RUIN PROBLEMS WITH COMPOUNDING ASSETS

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We consider a generalization of the classical model of collective risk theory. It is assumed that the cumulative income of a firm is given by a process $X$ with stationary independent increments, and that interest is earned continuously on the firm's assets. Then $Y(t)$, the assets of the firm at time $t$, can be represented by a simple path-wise integral with respect to the income process $X$. A general characterization is obtained for the probability $r(y)$ that assets will ever fall to zero when the initial asset level is $y$ (the probability of ruin). From this we obtain a general upper bound for $r(y)$, a general solution for the case where $X$ has no negative jumps, and explicit formulas for three particular examples.

In addition, an approximation theorem is proved using the weak convergence theory for stochastic processes. This shows that if the income process is well approximated by Brownian motion with drift, then the assets process $Y$ is well approximated by a certain diffusion process $Y_*$, and $r(y)$ is well approximated by a corresponding first passage probability $r_*(y)$. The diffusion $Y_*$, which we call compounding Brownian motion, is closely related to the classical Ornstein-Uhlenbeck process.

1. Introduction

Let $X = \{X(t), t \geq 0\}$ be a stochastic process with stationary, independent increments, finite variance, and $X(0) = 0$. This we call the income process. Given a positive level $y$ of initial assets and a positive interest rate $\beta$, we define the corresponding assets process $Y$ by

$$Y(t) = e^{\beta t}y + \int_0^t e^{\beta(t-s)} dX(s), \quad t \geq 0. \tag{1}$$

As we shall demonstrate, the Riemann–Stieltjes integral on the right side of (1) exists and is finite for all $t \geq 0$ and (almost) every sample path of $X$. Thus the assets process is a well defined path-wise functional of the income process.

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In order to give an interpretation for $Y$, we first consider an important special case. Let \( \{N(t), t \geq 0\} \) be a Poisson process with arrival rate \( \lambda \), and let \( W_1, W_2, \ldots \) be independent and identically distributed random variables with distribution function \( F \). Letting \( c \) be a finite constant, we take
\[
X(t) = ct - [W_1 + \ldots + W_{N(t)}], \quad t \geq 0.
\]
Thus \( X \) is compound Poisson with drift \( c \). Processes of this form, with \( F \) concentrated on \((0, \infty)\) and \( c \) positive, have traditionally been used in collective risk theory [6,19,21] to represent changes in the risk reserve of an insurance company. In that context, one interprets \( c \) as the rate at which premium payments are received from policyholders, \( N(t) \) as the cumulative number of claims incurred up to time \( t \), and \( W_k \) as the size of the \( k \)th claim. Suppose now that the risk reserve of the company is invested in a bank savings account, continuously earning interest at rate \( \beta \). (One dollar invested for a period of length \( t \) returns \( e^{\beta t} \) dollars at the end of the period.) Assume that premium payments are deposited to the account as they are received and that funds are withdrawn from the account to pay claims as they occur. Further assume that, should the contents of the account ever fall to zero, money can be borrowed from the bank at the same interest rate \( \beta \), interest compounding continuously on the debt just as it does on savings. Let \( y \) be the initial risk reserve, and denote by \( t_1, t_2, \ldots \) the times at which claims occur. Then the content of the account at time \( t \) is
\[
Y(t) = e^{\beta t} y + \int_0^t c e^{\beta (t-s)} \, ds - \sum_{k=1}^{N(t)} e^{\beta (t-t_k)} W_k,
\]
which coincides with our definition (1). A similar interpretation applies when \( F \) is arbitrary, this allowing for mixtures of annuity business and ordinary insurance business [19,21].

In general, we interpret \( y \) as the initial assets of a firm, \( \beta \) as the instantaneous rate of interest earned through (riskless) investment of the firm's assets, and \( X(t) \) as the firm's net operating profit during the interval \([0, t]\). As in the special case above, one can then interpret \( Y(t) \) as the assets of the firm at time \( t \), our general model allowing for income processes of a more complicated form than (2). Of course the interpretation is not valid for all \( t \) and all realizations of \( X \) unless one assumes that the firm can both lend and borrow at the interest rate \( \beta \). Let
\[
T = \inf \{t \geq 0 : Y(t) < 0\}.
\]
Our primary objective is to determine
\[
r(y) = P \{ T < \infty \mid Y(0) = y \}.
\]
We call \( T \) the time of ruin and \( r(\cdot) \) the ruin function. In addressing this problem, one restricts attention to the period of time during which \( Y \) remains positive, and hence \( \beta \) may be interpreted simply as the rate of interest earned on positive assets.

We begin in Section 2 by examining, without any special assumptions, the
behaviour of the assets process. The central result is a general characterization of the ruin function. The characterization is used in Section 3 to compute \( r(\cdot) \) for three particular cases. The first two correspond to an income process of the compound Poisson form (2). In one instance we assume that \( c \) is positive and \( F \) is an exponential distribution concentrated on \((0, \infty)\). In the other instance we assume that \( c \) is negative and \( F \) is an exponential distribution concentrated on \((-\infty, 0)\). The ruin functions for these two cases were previously determined by Segerdahl [20] in the context of insurance models with variable premium rates, but they are obtained here by a more direct method. The third example discussed in Section 3 takes the income process to be Brownian motion with drift. The corresponding assets process is the one-dimensional diffusion process called compounding Brownian motion by Emanuel, Harrison and Taylor [8]. The ruin function was computed in [8] but is again obtained here by a more direct probabilistic argument. To my knowledge, the three special cases discussed in Section 3 are the only processes of the form (1) for which results have previously been obtained.

Given an income process \( X \) of the type described above, let \( X_* \) be a Brownian motion having the same mean and variance, and denote by \( r_*(\cdot) \) the corresponding ruin function. We call \( X_* \) the natural diffusion approximation for \( X \). In the particular case where \( X \) is of the compound Poisson form (2), Emanuel, Harrison and Taylor [8] have suggested that \( r_*(\cdot) \) should be a good approximation for \( r(\cdot) \) if the jumps of \( X \) are small but frequent, and they have given some numerical comparisons which indicate that the agreement is very close in at least some circumstances. In Section 4 we prove a limit theorem that provides a rigorous justification for this and more general approximations. We consider a sequence of income processes \( X_1, X_2, \ldots \) all of which have the same natural diffusion approximation. Under assumptions which are necessary and sufficient for the one-dimensional distributions of \( X_n \) to converge to those of \( X_* \), it is shown that \( r_n(y) \to r_*(y) \) as \( n \to \infty \). A similar result is obtained concerning the probability of ruin before time \( t \).

The key to our analysis is the simple integral representation (1) for the assets process. This is virtually equivalent to the representation given by Gerber [10, 11] for discounted income streams, the properties of which were subsequently investigated by Whitt [23]. Whitt's results on the continuity of the discounting functional are central to the proof of our approximation theorems in Section 4.

2. The ruin function

We assume that the process \( X \) is defined on a probability space \((\Omega, \mathcal{F}, P)\) and has stationary, independent increments with \( \mathbb{E}[X(t)] = \mu t \) and \( \text{Var}[X(t)] = \sigma^2 t \), where \(-\infty < \mu < \infty\) and \( 0 < \sigma^2 < \infty \). We assume that \( X \) is continuous in probability and that its sample paths are right continuous and have left limits. For the characteristic function (c.f.) of \( X(t) \) we then have
where the exponent function \( \nu(\cdot) \) is given by the Levy–Khintchine representation (see Gikhman and Skorohod [12]). In the finite variance case, the representation can be (uniquely) specialized to the form

\[
\nu(u) = i\mu u + \sigma^2 \int_{\mathbb{R}} x^{-2} (e^{iux} - 1 - iux) G(dx),
\]

where \( G \) is a probability distribution on \( \mathbb{R} \) (see Gnedenko [13, pp. 323–327]). Finally, \( X \) is strong Markov (see Hunt [15]). We now rewrite (1) as

\[
Y(t) = e^{\mu t} [y + Z(t)], \quad t \geq 0,
\]

where

\[
Z(t) = \int_{0}^{t} e^{-\beta s} dX(s), \quad t \geq 0.
\]

The random variable \( Z(t) \) represents the present value (as viewed from time zero) of income earned during \([0, t]\). Let \( D[0, \infty) \) denote the set of real-valued functions on \([0, \infty)\) that are right continuous and have left limits, and define \( D[0, t] \) similarly for each \( t > 0 \).

**Proposition 2.1.** The Riemann–Stieltjes integrals (4) almost surely exist, are finite, and satisfy

\[
Z(t) = e^{-\beta t} X(t) + \beta \int_{0}^{t} e^{-\beta s} X(s) ds, \quad t \geq 0.
\]

Furthermore, \( Z \) is almost surely in \( D[0, \infty) \).

**Proof.** Define \( g(t) = \exp(-\beta t) \). From the lemma proved on p. 110 of Billingsley [3] and the Cauchy criterion for Riemann–Stieltjes integrability [1, p. 279] it follows easily that any function \( x(\cdot) \) in \( D[0, \infty) \) is integrable with respect to the monotone function \( g \) over \([0, t]\). From the integration by parts theorem [1, p. 282], \( g \) is then integrable with respect to any such \( x(\cdot) \) over \([0, t]\) and

\[
\int_{0}^{t} g(s) dx(s) = g(t)x(t) - g(0)x(0) - \int_{0}^{t} x(s) dg(s).
\]

Since \( -dg(s) = \beta \exp(-\beta s) ds \), the integral on the right side of (5) is a continuous function of \( t \). Thus the entire right side, viewed as a function of \( t \), is in \( D[0, \infty) \). The proposition now follows from the fact that \( X(\cdot) \) is almost surely in \( D[0, \infty) \) with \( X(0) = 0 \).

In order to relate our definition of the present value process \( Z \) to the general theory of stochastic integration, it is useful to define \( M(t) = X(t) - \mu t \) for \( t \geq 0 \) and then rewrite (4) as
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\[ Z(t) = \int_0^t e^{-\beta s} \, dM(s) + (\mu / \beta)(1 - e^{-\beta t}), \quad t \geq 0. \]  

Clearly \( M \) is a martingale, so (6) may be interpreted within the framework of the general theory of stochastic integration, as developed (for example) by Skorohod [22, pp. 29–34]. In the general theory one cannot interpret the integral as a path-wise function of the integrator process, since the approximating Riemann–Stieltjes sums need not converge almost surely. When the integral does exist path-wise, however, Skorohod’s limiting procedure yields a process having the same distribution as that defined by path-wise integration.

**Proposition 2.2.** The process \( Z = \{Z(t), t \geq 0\} \) has independent increments with

\[ E[Z(t)] = (\mu / \beta)(1 - e^{-\mu t}), \quad \text{Var}(Z(t)) = (\sigma^2 / 2\beta)(1 - e^{-2\beta t}) \]

for \( t \geq 0 \). Moreover, \( Z(\infty) = \lim Z(t) \) exists and is finite almost surely. The distribution of \( Z(\infty) \) is continuous, and its c.f. is

\[ E[e^{iuZ(t)}] = e^{\psi(u)}, \quad \text{where} \quad \psi(u) = \int_0^\infty \nu(ue^{-\beta t})dt. \]  

**Proof.** It is obvious that \( Z \) has independent increments, and the moments of \( Z(t) \) follow immediately from (6) and formulas (3.8) and (3.9) of Skorohod [22, p. 30]. To show that \( Z(t) \) converges almost surely, we shall assume \( \mu = 0 \), since the general case requires only a trivial extension of the argument. From the independence of its increments, we see that \( \{Z(t)\} \), together with the obvious family of increasing \( \sigma \)-algebras, is a martingale when \( \mu = 0 \). Moreover

\[ \sup E[(Z(t))^2] = \sigma^2 / 2\beta < \infty, \]

implying that \( \sup E[|Z(t)|] < \infty \). Thus, by the martingale convergence theorem [18, p. 96], \( Z(\infty) \) exists and is finite almost surely. By writing out the approximating Riemann–Stieltjes sums for the integral (4) and using the continuity theorem for characteristic functions, it is easily established that

\[ E[e^{iuZ(t)}] = \exp \int_0^\infty \nu(ue^{-\beta s})ds, \quad u \in \mathbb{R}, \]

and the c.f. of \( Z(\infty) \) follows by continuity.

To show that \( Z(\infty) \) has a continuous distribution, we assume the contrary and denote by \( p \) the largest probability associated with any single value of the random variable. Let \( C_1, \ldots, C_K \) be the values having discrete probability \( p \). Fix \( t > 0 \) and define

\[ V(t) = \int_t^\infty e^{-\beta(s-t)}dX(s) = e^{\beta t}[Z(\infty) - Z(t)]. \]

Thus \( Z(\infty) = Z(t) + e^{-\beta t}V(t) \), and \( V(t) \) has the same distribution as \( Z(\infty) \) and is
independent of \( Z(t) \). Letting \( H_i(\cdot) \) be the distribution function of \( Z(t) \), we then have

\[
p = \Pr[Z(\infty) = C_i] = \int \Pr[Z(\infty) = e^{\beta t}(C_i - z)]H_i(\text{d}z).
\]

But if this is to hold, \( H_i \) must concentrate all of its mass on the points \( z_k = C_i - C_k \exp(-\beta t), k = 1, \ldots, K \). This in turn implies that \( Z(t) \rightarrow C_i = Z(\infty) \) almost surely as \( t \rightarrow \infty \). But we have assumed \( X(t) \) non-degenerate (for each \( t \)), from which it follows that \( Z(\infty) \) is non-degenerate. Thus we have arrived at a contradiction, and the proposition is proved completely.

**Remark.** From Proposition 2.2 and the definitive relationship (3) it follows that \( \lim_{Y(t)} \rightarrow \infty \) almost surely, with

\[
\Pr[y + Z(\infty) > 0] = \Pr[Y(t) \rightarrow \infty] = 1 - \Pr[Y(t) \rightarrow -\infty].
\]

**Theorem 2.3.** For the ruin function we have

\[
r(y) = H(-y)/E[H(-Y(T)) | T < \infty],
\]

where \( H \) is the distribution function of \( Z(\infty) \), its c.f. being given by (6).

**Proof.** From (4) we see that \( T = \inf\{t \geq 0 : y + Z(t) < 0\} \), and thus the event \( \{T < \infty\} \) contains the event \( \{y + Z(\infty) < 0\} \). Now let \( \{V(t), t \geq 0\} \) be defined as in the proof of Proposition 2.2. Assuming \( T < \infty \), note that

\[
y + Z(\infty) = y + Z(T) + e^{-\beta T}V(T)
\]

\[
= e^{-\beta T} [e^{\beta T} (y + Z(T)) + V(T)] = e^{-\beta T} [Y(T) + V(T)].
\]

Thus we have

\[
\Pr[y + Z(\infty) < 0] = \Pr[T < \infty, Y(T) + V(T) < 0]
\]

\[
= \int_{\{T < \infty\}} \Pr[Y(T) + V(T) < 0 | X(s), 0 \leq s \leq T] \text{d}P.
\]

Observe that \( Y(T) \) depends only on \( \{X(s), 0 \leq s \leq T\} \), while \( V(T) \) depends only on \( \{X(s) - X(T), s > T\} \) and has distribution function \( H(\cdot) \). Since \( H(\cdot) \) is continuous by Proposition 2.2 and \( T \) is clearly a Markov time for \( X \), the independent increments and strong Markov property of \( X \) then give us

\[
\Pr[Y(T) + V(T) < 0 | X(s), 0 \leq s \leq T] = H(-Y(T))
\]

on \( \{T < \infty\} \). Combining (8) and (9), and again using the continuity of \( H \), we have

\[
H(-y) = \Pr[y + Z(\infty) < 0] = \int_{\{T < \infty\}} H(-Y(T)) \text{d}P,
\]

which is equivalent to the theorem statement.
Corollary 2.4. We have \( r(y) \leq H(-y)/H(0) \), with equality if \( X \) has no negative jumps.

**Proof.** Since \( Y(T) \leq 0 \) on \( \{T < \infty\} \) and \( H \) is non-decreasing, \( H(-Y(T)) \geq H(0) \) on \( \{T < \infty\} \). This establishes the inequality. If \( X \) has no negative jumps, the same is true of \( Y \), and hence \( Y(T) = 0 \) on \( \{T < \infty\} \), implying the equality.

For later use in Section 4, we define

\[
U(t) = \inf\{Z(s) : 0 \leq s \leq t\}, \quad t \geq 0.
\]

Since \( Z(\infty) \) is finite, the limit \( U(\infty) \) exists and is finite almost surely, and clearly

\[
r(y) = \mathbb{P}\{U(\infty) < 0 \mid Z(0) = y\}.
\]

3. Examples

If the income process is of the compound Poisson form (2), then its exponent function is

\[
v(u) = icu - \lambda \int_{\mathbb{R}} (1 - e^{-iu})F(dx), \quad u \in \mathbb{R}.
\]

**Example 3.1.** In (10), suppose that \( c \) is positive and that

\[
F(x) = 1 - e^{-x/m}, \quad x \geq 0 \quad (m > 0),
\]

with \( F(x) = 0 \) for \( x < 0 \). Thus the paths of the income process slope upward between downward jumps, and the absolute jump size has an exponential distribution with mean \( m \). The exponent function is

\[
v(u) = icu - \lambda imu/(1 + imu),
\]

and the reader may easily verify that

\[
\psi(u) = \int_{0}^{\infty} \nu(u e^{-mt})dt = icu/\beta - (\lambda/\beta)\ln(1 + imu).
\]

According to (7), the c.f. of \( H \) is then

\[
e^{\psi(u)} = e^{icu/\beta} \left(1 + imu\right)^{-\lambda/\beta}.
\]

Comparing this with the c.f. of a gamma distribution, we can easily invert to obtain

\[
H(z) = \left[\int_{c/\beta}^{\infty} x^{\lambda/\beta - 1} e^{-x/m} \, dx\right] / m^{\lambda/\beta} \Gamma(\lambda/\beta), \quad x \in \mathbb{R}.
\]

Now observe that the jumps of \( Y \) are exactly the jumps of \( X \), so the ‘memoryless’ property of the exponential distribution gives us the following: the amount by which assets fall below zero upon ruin, given that ruin occurs, has the same exponential distribution as a general jump of the income process. Thus
Combining (11) and (13), one finds that the convolution of $H$ with the conditional distribution of $Y(T)$ given $T < \infty$ is itself essentially a gamma distribution. To be specific,

$$E[H(-Y(T)) \mid T < \infty] = \left[ \int_{c/\beta}^{\infty} x^{\lambda/\beta} e^{-x/m} \, dx \right] / m^{\lambda/\beta+1} \Gamma(\lambda/\beta + 1).$$

From Theorem 2.3 we have

$$r(y) = H(-y)/E[H(-Y(T)) \mid T < \infty].$$

Substituting (12) and (14) into (15) yields an explicit solution for $r(\cdot)$ in terms of the incomplete gamma function. If one applies integration by parts to the denominator (14), this solution is found to agree with Segerdahl's [20] formula (56), repeated in his English survey paper [21].

**Example 3.2.** Now suppose in (10) that $c$ is negative and that

$$F(x) = e^{-x/m}, \quad x \leq 0 \quad (m > 0),$$

with $F(x) = 1$ for $x \geq 0$. Thus the paths of $X$ slope downward between upward jumps having an exponential distribution with mean $m$. Then we obtain

$$\nu(u) = \lambda \frac{imu}{1-imu} - i|c|u,$$

$$e^{\nu(u)} = e^{-i|c|/u/(1-imu)^{-\lambda/\beta}}.$$

Another easy inversion gives us the translated gamma distribution

$$H(z) = \begin{cases} 0 & \text{if } z \leq -c/\beta \\
\left[ \int_{c/\beta}^{\infty} x^{\lambda/\beta} e^{-x/m} \, dx \right] / m^{\lambda/\beta} \Gamma(\lambda/\beta) & \text{if } z > -c/\beta \end{cases}.$$

Since the income process has no negative jumps, Corollary 2.4 gives us $r(y) = H(-y)/H(0)$, an explicit solution in terms of the incomplete gamma function. (The reader should note that $r(y) = 0$ for $y \geq c/\beta$.) The corresponding formula (58) of Segerdahl [20] is incorrect due to a sign reversal, as Jung [17] has noted.

**Example 3.3.** Assume now that $X(t) = \sigma W(t) + \mu t$, where $W$ is a standard (zero drift and unit variance) Wiener process. Our basic representation (1) for the assets process yields

$$Y(t) = e^{\beta t} \left[ y + \sigma \int_0^t e^{-\beta s} \, dW(s) + (\mu/\beta)(1-e^{-\beta t}) \right]$$

$$= e^{\beta t}y + \sigma W(t) + \sigma \beta \int_0^t e^{\beta(t-s)} W(s) \, ds + (\mu/\beta)(e^{\beta t} - 1).$$

$$\text{E}[e^{i\omega Y(T)} \mid T < \infty] = (1 + imu)^{-1}.$$
One may readily verify that the present value process $Z(\cdot)$ is Gaussian and has
independent increments, the first two moments of $Z(t)$ being given by Proposition
2.2. Thus one immediately obtains a representation of the present value process as
a rescaling of Brownian motion,

$$Z(t) = (\sigma^2/2\beta)^{1/2} W(1 - e^{-\beta t}) + (\mu/\beta)(1 - e^{-\beta t}), \quad t \geq 0.$$  \hfill (17)

Combining (3) and (17) we then see that the assets process has the same distribution
as

$$Y(t) = (\sigma^2/2\beta)^{1/2} W(e^{\beta t} - 1) + ye^{\beta t} + (\mu/\beta)(e^{\beta t} - 1), \quad t \geq 0.$$ \hfill (18)

Thus $Y$ is Gaussian and has continuous sample paths. It further follows from (18)
that $Y$ is a strong Markov process with stationary transition probabilities, so it is a
diffusion, and an easy direct computation shows its infinitesimal mean and variance
to be

$$\mu(y) = \mu + \beta y, \quad \sigma^2(y) = \sigma^2, \quad y \in \mathbb{R}. \hfill (19)$$

The classical Ornstein–Uhlenbeck process is a diffusion with $\sigma^2(y) = \sigma^2$ and
$\mu(y) = \mu - \gamma y$, where $\gamma$ is positive. It is often (roughly) characterized as being
Brownian motion plus an elastic force that pulls the process back toward zero with
a strength directly proportional to the current absolute position. Of course $\gamma$ is the
constant of proportionality. From (19) one sees that our assets process $Y$, which we
shall call compounding Brownian motion, is characterized similarly by a repulsive
force pushing the process away from zero with a strength directly proportional to
its current absolute position. The two processes are of course very different in some
regards. In particular, the O-U process has a non-defective (Gaussian) limiting
distribution, whereas $|Y(t)| \to \infty$ almost surely as $t \to \infty$. Still many arguments
pertaining to the O-U process carry over to compounding Brownian motion with
little or no change. Our basic representation (16) of compounding Brownian motion
differs only trivially from the integral representation for the O-U process given by
Breiman [5, pp. 347–350]. Similarly, (18) is a precise analog to the characterization
of the O-U process given by Feller [9, p. 336].

Although the same result can be gotten easily from Proposition 2, we may use
(17) directly to conclude that $Z(\infty)$ has the Gaussian distribution

$$H(z) = \Phi[(z - \mu/\beta)/(\sigma^2/2\beta)^{1/2}] = 1 - \Phi(b - az),$$

where $a = (2\beta/\sigma^2)^{1/2}$, $b = a\mu/\beta$, and $\Phi(\cdot)$ is the standardized normal distribution
function. Since the sample paths of Brownian motion are continuous, Corollary 2.4
gives us

$$r(y) = H(-y)/H(0) = [1 - \Phi(ay + b)]/[1 - \Phi(b)], \hfill (20)$$

which is the same result obtained by Emanuel, Harrison and Taylor [8].
4. The compound Brownian approximation

Throughout this section we take the initial asset level $y$ and the interest rate $\beta$ to be fixed. Let $X_1, X_2, \ldots$ be income processes defined on a common probability space $(\Omega, \mathcal{F}, P)$. All of the notation established earlier will be carried forward with a subscript $n$ added to indicate a quantity associated with the $n$th income process. Let $X_*$ be a Brownian motion with mean $\mu_*$ and variance $\sigma_*^2$, also defined on $(\Omega, \mathcal{F}, P)$. In the obvious way, we add an asterisk to our previous notation to indicate a quantity associated with $X_*$. We assume throughout that

$$\mu_1 = \mu_2 = \ldots = \mu_* \quad \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_*^2.$$  \hspace{1cm} (21)

Thus $X_*$ is the natural diffusion approximation for each of the income processes $X_n$. One could alternately assume that $\mu_n \to \mu_*$ and $\sigma_n^2 \to \sigma_*^2$, but this complicates matters slightly and does not really give any greater generality from the standpoint of justifying approximations.

We shall use the symbol $\Rightarrow$ to denote weak convergence for a sequence of probability measures on a metric space (or equivalently, a sequence of random elements of the metric space). In the case of random variables (random elements of $\mathbb{R}$), weak convergence is equivalent to convergence of distribution functions at continuity points of the limit. For a general definition and thorough discussion of weak convergence, see Billingsley [3]. When we speak of weak convergence in $D[0, t]$, it is understood that the function space is endowed with Skorohod's metric topology (see [3, pp. 111-114]). Observe that the restriction to $[0, t]$ of any process discussed in Section 2 is a random element of $D[0, t]$.

The following proposition is an immediate consequence of the basic limit theorem for infinitely divisible distributions (see Gnedenko [13, p. 328]).

Theorem 4.1. The one-dimensional distributions of $X_n$ converge to those of $X_*$ if and only if

$$G_n(\cdot) \Rightarrow \delta(\cdot) \text{ as } n \to \infty,$$  \hspace{1cm} (22)

where $\delta(\cdot)$ is the distribution function degenerate at zero.

Hereafter we assume that (22) holds. If the $X_n$ are of the compound Poisson form (2) with drift rates $c_n$, jump rates $\lambda_n$ and jump distributions $F_n$, then (21) and (22) specialize to

$$c_n - \lambda_n \int_{\mathbb{R}} x F_n(dx) = \mu_* \text{ for all } n,$$  \hspace{1cm} (23)

$$\lambda_n \int_{\mathbb{R}} x^2 F_n(dx) = \sigma_*^2 \text{ for all } n,$$  \hspace{1cm} (24)

$$\lambda_n \int_{|x| \geq \varepsilon} x^2 F_n(dx) \to 0 \text{ as } n \to \infty \text{ for all } \varepsilon > 0.$$  \hspace{1cm} (25)
Note that (24) and (25) together imply that $\lambda_n \to -\infty$ and $F_n(\cdot) \Rightarrow \delta(\cdot)$ as $n \to \infty$. In the case where $F_n$ is concentrated on $(0, \infty)$ for all $n$, (23) further implies that $c_n \to \infty$. Thus, for large $n$, the paths of $X_n$ must increase rapidly between jumps that are small but frequent. This gives us approximately continuous paths with large local variations, so (23)-(25) are intuitively consistent with the continuity and unbounded local variation of Brownian paths.

**Theorem 4.2.** For each $t > 0$, $X_n \Rightarrow X_*$, $Y_n \Rightarrow Y_*$, $Z_n \Rightarrow Z_*$ and $U_n \Rightarrow U_*$ in $D[0, t]$ as $n \to \infty$.

**Corollary 4.2.** $P\{T_n < t\} \to P\{T_\ast < t\}$ as $n \to \infty$ for $t > 0$.

**Proof of Theorem 4.2.** Throughout the proof let $t > 0$ be fixed. In showing that $X_n \Rightarrow X_*$ we shall assume that $\mu_\ast = 0$. The general case requires only a trivial extension of the argument. We first observe that the finite dimensional distributions of $X_n$ converge to those of $X_*$ as $n \to \infty$. This is immediate from Theorem 4.1 and our assumption of stationary, independent increments. Next, if $u_1 < u < u_2$, we have

$$E\{[X_n(u) - X_n(u_1)]^2[X_n(u_2) - X_n(u)]^2\}$$

$$= [(u-u_1)^2][(u_2-u)^2] \leq \sigma_\ast^2(u_2-u_1)^2/4.$$ 

Combining these facts with [3, Theorem 15.6], we have that $X_n \Rightarrow X_*$ in $D[0, t]$ as $n \to \infty$. We now define mappings $f$, $g$, and $h$ from $D[0, t]$ into itself as follows. For $x(\cdot)$ in $D[0, t]$ let

$$f(x)(s) = e^{-\alpha s} x(s) + \beta \int_0^s e^{-\alpha u} x(s) ds, \quad 0 \leq s \leq t,$$

$$g(x)(s) = e^{\beta t} [y + x(s)], \quad 0 \leq s \leq t,$$

$$h(x)(s) = \inf \{x(u) : 0 \leq u < s\}, \quad 0 < s < t.$$ 

It is shown by Whitt [23] that $f$ is continuous in the Skorohod topology, and it is easy to prove that $g$ and $h$ are continuous as well. The continuous mapping theorem (see Billingsley [3, pp. 29-31]) then gives us

$$Z_n = f(X_n) \Rightarrow f(X_\ast) = Z_\ast$$

as $n \to \infty$, and hence

$$Y_n = g(Z_n) \Rightarrow g(Z_\ast) = Y_\ast$$

$$U_n = h(Z_n) \Rightarrow h(Z_\ast) = U_\ast$$

as $n \to \infty$.

To prove the corollary, we observe that the projection at $t$ is a continuous mapping $D[0, t] \to \mathbb{R}$, and hence $U_n \Rightarrow U_\ast$ implies $U_n(t) \Rightarrow U_\ast(t)$ as $n \to \infty$. Since $P\{T_n \leq t\} = P\{U_\ast(t) < 0\}$ and the distribution of $U_\ast(t)$ is clearly continuous, the corollary follows immediately.
Theorem 4.4. \( r_n(y) \to r_\ast(y) \) as \( n \to \infty \), where \( r_\ast(\cdot) \) is given by formula (20).

Proof. We begin by showing that, for each positive \( \varepsilon \) and \( \delta \), there exists \( t_0 \) such that

\[
P\{ | U_\ast(t_0) - U_\ast(\infty) | \geq \varepsilon \} \leq \delta,
\]

(26)

\[
P\{ | U_n(t_0) - U_n(\infty) | \geq \varepsilon \} \leq \delta \quad \text{for all } n.
\]

(27)

We shall prove this only for the case \( \mu_\ast = 0 \), since the general case requires only minor alterations in the argument. As we have observed in the proof of Proposition 2.2, the process \( Z_\ast \) is a martingale when \( \mu_\ast = 0 \), and hence \( Z_\ast(\cdot) \) is a submartingale with right continuous paths (see Meyer [15, p. 79]). From the fundamental supermartingale inequality (see [18, p. 93]) one then easily obtains

\[
P\left\{ \sup_{0 \leq t \leq \tau} Z_\ast(t) \geq \varepsilon \right\} \leq \varepsilon^2 \mathbb{E}[Z_\ast(\tau)] = (\sigma_\ast^2/2\beta \varepsilon^2)(1 - e^{-2\beta \varepsilon}),
\]

a generalization of Kolmogorov's inequality to our continuous parameter martingale. Passing \( \tau \to \infty \) then yields

\[
P\left\{ \sup_{t \geq 0} | Z_\ast(t) | \geq \varepsilon \right\} \leq \sigma_\ast^2/2\beta \varepsilon^2.
\]

(28)

Now observe that, for any \( \tau > 0 \),

\[
\sup_{t \geq 0} | Z_\ast(\tau + t) - Z_\ast(\tau) | \sim e^{\beta \tau} \sup_{t \geq 0} | Z_\ast(t) | ,
\]

(29)

where \( \sim \) denote equivalence in distribution. Moreover,

\[
| U_\ast(\tau) - U_\ast(\infty) | \leq \sup_{t \geq 0} | Z_\ast(\tau + t) - Z_\ast(\tau) | .
\]

(30)

Combining (28)-(30), one sees that (26) holds for any \( t_0 \) such that \( \exp(-2\beta t_0) \leq 2 \varepsilon^2 \beta \delta / \sigma_\ast^2 \). By an identical argument, (27) holds for all such \( t_0 \) as well.

We have shown in the proof of Theorem 4.2 and Corollary 4.3 that \( U_n(t) \Rightarrow U_\ast(t) \) for each \( t > 0 \). Combining this with (26) and (27), it follows easily that \( U_n(\infty) \Rightarrow U_\ast(\infty) \) as \( n \to \infty \). Since \( r_n(y) = P[U_n(\infty) < 0] \) and \( U_\ast(\infty) \) has a continuous distribution, this implies the desired result.

A number of limit theorems have been proved previously to justify the ordinary Brownian approximation for the income process \( X \). In particular, Iglehart [16] considered a sequence of such processes, allowing each to be a compound renewal process plus a linear drift, rather than just a compound Poisson process plus drift. If one specializes Iglehart's treatment to the case of the Poisson jump arrival processes, and then incorporates his normalization factors into the jump size distributions and jump arrival rates, his assumptions are found to be somewhat stronger than (23)-(25). It is my feeling, incidentally, that the approach taken here,
showing convergence of the income processes themselves, is much more natural for risk theoretic applications than an approach using explicit normalization constants. Although the two approaches are virtually equivalent mathematically, the presence of normalization constants makes much more difficult the task of translating a limit theorem into a corresponding approximation procedure.

A limit theorem justifying the Brownian approximation in a discrete-time model has been proved by Bohman [4] and Grandell [14]. This result, as well as that of Iglehart [16], is presented in the recent book by Beekman [2, Chapter 5].

References