alpha = n/2. \qquad \square$$

It follows that $\mathcal{I}_{f,\sigma}$ shifts θ_y by $n = \dim_{\mathbb{C}} X$.

Remark 7.4. The fact that the natural grading for a torus fiber is $n/2$ has been observed previously by Douglas in the context of Π-stability [11]. See also [2].

We close this section with a lemma that shows that assumption (57) follows from the other assumptions under mild conditions.

Lemma 7.5. *Let (X, ω) be a symplectic Calabi–Yau manifold with ω-tame almost complex structure J and nowhere-vanishing $(n, 0)$-form Ω. Let ϕ be an anti-symplectic involution of X that is also anti-J-holomorphic. If X is simply connected or if Ω is closed, then there exists a smooth complex valued function g on X such that $\widehat{\Omega} = g\Omega$ satisfies condition* (57)*. If $f : X \to B$ is* *a special Lagrangian fibration with respect to Ω and $f \circ \phi = f$, then f is also a special Lagrangian fibration with respect to $\widehat{\Omega}$.*

Proof. Since $\Lambda^{0,3}(T^*X)$ is a line bundle, there exists a complex valued function h such that $\phi^*\Omega = h\bar{\Omega}$. It follows from the fact that ϕ is an involution that

$$h \circ \phi = h^{-1}.$$

So, if h has a square root $h^{1/2}$ we can take $g = h^{1/2}$. Clearly, if X is simply connected, then g has a square-root. Alternatively, if Ω is closed, then J is integrable and Ω is holomorphic [21]. So, both $\phi^*\Omega$ and $\bar{\Omega}$ are anti-holomorphic and therefore so is h. It follows that h is constant and hence has a square-root.

To prove the final claim, we show that the phase of h is constant on fibers of f. It follows that if f is special Lagrangian with respect to Ω then it is also special Lagrangian with respect to $\widehat{\Omega} = h^{1/2}\Omega$. Indeed, let L_y be a fiber of f. In the notation of Eq. (56), using the fact that θ_y is constant, we have

$$(-1)^n(\psi_y \circ \phi)e^{i\pi\theta_y} vol_g = \phi^*\Omega|_{L_y} = h\bar{\Omega}|_{L_y} = h|_{L_y}\psi_y e^{-i\pi\theta_y} vol_g.$$

It follows that

$$h|_{L_y} = (-1)^n(\psi_y \circ \phi)\psi_y^{-1} e^{2\pi i\theta_y},$$

which has constant phase. □

8. Coherent sheaves

In this section, we prove Theorems 1.4 and 1.6. The proof uses Theorem 8.1 below, which was proven by D. Orlov [26].

Let M and X be smooth projective varieties over a field k. For any object E of $\mathcal{D}^b\,Coh(M \times X)$ we can define a functor

$$\Phi_E : \mathcal{D}^b\,Coh(M) \to \mathcal{D}^b\,Coh(X)$$

as follows. Let $p : M \times X \to M$ and $\pi : M \times X \to X$ denote the projections to M and X respectively. Define

$$\Phi_E(\bullet) = R\pi_*\big(E \overset{L}{\otimes} p^*(\bullet)\big). \tag{59}$$

Theorem 8.1. *Let F be an exact functor from $\mathcal{D}^b\,Coh(M)$ to $\mathcal{D}^b\,Coh(X)$, where M and X are smooth projective varieties. Suppose F is full and faithful and has a right (and, consequently, a left) adjoint functor. Then there exists an object E of $\mathcal{D}^b\,Coh(M \times X)$ such that F is isomorphic to the functor Φ_E defined by the rule* (59)*, and this object is unique up to isomorphism.*

The following corollary parallels Proposition 1.7. As above, $\mathcal{O}_y$ denotes the skyscraper sheaf at a point $y \in Y$.

Corollary 8.2. *Let Y be a smooth projective variety. Let $F : \mathcal{D}^b\,Coh(Y) \to \mathcal{D}^b\,Coh(Y)$ be an auto-equivalence such that*

$$F(\mathcal{O}_y) \simeq \mathcal{O}_y,\ \forall y \in Y, \qquad F(\mathcal{O}_Y) \simeq \mathcal{O}_Y.$$

Then, F is isomorphic to the identity functor.

Proof. We apply Theorem 8.1 in the case that $M = X = Y$. Let E be the object of $\mathcal{D}^b\,Coh(Y \times Y)$ associated to F by Theorem 8.1. Let $\Delta : Y \to Y \times Y$ denote the diagonal map. We also use Δ to denote the diagonal subvariety.

Given an object C of $\mathcal{D}^b\,Coh(Y)$ we define $\operatorname{supp}(C)$ to be the closed subset of the underlying topological space of Y that is the union of the supports of all the cohomology sheaves of C.

First, we prove that $\operatorname{supp}(E) \subset \Delta$. Indeed, define

$$i_y : Y \to Y \times Y$$

to be the inclusion of $y \times Y$ into $Y \times Y$. Since

$$p^*\mathcal{O}_y \simeq \mathcal{O}_{y\times Y}$$

we have

$$E \overset{L}{\otimes} p^*\mathcal{O}_y \simeq E|_{y\times Y},$$

where the restriction is in the derived sense. Since $\pi|_{y\times Y}$ is the identity map,

$$F(\mathcal{O}_y) = R\pi_*\big(E \overset{L}{\otimes} p^*\mathcal{O}_y\big) \simeq Li_y^*E.$$

So, by assumption,

$$Li_y^*E \simeq \mathcal{O}_y. \tag{60}$$

So, the only fibers of E which are not zero are on Δ.

Next, we prove that E can be represented by a complex concentrated in degree zero. Since $\operatorname{supp}(E) \subset \Delta$, we know that E is the push-forward of a complex of sheaves supported on the diagonal with some possibly non-reduced scheme structure. So, it suffices to work in a neighborhood of the diagonal. Locally, in a neighborhood of the diagonal, π is affine. Indeed, if $U \subset Y$ is an open affine, then

$$\Delta \cap \pi^{-1}(U) \subset U \times U \subset \pi^{-1}(U).$$

We call this property (LA). Since

$$p^*\mathcal{O}_Y = \mathcal{O}_{Y\times Y},$$

we have by assumption,

$$R\pi_*(E) = F(\mathcal{O}_Y) \simeq \mathcal{O}_Y.$$

But, by property (LA), $R\pi_*$ coincides with π_*. Moreover, affine push-forward cannot send a non-trivial sheaf to zero. Therefore, like its push-forward, E must be concentrated in degree 0.

Next, we construct an isomorphism

$$\mathcal{O}_\Delta \xrightarrow{\sim} E|_\Delta. \tag{61}$$

Indeed, by assumption, we have an isomorphism

$$\mathcal{O}_Y \to F(\mathcal{O}_Y) = \pi_*(E).$$

By adjunction, we have a morphism

$$\mathcal{O}_{Y\times Y} \simeq \pi^*\mathcal{O}_Y \to E.$$

Let s denote the restriction of this morphism to the diagonal. We claim that s is the desired isomorphism. First, we prove it is an isomorphism on fibers. Let

$$r : E \to E|_{Y\times y}$$

denote restriction. By property (LA), we have

$$\pi_*(E|_{Y\times y}) \simeq \pi_*(E)_y.$$

By property (LA) and the exactness of affine push-forward,

$$\pi_*(r) : \pi_*(E) \to \pi_*(E|_{Y\times y})$$

is surjective. So, the composition

$$\mathcal{O}_Y \to \pi_*(E) \xrightarrow{\pi_*(r)} \pi_*(E|_{Y\times y}) \tag{62}$$

is surjective onto the non-zero sheaf $\pi_*(E)_y \simeq \mathcal{O}_y$, and in particular is not zero. By naturality of adjunction, the composition (62) is adjoint to the composition

$$\mathcal{O}_{Y\times Y} \to E \xrightarrow{r} E|_{Y\times y}.$$

In particular, the latter cannot be zero. By Eq. (60) and the fact that E is concentrated in one degree, $E|_{Y\times y}$ has only one non-vanishing fiber,

$$(E|_{Y\times y})_y \simeq E_{y\times y} \simeq \mathcal{O}_{y\times y}.$$

We conclude that the composition

$$\mathcal{O}_{Y\times Y} \to E \to E|_{y\times y}$$

must be surjective for all $y \in Y$. It follows that s is an isomorphism on fibers. By Nakayama's lemma, s is surjective. Since Δ is reduced, there are no nilpotents among the sections of $\mathcal{O}_\Delta$. So, we could detect any non-trivial section in the kernel of s at some fiber. We conclude that s is an isomorphism.

Finally, we prove that the restriction map

$$q : E \to E|_\Delta$$

is an isomorphism. We know that q is surjective. By property (LA) and exactness of affine push-forward, we know that the map

$$\mathcal{O}_Y \simeq \pi_*(E) \xrightarrow{\pi_*(q)} \pi_*(E|_\Delta)$$

is surjective. By isomorphism (61),

$$\pi_*(E|_\Delta) \simeq \pi_*(\mathcal{O}_\Delta) \simeq \mathcal{O}_Y.$$

So, $\pi_*(q)$ is a surjective map from O_Y to itself. So, $\pi_*(q)$ is multiplication by a non-vanishing function, and hence is an isomorphism. Now, suppose q has a kernel K. Since $\pi_*(q)$, is an isomorphism, again using property (LA) and exactness of affine push-forward, we conclude that $\pi_* K$ vanishes. Using property (LA) and the fact that affine push-forward cannot send a non-trivial sheaf to the zero sheaf, we conclude that K vanishes. So, q is an isomorphism.

Composing the isomorphism q with the inverse of isomorphism (61), we have

$$E \simeq \mathcal{O}_\Delta.$$

So, $F \simeq \mathrm{Id}$, as claimed. □

We now prove Theorems 1.4 and 1.6.

Proof of Theorem 1.4. The auto-equivalence $\mathcal{D} \circ \mathcal{D}'^{op}$ of $\mathcal{D}^b\, Coh(Y)$ satisfies the hypothesis of Corollary 8.2. So, we have

$$\mathcal{D}' \simeq \left(\mathcal{D}^{-1}\right)^{op} \simeq \mathcal{D}. \qquad \square$$

Proof of Theorem 1.6. The auto-equivalence $\mathcal{T}^{-1} \circ \mathcal{T}'$ of $\mathcal{D}^b\, Coh(Y)$ satisfies the hypothesis of Corollary 8.2. The theorem follows. □

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