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Uniform convergence of the empirical spectral distribution function

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Abstract

Let X be a linear process having a finite fourth moment. Assume $\mathscr{F}$ is a class of square-integrable functions. We consider the empirical spectral distribution function $J_{n,X}$ based on X and indexed by $\mathscr{F}$. If $\mathscr{F}$ is totally bounded then $J_{n,X}$ satisfies a uniform strong law of large numbers. If, in addition, a metric entropy condition holds, then $J_{n,X}$ obeys the uniform central limit theorem. © 1997 Elsevier Science B.V.

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1. Introduction

Let $X = (X_t)_{t \in \mathbb{Z}}$ be a linear process defined by

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $Z = (Z_t)_{t \in \mathbb{Z}}$ is a sequence of iid random variables with $EZ_1 = 0$, $\sigma^2 = \mathrm{var}(Z_1) < \infty$ and (c_j) is a square-summable sequence of real numbers. Let $\mathscr{F}$ be a class of square integrable functions $f \in \mathscr{L}^2(\Pi)$, $\Pi = [-\pi, \pi]$. For the stationary process X we study some limit theory for the empirical spectral distribution function $J_{n,X} = (J_{n,X}(f))_{f \in \mathscr{F}}$, $n \geqslant 1$, where

$$J_{n,X}(f) = \int_\Pi f(x) I_{n,X}(x)\,\mathrm{d}x = \int_\Pi f(x)\,\mathrm{d}\tilde{J}_{n,X}(x), \quad f \in \mathscr{F}.$$

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Here

$$\tilde{J}_{n,X}(\lambda) = \int_{-\pi}^{\lambda} I_{n,X}(x)\,\mathrm{d}x \quad \text{and} \quad I_{n,X}(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} \mathrm{e}^{-\mathrm{i}\lambda t} X_t\right|^2, \quad \lambda \in \Pi,$$

denote the ordinary empirical spectral distribution function and the raw periodogram of the first n observations of the time series X, respectively.

The periodogram $I_{n,X}$, as an estimator of the spectral density of X, has been studied for a long time. The same concerns its integrated version $\tilde{J}_{n,X}$ and various weighted modifications of the latter as estimators of the corresponding spectral distribution function. Accounts of the general theory are given e.g. in Brillinger (1981), Brockwell and Davis (1991), Grenander and Rosenblatt (1957) and Priestley (1981). For fixed functions f, the asymptotic behaviour of $J_{n,X}(f)$ has been considered in detail, e.g., in connection with the Whittle estimator, one of the most popular parameter estimates for ARMA and fractional ARIMA processes (see e.g. Brockwell and Davis, 1991, Section 10.8).

The need to study $J_{n,X}$ indexed by a class of functions arises e.g. when we consider a goodness-of-fit test statistic based on the process $(J_{n,X}(I_{[0,\lambda]}))_{\lambda \in \Pi}$ or on $J_{n,X}(I_{[0,\lambda]}/|C(\lambda)|^2))_{\lambda \in \Pi}$ where

$$C(\lambda) = \sum_{j=0}^{\infty} c_j \mathrm{e}^{-\mathrm{i}\lambda j}, \quad \lambda \in \Pi,$$

denotes the transfer function of the linear filter (c_j). Then $|C(\lambda)|^2$ is the so-called power transfer function which, up to a constant multiple, is nothing but the spectral density of X. Classical goodness-of-fit tests such as Bartlett's and the Grenander–Rosenblatt type tests in the frequency domain are based on these processes (see e.g. Anderson, 1993; Bartlett, 1954, 1978; Dzhaparidze, 1986; Grenander and Rosenblatt, 1957; Priestley, 1981). The consideration of function indexed $J_{n,X}$ is also motivated by goodness-of-fit tests based on $J_{n,X}(I_{[0,\lambda]}/|C(\lambda)|^2))_{\lambda \in \Pi}$ with estimated coefficients (c_j).

First work on uniform convergence theory for the empirical spectral distribution function is due to Dahlhaus (1988) who closely followed the theory of function indexed empirical processes developed in Pollard (1984). Moreover, Dahlhaus assumed that an exponential moment of the stationary (vector-valued) sequence (X_t) exists which allowed him to derive uniform exponential estimates for the spectral distribution function using suitable cumulant techniques. Inspired by results from the theory of empirical processes Hosoya (1989) applies a uniform limit theorem with bracketing condition to the problems of quasi-likelihood estimation.

In the present paper we derive some asymptotic theory for $J_{n,X}$ indexed by a class of functions $f \in \mathscr{F}$ such that $f|C|^2 \in \mathscr{L}^2(\Pi)$. We endow $\mathscr{F}$ with a pseudometric space structure inherited from $\mathscr{L}^2$. The set $\mathscr{C}_u(\mathscr{F})$ of real-valued, uniformly continuous and bounded functions ϕ defined on $\mathscr{F}$ and equipped with norm

$$\|\phi\|_{\mathscr{F}} = \sup_{f \in \mathscr{F}} |\phi(f)|$$

is a Banach space. If, in addition, $\mathscr{F}$ is totally bounded then $\mathscr{C}_u(\mathscr{F})$ is separable. Then the process $J_{n,X}$ has almost all sample paths in $\mathscr{C}_u(\mathscr{F})$ (see decomposition (2.1)). We prove a central limit theorem (CLT) in $\mathscr{C}_u(\mathscr{F})$ only under a fourth moment condition and under an entropy assumption weaker than in Dahlhaus (1988). This means we show the convergence

$$\sqrt{n}(J_{n,X}(f) - J_C(f))_{f\in\mathscr{F}} \xrightarrow{d} (G_C(f))_{f\in\mathscr{F}} \text{ in } \mathscr{C}_u(\mathscr{F}),$$

where

$$J_C(f) = J(f|C|^2) = \sigma^2 \int_\Pi f(x)|C(x)|^2\,\mathrm{d}x, \quad f\in\mathscr{F},$$

denotes the spectral distribution function indexed by $\mathscr{F}$ and

$$G_C(f) = G(f|C|^2), \quad f\in\mathscr{F},$$

is a Gaussian process with almost all sample paths in $\mathscr{C}_u(\mathscr{F})$.

Under a fourth moment condition and under the total boundedness of $\mathscr{F}$, we prove a strong law of large numbers (SLLN) for $J_{n,X}$ in $\mathscr{C}_u(\mathscr{F})$, i.e.

$$\|J_{n,X} - J_C\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0.$$

Our proofs strongly depend on methods developed in probability theory in Banach spaces (see e.g. Ledoux and Talagrand, 1991) and for random quadratic forms (see e.g. Kwapień and Woyczyński, 1992). We also make use of a decomposition of the form

$$J_{n,X}(f) \approx J_{n,Z}(f|C|^2)$$

uniformly for $f\in\mathscr{F}$ which allows to reduce the limit theory for the empirical spectral distribution function of the stationary sequence X to the limit theory for the empirical spectral distribution function of the iid sequence Z.

The paper is organized as follows. In Section 2 we introduce some notation and assumptions used throughout. Our main results (CLT and SLLN for $J_{n,X}$) are presented in Section 3. Some auxiliary results are collected in Section 4. In Sections 5–7 we prove the CLT, the weak law of large numbers (WLLN) and the SLLN for the empirical spectral distribution function.

2. Notation and assumptions

Throughout $\mathscr{F}$ is a subset of the weighted $\mathscr{L}^2$ spaces

$$\mathscr{L}^2(\Pi, C) = \left\{f\colon \Pi \to \mathscr{R};\ \|f\|_{2,C} = \left(\int_\Pi f^2(x)|C(x)|^4\,\mathrm{d}x\right)^{1/2} < \infty\right\}.$$

We also write $\mathscr{L}^2(\Pi) = \mathscr{L}^2(\Pi, 1)$ and $\|\cdot\|_2 = \|\cdot\|_{2,1}$. We note that elements of $\mathscr{L}^2(\Pi, C)$ are functions rather than equivalence classes. So $\|\cdot\|_{2,C}$ induces a pseudometric $d_{2,C}(f, g) = \|f - g\|_{2,C}$ on $\mathscr{F}$. We also write $d_2 = d_{2,1}$.

Given a pseudometric space $(\mathscr{F}, d)$ and $\varepsilon > 0$, the ε-covering number $N(\varepsilon, \mathscr{F}, d)$ of $(\mathscr{F}, d)$ is defined as the minimal integer m such that there exist functions $f_1, \ldots, f_m \in \mathscr{F}$

with $\sup_{f \in \mathscr{F}} \min_{i=1,\dots,m} d(f, f_i) < \varepsilon$. Recall that $(\mathscr{F}, d)$ is totally bounded if and only if $N(\varepsilon, \mathscr{F}, d) < \infty$ for each $\varepsilon > 0$.

Define functions $a_{t,C} = (a_{t,C}(f))_{f \in \mathscr{F}}$ by

$$a_{t,C}(f) = \int_{\Pi} f(x)|C(x)|^2 \cos(xt)\,\mathrm{d}x, \quad t \in \mathscr{Z},\ f \in \mathscr{F},$$

and set $a_t = a_{t,1}$. We note that a_t and $a_{t,C}$ belong to $\mathscr{C}_u(\mathscr{F})$ when $\mathscr{F}$ is equipped with d_2 and $d_{2,C}$, respectively. Let

$$\gamma_{n,X}(t) = \frac{1}{n} \sum_{s=1}^{n-|t|} X_s X_{s+|t|}, \quad t \in \mathscr{Z},$$

denote the sample covariances of the vector $(X_1, \dots, X_n)$ with the convention that $\gamma_{n,X}(t) = 0$ for $|t| \geqslant n$.

We frequently make use of the following decomposition of the empirical spectral distribution function $J_{n,X}$:

$$J_{n,X} = \gamma_{n,X}(0) a_0 + Q_{n,X}, \tag{2.1}$$

where

$$Q_{n,X} = \frac{1}{n} \sum_{1 \leqslant s \neq t \leqslant n} a_{t-s} X_t X_s = 2 \sum_{t=1}^{n-1} a_t \gamma_{n,X}(t).$$

We also write

$$Q_n(A, B) = \frac{1}{n} \sum_{1 \leqslant s \neq t \leqslant n} a_{t-s} A_t B_s$$

for any sequences of random variables $A = (A_t)$, $B = (B_t)$. Since $a_t \in \mathscr{C}_u(\mathscr{F})$ for each t, the decomposition (2.1) shows that $J_{n,X}$ has almost all sample paths in $\mathscr{C}_u(\mathscr{F})$.

By c we denote a generic constant whose value may change from line to line or even from formula to formula.

3. Main results

In this section we present our main results, a CLT and an SLLN for the empirical spectral distribution function $J_{n,X}$ indexed by a class of functions $\mathscr{F} \subset \mathscr{L}^2(\Pi, C)$.

Theorem 3.1 (Central limit theorem). *Let X be a linear process* (1.1) *with i.i.d. innovations Z such that $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2$, $EZ_1^4 < \infty$ and with coefficients (c_j) satisfying $\sum_{j=1}^{\infty} c_j^2 j^{3/2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Consider the empirical spectral distribution function $J_{n,X}$ indexed by a subset $\mathscr{F}$ of $\mathscr{L}^2(\Pi, C)$. Assume that $\sup_{f \in \mathscr{F}} \|f\|_2 < \infty$ and*

$$\int_0^1 \ln N(\varepsilon, \mathscr{F}, d_{2,C})\,\mathrm{d}\varepsilon < \infty. \tag{3.1}$$

Then $J_{n,X}$ satisfies the CLT in $\mathscr{C}_u(\mathscr{F})$, i.e.

$$\sqrt{n}(J_{n,X} - J_C) \xrightarrow{d} G_C,$$

with Gaussian limit process $G_C = (G_C(f))_{f\in\mathscr{F}}$ given by

$$G_C = G_0(\mathrm{var}(Z_1^2))^{1/2} a_{0,C} + 2\sigma^2 \sum_{t=1}^{\infty} G_t a_{t,C}, \tag{3.2}$$

where (G_t) are i.i.d. standard normal random variables.

Remarks. (1) If $EZ_1^4 = \infty$ the CLT of Theorem 3.1 does not remain valid for $X = Z$ even for fixed f (see Klüppelberg and Mikosch, 1996).

(2) The condition $\sum_{j=1}^{\infty} c_j^2 j^{3/2+\varepsilon} < \infty$ rules out long-range dependence as e.g. for fractional ARIMA(p, d, q) processes where $c_j = j^{d-1}(1 + o(1))$ as $j \to \infty$ for some $d \in (0, 1/2)$. For $f = 1/|C|^2$, Kokoszka and Mikosch (1997) derived the limit theory of $J_{n,X}(f)$ both in the finite and infinite variance cases.

(3) The case $f = 1$ was treated in Anderson (1993) who also derived the asymptotic distribution of various goodness-of-fit test statistics. Dahlhaus (1988) obtained Theorem 3.1 for a vector-valued stationary process assuming an exponential moment condition and an entropy condition which corresponds to square integrability of $\ln N(\varepsilon, \mathscr{F}, d_{2,C})$. He also mentioned (his Remark 2.6) that, for Gaussian processes, his condition can be replaced by integrability of $\ln(\varepsilon^{-1} N(\varepsilon, \mathscr{F}, d_{2,C}))$. The cases $f = 1$ and $f = 1/|C|^2$ were treated in relation to goodness-of-fit tests by Bartlett (1954), see also Bartlett (1978), and by Grenander and Rosenblatt (1957). We refer to Priestley (1981) for a survey on these results.

(4) We reformulate Theorem 3.1 for the particular case that $J_{n,X}$ is indexed by functions of the form $f/|C|^2$. Let $\mathscr{F} \subset \mathscr{L}^2(\Pi)$ and $\mathscr{F}_C = \{f/|C|^2 \colon f \in \mathscr{F}\}$ where we assume that $|C|^2 > 0$ on Π. Then $\mathscr{F}_C \subset \mathscr{L}^2(\Pi, C)$. We also assume that $\sup_{g\in\mathscr{F}_C} \|g\|_2 < \infty$ and that $\int_0^1 \ln N(\varepsilon, \mathscr{F}_C, d_2)\,d\varepsilon < \infty$. Then $\sqrt{n}(J_{n,X} - J) \xrightarrow{d} G$ in $\mathscr{C}_u(\mathscr{F}_C)$.

(5) We write

$$\tilde{G}_C(f) = \tilde{G}(f|C|^2) = 2\sigma^2 \sum_{t=1}^{\infty} G_t a_{t,C}(f), \quad f \in \mathscr{F}. \tag{3.3}$$

Then $\tilde{G}_C$ is a Gaussian process with covariance structure

$$E\tilde{G}_C(f)\tilde{G}_C(g) = 4\sigma^4 \sum_{t=1}^{\infty} a_{t,C}(f) a_{t,C}(g), \quad f, g \in \mathscr{F}.$$

Applying Parseval's formula (see Zygmund, 1988, formula (1.13) on p. 37) and writing $\tilde{f}(x) = \frac{1}{2}(f(x) + f(-x))$, we obtain

$$\begin{aligned} &E\tilde{G}_C(f)\tilde{G}_C(g) \\ &= 4\pi\sigma^4 \int_\Pi \tilde{f}(x) g(x) |C(x)|^4 \,dx - 2\sigma^4 \int_\Pi f(x)|C(x)|^2\,dx \int_\Pi g(x)|C(x)|^2\,dx \\ &= 8\sigma^4\pi^2 \left(\frac{1}{2\pi} \int_\Pi \tilde{f}(x) g(x) |C(x)|^4\,dx \right. \\ &\quad \left. - \frac{1}{2\pi}\int_\Pi f(x)|C(x)|^2\,dx \frac{1}{2\pi}\int_\Pi g(x)|C(x)|^2\,dx \right). \end{aligned}$$

If $\mathscr{F}$ is a class of even functions this is the covariance structure (up to a constant multiple) of a generalised Brownian bridge. Moreover,

$$EG_C(f)G_C(g)$$
$$= (\operatorname{var}(Z_1^2) - 2\sigma^4)\int_\Pi f(x)|C(x)|^2\,dx\int_\Pi g(x)|C(x)|^2\,dx$$
$$+ 4\pi\sigma^4\int_\Pi \tilde{f}(x)g(x)|C(x)|^4\,dx.$$

If $\mathscr{F}$ is a class of even functions and Z_1 is Gaussian the first summand in the latter relation vanishes and G_C has then the covariance structure of a generalized Wiener process.

(6) The entropy condition (3.1) is satisfied for many classes of functions. For example, if $\mathscr{F}$ is a Vapnik-Červonenkis class (see Dudley, 1984; Pollard, 1984, p. 27) then it follows that $N(\varepsilon, \mathscr{F}, d_{2,c}) \leqslant c\varepsilon^{-w}$ for some positive $w > 0$ and hence (3.1) holds. For a fixed order (p, q), the spectral densities of causal, invertible ARMA(p, q), processes form a VC-class since their graphs actually constitute a finite-dimensional vector space. A collection of applications of Theorem 3.1 is provided by Dahlhaus (1988). These include the CLT for Whittle's estimate and the limit distribution for goodness-of-fit test statistics of Grenander–Rosenblatt type based on the spectral density with estimated parameters. We mention that the results below also prove the weak and strong consistency of these statistics.

(7) It is common use to give sufficient conditions for the uniform CLT over a class $\mathscr{F}$ in terms of the intrinsic metric $\rho(f, g) = (E(G_C(f - g))^{1/2}$ for $f, g \in \mathscr{F}$. For the pseudometric $d_{2,C}$, using Parseval's formula (see Remark 5 above) and the representation (3.2), we have

$$\min((\operatorname{var}(Z_1^2))^{1/2}, \sigma^2)\sqrt{2\pi}d_{2,C}(f, g) \leqslant \rho(f, g)$$
$$\leqslant \max((\operatorname{var}(Z_1^2))^{1/2}, \sigma^2)\sqrt{2\pi}d_{2,C}(f, g),$$

where the first inequality holds for even functions $f, g \in \mathscr{L}^2(\Pi, C)$ and the second one is satisfied for any functions $f, g \in \mathscr{L}^2(\Pi, C)$. Therefore, by Dudley's theorem (cf. Theorem 11.17 in Ledoux and Talagrand (1991)), if the class $\mathscr{F} \subset \mathscr{L}^2(\Pi, C)$ satisfies

$$\int_0^1 (\ln N(\varepsilon, \mathscr{F}, d_{2,C}))^{1/2} < \infty$$

then the Gaussian process G_C has a version with almost all sample paths in $C_u(\mathscr{F})$.

In applications, σ^2 is in general not known and has to be replaced by an estimator. In this case the CLT of Theorem 3.1 has to be modified. For the construction of the centring process in the CLT we closely follow Klüppelberg and

Mikosch (1996). There it is shown that this construction also works for infinite variance processes.

Theorem 3.2. *In addition to the conditions of Theorem* 3.1 *assume that* $1/|C|^2 \in \mathscr{L}^2(\Pi)$. *Then the empirical spectral distribution function* $J_{n,X}$ *satisfies the CLT in* $\mathscr{C}_u(\mathscr{F})$, *i.e.*

$$\sqrt{n}\left(J_{n,X} - \frac{T_n}{\sigma^2} J_C\right) \xrightarrow{d} \tilde{G}_C, \tag{3.4}$$

where

$$T_n = (2\pi)^{-1} J_{n,X}(1/|C|^2)$$

and $\tilde{G}_C$ *is the Gaussian limit process defined in* (3.3).

The limit $\tilde{G}_C$ still depends on the variance σ^2. It follows as in the proof of Theorem 3.2 that $T_n \xrightarrow{P} \sigma^2$. Thus we may conclude that the relation

$$\sqrt{n}(T_n^{-1} J_{n,X} - \sigma^{-2} J_C) \xrightarrow{d} \sigma^{-2}\tilde{G}_C \quad \text{in } \mathscr{C}_u(\mathscr{F})$$

holds. The quantity $T_n^{-1} J_{n,X}$ can be interpreted as self-normalised empirical spectral distribution function. Notice that the quantity $T_n^{-1} J_{n,X} - \sigma^{-2} J_C$ and the limit $\sigma^{-2}\tilde{G}_C$ do not explicitly depend on the variance σ^2.

Next we give the laws of large numbers for the empirical spectral distribution function.

Proposition 3.3 (Weak law of large numbers). *Let* X *be a linear process* (1.1) *with iid innovations* Z *such that* $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2 < \infty$ *and with coefficients* (c_j) *satisfying* $\sum_{j=1}^{\infty} c_j^2 j < \infty$. *Consider the empirical spectral distribution function* $J_{n,X}$ *indexed by a subset* $\mathscr{F}$ *of* $\mathscr{L}^2(\Pi, C)$. *Assume* $\sup_{f \in \mathscr{F}} \|f\|_2 < \infty$ *and* $\mathscr{F}$ *is totally bounded. Then* $J_{n,X}$ *satisfies the WLLN in* $\mathscr{C}_u(\mathscr{F})$, *i.e.*

$$\|J_{n,X} - J_C\|_{\mathscr{F}} \xrightarrow{P} 0.$$

Theorem 3.4 (Strong law of large numbers). *Let* X *be a linear process* (1.1) *with iid innovations* Z *such that* $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2$ *and* $EZ_1^4 < \infty$ *and with coefficients* (c_j) *satisfying* $\sum_{j=1}^{\infty} c_j^2 j^{3/2} < \infty$. *Consider the empirical spectral distribution function* $J_{n,X}$ *indexed by a subset* $\mathscr{F}$ *of* $\mathscr{L}^2(\Pi, C)$. *Assume* $\sup_{f \in \mathscr{F}} \|f\|_2 < \infty$ *and* $\mathscr{F}$ *is totally bounded. Then* $J_{n,X}$ *satisfies the SLLN in* $\mathscr{C}_u(\mathscr{F})$, *i.e.*

$$\|J_{n,X} - J_C\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0.$$

Remarks (8). The SLLN of Theorem 3.4 seems to be new. The fourth moment condition on Z is certainly not optimal. We guess that the existence of the second moment is sufficient.

(9) A WLLN for $J_{n,X}$ under an entropy condition is a consequence of Dahlhaus' (1988) results. However, it follows from Proposition 3.3 that total boundedness of $\mathscr{F}$ is

sufficient. This agrees with the uniform law of large numbers for sums of independent random variables (see Dudley, 1984).

(10) A careful study of the proofs in Sections 5–7 shows that, in the results formulated above, we may replace the periodogram $I_{n,X}$ by

$$\tilde{I}_{n,X}(\lambda) = \frac{1}{n}\left|\sum_{t=1}^{n} e_t(\lambda) X_t\right|^2$$

where (e_t) is any orthonormal class of complex-valued functions on Π. Indeed, the proofs depend only on the use of Parseval's (or Bessel's) formula and on the orthogonality of the cosine functions. The coefficients in the limit processes have then to be replaced by $\int_\Pi f(x)\mathrm{Re}(e_t(x))\,\mathrm{d}x$.

(11) The proofs in Sections 5–7 are based on limit results for the $\mathscr{C}_u(\mathscr{F})$-valued random quadratic forms $Q_{n,Z} = n^{-1}\sum_{1 \leqslant s \neq t \leqslant n} a_{t-s} Z_s Z_t$. The CLT, the WLLN and the SLLN for this structure might be of independent interest.

4. Tools

We collect some auxiliary results which are needed for the proofs in the following sections.

Lemma 4.1. *Let (C_i) be a sequence of i.i.d. symmetric random variables and $y > 0$.*
If $E|C_1| < \infty$ then there exist constants $c_1 = c_1(y)$ and x_0 such that

$$P\left(\frac{1}{n}\sum_{i=1}^{n} C_i I_{\{|C_i| \leqslant yn\}} > x\right) \leqslant \mathrm{e}^{-c_1 x}, \quad x \geqslant x_0 > 0. \tag{4.1}$$

If $EC_1^2 < \infty$ then there exist constants $c_2 = c_2(y)$ and x_0 such that

$$P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} C_i I_{\{|C_i| \leqslant y\sqrt{n}\}} > x\right) \leqslant \mathrm{e}^{-c_2 x}, \quad x \geqslant x_0 > 0. \tag{4.2}$$

Proof. An application of Prokhorov's exponential inequality (cf. Petrov, 1995, p. 77) yields the estimate

$$P\left(\frac{1}{n}\sum_{i=1}^{n} C_i I_{\{|C_i| \leqslant yn\}} > x\right) \leqslant \exp\left\{-\frac{x}{2y}\operatorname{arsinh}\left(\frac{yxn^2}{2B_n^{(1)}}\right)\right\},$$

where $B_n^{(1)} = n\,\mathrm{var}(C_1 I_{\{|C_1| \leqslant yn\}})$, but $B_n^{(1)} \leqslant yn^2 E|C_1|$. Similarly,

$$P\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} C_i I_{\{|C_i| \leqslant y\sqrt{n}\}} > x\right) \leqslant \exp\left\{-\frac{x}{2y}\operatorname{arsinh}\left(\frac{yxn}{2B_n^{(2)}}\right)\right\},$$

where $B_n^{(2)} = n\,\mathrm{var}(C_1 I_{\{|C_1| \leqslant y\sqrt{n}\}}) \leqslant nEC_1^2$. This proves the lemma. □

The following chaining lemma is due to Pisier (1983):

Lemma 4.2. *Let ψ be a convex, increasing function on $[0, \infty)$ such that $\lim_{t\to\infty}\psi(t) = \infty$ and $\psi(0) = 0$. Let S be a separable process on a pseudometric space $(\mathscr{F}, d)$ with property*

$$E\psi\left(\frac{|S(f) - S(g)|}{d(f, g)}\right) \leqslant 1, \quad \forall f, g \in \mathscr{F} \text{ with } d(f, g) \neq 0. \tag{4.3}$$

Then

$$E \sup_{f, g\in\mathscr{F}} |S(f) - S(g)| \leqslant 8 \int_0^D \psi^{-1}(N(\varepsilon, \mathscr{F}, d))\,\mathrm{d}\varepsilon,$$

where $D = \sup\{d(f, g) : f, g \in \mathscr{F}\}$ and ψ^{-1} denotes the inverse of ψ.

Now we give exponential tail inequalities for quadratic forms.

Lemma 4.3. (A) *Let (ε_t) be a Rademacher sequence and $(b_{ts})_{t,s=1,2,\ldots}$ be a double sequence from $\mathscr{C}_u(\mathscr{F})$ such that $b_{tt} = 0$ for each t and $b_{ts} = b_{st}$. Let*

$$\sigma^2 = \left\| \sum_{t=1}^{n} \sum_{s=1}^{n} b_{ts}^2 \right\|_{\mathscr{F}}$$

and let M, m be positive constants such that

$$P\left(\left\| \sum_{t=1}^{n} \sum_{s=1}^{n} b_{ts}\varepsilon_t\varepsilon_s \right\|_{\mathscr{F}} > M\right) < \frac{1}{64}, \qquad P\left(\left\| \sum_{t=1}^{n} \left(\sum_{s=1}^{n} b_{ts}\varepsilon_s\right)^2 \right\|_{\mathscr{F}} > m^2\right) < \frac{1}{16}.$$

Then, for each $y > 0$,

$$P\left(\left\| \sum_{t=1}^{n} \sum_{s=1}^{n} b_{ts}\varepsilon_t\varepsilon_s \right\|_{\mathscr{F}} > M + my + \sigma y^2\right) \leqslant 20\mathrm{e}^{-y^2/144}.$$

(B) *Let (ε_t) be a Rademacher sequence and $(b_{ts})_{t,s=1,2,\ldots}$ be a double sequence of real numbers such that $b_{tt} = 0$ for each t and $b_{ts} = b_{st}$. Then there exist positive constants q_1, q_2 such that, for each $y > 0$,*

$$P\left(\left(\sum_{1\leqslant s,t\leqslant n} b_{ts}^2\right)^{-1/2} \left| \sum_{t=1}^{n} \sum_{s=1}^{n} b_{ts}\varepsilon_t\varepsilon_s \right| > y\right) \leqslant q_1 \mathrm{e}^{-q_2 y}.$$

Part A is given in Ledoux and Talagrand (1991, Theorem 4.11). The proof of part B can be found in Pisier and Zinn (1977, p. 292).

Chow and Lai (1973) proved the following lemma for real-valued random variables. The proof remains the same for random elements with values in a Banach space (see e.g. Lemma 3.4 in Mikosch and Norvaiša, 1987).

Lemma 4.4 *Let (U_n) and (W_n) be two sequences of random elements with values in a Banach space. Assume that $U_n + W_n \overset{\text{a.s.}}{\to} 0$. If $(U_1, \ldots, U_n)$ and W_n are independent for every n and $W_n \overset{P}{\to} 0$ then $W_n \overset{\text{a.s.}}{\to} 0$.*

5. Proofs of Theorem 3.1 and 3.2

First we reformulate and prove Theorem 3.1 for the i.i.d. sequence Z:

Proposition 5.1. *Assume that Z is a sequence of iid random variables with $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2$ and $EZ_1^4 < \infty$. Let $\mathscr{F} \subset \mathscr{L}^2(\Pi)$ be such that*

$$\int_0^1 \ln N(\varepsilon, \mathscr{F}, d_2)\,\mathrm{d}\varepsilon < \infty.$$

Then $J_{n,Z}$ satisfies the CLT in $\mathscr{C}_u(\mathscr{F})$, i.e. $\sqrt{n}(J_{n,Z} - J) \xrightarrow{d} G$ with Gaussian limit process G given by

$$G = G_0(\mathrm{var}(Z_1^2))^{1/2} a_0 + 2\sigma^2 \sum_{t=1}^{\infty} G_t a_t,$$

where (G_t) are iid standard normal random variables.

For ease of notation we assume $\sigma^2 = 1$ in the sequel. We prove this proposition by a series of lemmas. The first one characterises convergence in distribution in the Banach space $\mathscr{C}_u(\mathscr{F})$. A proof for the path space $\ell_\infty(\mathscr{F})$ can be found in Andersen and Dobrič (1987) (their implication (2.12.3)$\Rightarrow$(2.12.1) of Theorem 2.12), and the same arguments work for the path space $C_u(\mathscr{F})$. Andersen and Dobrič (1987) attribute this result to Hoffmann-Jørgensen; a textbook treatment can be found in van der Vaart and Wellner (1996).

Lemma 5.2. *Let $(\mathscr{F}, d)$ be a totally bounded pseudometric space. Assume that the processes $S_n = (S_n(f))_{f\in\mathscr{F}}$ have almost all sample paths in $\mathscr{C}_u(\mathscr{F})$. Suppose the following two conditions hold:*

(a) *The finite-dimensional distributions of (S_n) converge.*
(b) *(S_n) is eventually uniformly d-equicontinuous, i.e. for each $\varepsilon > 0$*

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(\sup_{\mathscr{F}(\delta)} |S_n(f) - S_n(g)| > \varepsilon\right) = 0,$$

where $\mathscr{F}(\delta) = \{(f, g)\colon f,\ g \in \mathscr{F},\ d(f, g) < \delta\}$. Then (S_n) converges in distribution to a process with sample paths in $\mathscr{C}_u(\mathscr{F})$.

In view of this lemma we first have to check the convergence of the finite-dimensional distributions of the process $\sqrt{n}(J_{n,Z} - J)$.

Lemma 5.3. *The finite-dimensional distributions of $\sqrt{n}(J_{n,Z} - J)$ converge to the corresponding ones of G.*

Proof. By the Cramér–Wold device (see Billingsley, 1968, Theorem 7.7) and by linearity of the coefficients a_t it is sufficient to prove, for each $f \in \mathscr{F}$,

$$\sqrt{n}(J_{n,Z}(f) - J(f)) \xrightarrow{d} G(f) = G_0(\mathrm{var}(Z_1^2))^{1/2} a_0(f) + 2 \sum_{t=1}^{\infty} G_t a_t(f), \tag{5.1}$$

where (G_t) are iid standard normal random variables. Choose any $f \in \mathscr{F}$. To show (5.1) we use the decomposition (2.1). By the CLT for vector-valued martingales (cf. Theorem 3.33 on p. 437 of Jacod and Shiryaev, 1987) we obtain, for each fixed $M \geqslant 1$,

$$\sqrt{n}(\gamma_{n,Z}(0) - 1, (\gamma_{n,Z}(t))_{t=1,\dots,M}) \xrightarrow{d} ((\mathrm{var}(Z_1^2))^{1/2} G_0, (G_t)_{t=1,\dots,M}),$$

given that $EZ_1^4 < \infty$, $EZ_1^2 = 1$, $EZ_1 = 0$. For every $M \geqslant 1$ we conclude with the continuous mapping theorem (see Billingsley, 1968, Theorem 5.1) that

$$\begin{aligned} &\sqrt{n}(\gamma_{n,Z}(0) - 1) a_0(f) + 2 \sum_{t=1}^{M} (\sqrt{n}\gamma_{n,Z}(t)) a_t(f) \\ &\xrightarrow{d} G_0(\mathrm{var}(Z_1^2))^{1/2} a_0(f) + 2 \sum_{t=1}^{M} G_t a_t(f). \end{aligned} \tag{5.2}$$

In view of the representation (2.1) and by (5.2) it now suffices to apply a Slutsky argument (see Billingsley, 1968, Theorem 4.2), i.e. we have to show that, for every $\varepsilon > 0$,

$$\lim_{M \to \infty} \limsup_{n \to \infty} P\left(\sqrt{n} \left| \sum_{t=M+1}^{n-1} \gamma_{n,Z}(t) a_t(f) \right| > \varepsilon \right) = 0. \tag{5.3}$$

By Čebyshev's inequality and by orthogonality of the $\gamma_{n,Z}(t)$, the probability in the latter relation is bounded by

$$\frac{1}{\varepsilon^2 n} \sum_{t=M+1}^{n-1} (n-t) a_t^2(f) \leqslant \frac{1}{\varepsilon^2} \sum_{t=M+1}^{n-1} a_t^2(f) \leqslant \frac{1}{\varepsilon^2} \sum_{t=M+1}^{\infty} a_t^2(f).$$

An application of Parseval's formula yields that the right-hand side of the latter inequality converges to zero as $M \to \infty$. This proves (5.3) and hence (5.1). This shows the convergence of the finite-dimensional distributions. □

We introduce some further notation: Let $(\tilde{a}_t)_{t \in \mathscr{Z}}$ be a sequence of real numbers such that

$$\sum_{t \in \mathscr{Z}} \tilde{a}_t^2 \leqslant 1, \qquad \tilde{a}_t = \tilde{a}_{-t},\ t \in \mathscr{Z} \quad \text{and} \quad \tilde{a}_0 = 0.$$

Moreover, we write

$$Y_t = Y_t^{(n)} = Z_t I_{\{|Z_t| \leqslant n^{1/4}\}} - EZ_1 I_{\{|Z_1| \leqslant n^{1/4}\}}, \quad t = 1, \dots, n, \tag{5.4}$$

$$\bar{Y}_t = \bar{Y}_t^{(n)} = Z_t I_{\{|Z_t| > n^{1/4}\}} - EZ_1 I_{\{|Z_1| > n^{1/4}\}}, \quad t = 1, \dots, n. \tag{5.5}$$

For a sequence of real numbers $b = (b_t)$ we write

$$\tilde{Q}_n(b) = \frac{1}{\sqrt{n}} \sum_{1 \leqslant s,t \leqslant n} \tilde{a}_{t-s} b_s b_t, \qquad \tilde{Q}_n^2(b) = \frac{1}{n} \sum_{1 \leqslant s,t \leqslant n} \tilde{a}_{t-s}^2 b_s b_t.$$

For fixed $c > 0$, we define a non-negative, increasing, convex function ψ with $\psi(0) = 0$: On $[c^{-1}, \infty)$, we have $\psi(x) = e^{cx} - e^{-1}$ and, on $[0, c^{-1}]$, we construct ψ in the same way as ψ_α before Lemma 2.1 in Arcones and Giné (1995).

The following lemma deals with (4.3) for $Q_n(Y)$ where $Y = (Y_t)$. It will imply the eventual uniform equicontinuity of $\sqrt{n}(J_{n,Z} - J)$ via an application of Lemma 4.2.

Lemma 5.4. *There exist constants $q_1, q_2 > 0$ such that*

$$E\exp\{q_1 |\tilde{Q}_n(Y)|\} \leqslant q_2 < \infty, \quad n \geqslant 1. \tag{5.6}$$

Proof. Let Y' be an independent copy of Y and $\tilde{Y} = Y - Y'$. We apply a symmetrisation argument for quadratic forms. Notice that $E(\tilde{Q}_n(\tilde{Y}) | Y') = E\tilde{Q}_n(Y)$. Then the convexity of the function ψ implies

$$E\exp\{q_1 |\tilde{Q}_n(Y)|\} \leqslant c + cE\psi(c|\tilde{Q}_n(\tilde{Y})|).$$

Let $\varepsilon = (\varepsilon_t)$ be a sequence of iid Rademacher random variables independent of $\tilde{Y}$. Then $\tilde{Y}$ and $\varepsilon\tilde{Y} = (\varepsilon_t \tilde{Y}_t)$ have the same distribution. Hence, to show (5.6), it suffices to prove that

$$I(x) = P(|\tilde{Q}_n(\varepsilon\tilde{Y})| > x) \leqslant ce^{-cx}, \quad x > 0,$$

for some constant $c > 0$. Writing $\tilde{Y}^2 = (\tilde{Y}_t^2)$ we obtain

$$\begin{aligned} I(x) &= P(|\tilde{Q}_n(\varepsilon\tilde{Y})| > x, \tilde{Q}_n^2(\tilde{Y}^2) \leqslant x) + P(|\tilde{Q}_n(\varepsilon\tilde{Y})| > x, \tilde{Q}_n^2(\tilde{Y}^2) > x) \\ &= I_1(x) + I_2(x). \end{aligned}$$

Since $\tilde{Q}_n^2(\tilde{Y}^2)$ is uniformly bounded by a constant c', $I_2(x) = 0$ for $x \geqslant x_0$, say. Conditioning on $\tilde{Y}$ and applying part (B) of Lemma 4.3, we conclude that for $x \geqslant x_0$,

$$\begin{aligned} I_1(x) &= E_{\tilde{Y}}(I_{\{\tilde{Q}_n^2(\tilde{Y}^2) \leqslant x\}} P(|\tilde{Q}_n(\varepsilon\tilde{Y})| > x \mid \tilde{Y})) \\ &\leqslant cE_{\tilde{Y}}\left(I_{\{\tilde{Q}_n^2(\tilde{Y}^2) \leqslant c'\}} \exp\left\{-\frac{cx}{(\tilde{Q}_n^2(\tilde{Y}^2))^{1/2}}\right\}\right) \\ &\leqslant ce^{-cx}, \quad x \geqslant x_0. \end{aligned}$$

This proves the lemma. □

Recall the definition of $Y = (Y_t)$ and $\bar{Y} = (\bar{Y}_t)$ from (5.4) and (5.5).

Lemma 5.5. *The following relations hold as $n \to \infty$:*

$$\|\sqrt{n} Q_n(Y, \bar{Y})\|_{\mathscr{F}} \xrightarrow{P} 0 \quad \text{and} \quad \|\sqrt{n} Q_n(\bar{Y}, \bar{Y})\|_{\mathscr{F}} \xrightarrow{P} 0.$$

Proof. For each $\varepsilon > 0$ we have

$$P(\|\sqrt{n}Q_n(Y, \bar{Y})\|_{\mathscr{F}} > \varepsilon)$$
$$= P\left(\left\|\sum_{s=1}^{n} \bar{Y}_s\left(\sum_{t=1, t\neq s}^{n} a_{t-s}Y_t\right)\right\|_{\mathscr{F}} > \sqrt{n}\varepsilon\right)$$
$$\leqslant P\left(\left\|\sum_{s=1}^{n} Z_s I_{\{|Z_s| > n^{1/4}\}}\left(\sum_{t=1, t\neq s}^{n} a_{t-s}Y_t\right)\right\|_{\mathscr{F}} > \sqrt{n}\varepsilon/2\right)$$
$$+ P\left(|EZ_1 I_{\{|Z_1| > n^{1/4}\}}|\left\|\sum_{s=1}^{n}\left(\sum_{t=1, t\neq s}^{n} a_{t-s}Y_t\right)\right\|_{\mathscr{F}} > \sqrt{n}\varepsilon/2\right)$$
$$= I_1 + I_2.$$

Then

$$I_1 \leqslant P(|Z_t| I_{\{|Z_t| > n^{1/4}\}} > 0 \quad \text{for some } t \in \{1, \dots, n\})$$
$$\leqslant nP(|Z_1| > n^{1/4}) = o(1),$$

and, by the Cauchy–Schwartz and the Markov inequality,

$$I_2 \leqslant P\left(cn^{-5/4}\left\|\sum_{t=1}^{n-1} a_t \sum_{s=1}^{n-t}(Y_{s+t} + Y_s)\right\|_{\mathscr{F}} > \varepsilon/2\right)$$
$$\leqslant P\left(cn^{-5/4}\left\|\sum_{t=1}^{n-1} a_t^2\right\|_{\mathscr{F}}^{1/2}\left(\sum_{t=1}^{n-1}\left(\sum_{s=1}^{n-t}(Y_{s+t} + Y_s)\right)^2\right)^{1/2} > \varepsilon/2\right)$$
$$\leqslant \varepsilon^{-2}cn^{-5/2}\sum_{t=1}^{n-1} E\left(\sum_{s=1}^{n-t}(Y_{s+t} + Y_s)\right)^2$$
$$\leqslant \varepsilon^{-2}cn^{-1/2}.$$

This proves the first statement; the second one can be proved analogously. □

Proof of Proposition 5.1. By Lemmas 5.2 and 5.3, it suffices to check the eventual uniform equicontinuity of $v_{n,Z} = \sqrt{n}(J_{n,Z} - J)$. By (2.1), for $X = Z$ we have for each $f, g \in \mathscr{F}$

$$|v_{n,Z}(f) - v_{n,Z}(g)|$$
$$= \sqrt{n}|\gamma_{n,Z}(0) - 1|\sqrt{2\pi}\|f - g\|_2 + \sqrt{n}|Q_{n,Z}(f) - Q_{n,Z}(g)|. \tag{5.7}$$

With Y and $\bar{Y}$ as in (5.4) and (5.5) we obtain

$$Q_{n,Z} = Q_n(Y, Y) + 2Q_n(\bar{Y}, Y) + Q_n(\bar{Y}, \bar{Y}).$$

In view of Lemma 5.5 and by (5.7) it suffices to show that, for each $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\left(\sqrt{n}\sup_{f, g\in\mathscr{F}: d_2(f,g) < \delta}|Q_{n,Y}(f) - Q_{n,Y}(g)| > \varepsilon\right) = 0.$$

We apply the chaining Lemma 4.2 to $S_{f,g} = \sqrt{n}(Q_{n,Y}(f) - Q_{n,Y}(g))$ with

$$f, g \in \mathscr{F}'(\delta) = \{f - g : f, g \in \mathscr{F}, d_2(f, g) < \delta\}.$$

Note that, for each $\varepsilon > 0$,

$$N(\varepsilon, \mathscr{F}'(\delta), d_2) \leqslant N^2(\varepsilon/2, \mathscr{F}, d_2).$$

The function ψ introduced before Lemma 5.4 satisfies the conditions of Lemma 4.2. By Lemma 5.4, (4.3) holds for $S_{f,g}$. This concludes the proof of Proposition 5.1. □

To prove Theorem 3.1 we need the following decomposition for the periodogram of the linear process (1.1):

$$I_{n,X}(\lambda) = |C(\lambda)|^2 I_{n,Z}(\lambda) + n^{-1} R_n(\lambda), \quad \lambda \in \Pi, \tag{5.8}$$

where

$$R_n(\lambda) = C(\lambda) L_n(\lambda) K_n(-\lambda) + C(-\lambda) L_n(-\lambda) K_n(\lambda) + |K_n(\lambda)|^2, \tag{5.9}$$

$$L_n(\lambda) = \sum_{t=1}^{n} Z_t \mathrm{e}^{-\mathrm{i}\lambda t}, \qquad K_n(\lambda) = \sum_{j=0}^{\infty} c_j \mathrm{e}^{-\mathrm{i}\lambda j} U_{nj}(\lambda),$$

$$U_{nj}(\lambda) = \sum_{t=1-j}^{n-j} Z_t \mathrm{e}^{-\mathrm{i}\lambda t} - \sum_{t=1}^{n} Z_t \mathrm{e}^{-\mathrm{i}\lambda t}.$$

Relation (5.8) follows by noting that $n^{-1}|L_n(\lambda)|^2 = I_{n,Z}(\lambda)$ and

$$n I_{n,X}(\lambda) = |K_n(\lambda) + C(\lambda) L_n(\lambda)|^2.$$

By (5.8) we may write for each f

$$\sqrt{n}(J_{n,X}(f) - J_C(f)) = \sqrt{n}(J_{n,Z}(f|C|^2) - J(f|C|^2)) + n^{-1/2} \int_\Pi f(x) R_n(x)\,\mathrm{d}x. \tag{5.10}$$

Therefore to prove Theorem 3.1 it is sufficient to use Proposition 5.1 and the relation

$$\frac{1}{\sqrt{n}} \sup_{f \in \mathscr{F}} \left| \int_\Pi f(x) R_n(x)\,\mathrm{d}x \right| \xrightarrow{P} 0, \tag{5.11}$$

whose proof is given by the following two lemmas.

Lemma 5.6. *Assume* $\sup_{f \in \mathscr{F}} \|f\|_2 < \infty$ *and* $l_1 = \sum_{j=1}^{\infty} c_j^2 j^{3/2} < \infty$. *Then*

$$E \sup_{f \in \mathscr{F}} \left| \int_\Pi f(x) |K_n(x)|^2\,\mathrm{d}x \right| \leqslant c, \quad n \geqslant 1.$$

In particular,

$$\sup_{f \in \mathscr{F}} \left| \int_\Pi f(x) |K_n(x)|^2\,\mathrm{d}x \right| = \mathrm{O_P}(1).$$

Proof. We have by the Cauchy–Schwartz inequality

$$\left|\int_{\Pi} f(x)|K_n(x)|^2\,\mathrm{d}x\right| \leqslant \int_{\Pi} f^2(x)\,\mathrm{d}x \int_{\Pi} |K_n(x)|^4\,\mathrm{d}x.$$

Since $\sup_{f\in\mathscr{F}} \|f\|_2 < \infty$ it suffices to show that

$$E\int_{\Pi} |K_n(x)|^4\,\mathrm{d}x \leqslant c, \quad n \geqslant 1. \tag{5.12}$$

Observe that

$$\begin{aligned} E\int_{\Pi} |K_n(x)|^4\,\mathrm{d}x &\leqslant c\Bigg(\int_{\Pi} E\left|\sum_{j=1}^{n} c_j \mathrm{e}^{-\mathrm{i}xj} \sum_{s=1-j}^{0} Z_s \mathrm{e}^{-\mathrm{i}xs}\right|^4 \mathrm{d}x \\ &\quad + \int_{\Pi} E\left|\sum_{j=1}^{n} c_j \mathrm{e}^{-\mathrm{i}xj} \sum_{s=n-j+1}^{n} Z_s \mathrm{e}^{-\mathrm{i}xs}\right|^4 \mathrm{d}x + \int_{\Pi} E\left|\sum_{j=n+1}^{\infty} c_j \mathrm{e}^{-\mathrm{i}xj} \sum_{s=1}^{n} Z_s \mathrm{e}^{-\mathrm{i}xs}\right|^4 \mathrm{d}x \\ &\quad + \int_{\Pi} E\left|\sum_{j=n+1}^{\infty} c_j \mathrm{e}^{-\mathrm{i}xj} \sum_{s=1-j}^{n-j} Z_s \mathrm{e}^{-\mathrm{i}xs}\right|^4 \mathrm{d}x\Bigg) \\ &= c(I_1 + I_2 + I_3 + I_4). \end{aligned}$$

We frequently make use of the identity

$$\left|\sum_{j=s+1}^{n} c_j \mathrm{e}^{-\mathrm{i}xj}\right|^2 = \sum_{j=s+1}^{n} c_j^2 + 2\sum_{j=1}^{n-s-1} \cos(xj) \sum_{r=s+1}^{n-j} c_r c_{r+j}, \tag{5.13}$$

By the Marcinkiewicz–Zygmund and the Minkowski inequality, using the orthogonality of the cosine functions and (5.13),

$$\begin{aligned} I_1 &= \int_{\Pi} E\left|\sum_{s=1-n}^{0} Z_s \mathrm{e}^{-\mathrm{i}xs} \sum_{j=1-s}^{n} c_j \mathrm{e}^{-\mathrm{i}xj}\right|^4 \mathrm{d}x \leqslant c\int_{\Pi}\left(\sum_{s=0}^{n-1}\left|\sum_{j=s+1}^{n} c_j \mathrm{e}^{-\mathrm{i}xj}\right|^2\right)^2 \mathrm{d}x \\ &\leqslant c\left(\sum_{j=1}^{\infty} c_j^2 j\right)^2 + c\int_{\Pi}\left(\sum_{j=1}^{n-1} \cos(xj) \sum_{s=0}^{n-j-1} \sum_{t=s+1}^{n-j} c_t c_{t+j}\right)^2 \mathrm{d}x \\ &\leqslant cl_1^2 + c\sum_{j=1}^{n-1}\left(\sum_{s=0}^{n-j-1} \sum_{t=s+1}^{n-j} c_t c_{t+j}\right)^2 \\ &\leqslant cl_1^2 + c\sum_{j=1}^{n-1}\left(\sum_{t=1}^{n-j} c_t c_{t+j} t\right)^2. \end{aligned}$$

An application of the Cauchy–Schwartz inequality and the assumption $l_1 < \infty$ yield

$$\begin{aligned} I_1 &\leqslant cl_1^2 + c\sum_{j=1}^{n} \sum_{t=1}^{n-j} c_t^2 t^{3/2} \sum_{t=1}^{n-j} c_{t+j}^2 (t+j)^{1/2} \\ &\leqslant cl_1^2 + cl_1 \sum_{j=1}^{n} \sum_{t=j}^{n} c_t^2 t^{1/2} \leqslant cl_1^2 < \infty. \end{aligned}$$

The same ideas apply to the estimation of I_2, I_3, I_4. We omit details. □

Lemma 5.7. *Assume that* $\sup_{f\in\mathscr{F}}\|f\|_2<\infty$ *and* $\sum_{j=1}^{\infty}c_j^2 j^{3/2+\varepsilon}<\infty$ *for some* $\varepsilon>0$. *Then* (5.11) *holds.*

Proof. Lemma 5.6 and by (5.9), it is sufficient to prove that

$$\frac{1}{\sqrt{n}}\sup_{f\in\mathscr{F}}\left|\int_{\Pi}f(x)C(x)L_n(x)K_n(-x)\,\mathrm{d}x\right|\stackrel{P}{\to}0.$$

Since $(\mathscr{F},d_{2,C})$ is totally bounded, for every $\delta>0$, we find functions $f_1,\ldots,f_m\in\mathscr{F}$ such that

$$\sup_{f\in\mathscr{F}}\min_{i=1,\ldots,m}d_{2,C}(f,f_i)<\delta.$$

We have by the Cauchy–Schwartz inequality

$$\left\|\int_{\Pi}f(x)C(x)L_n(x)K_n(-x)\,\mathrm{d}x\right\|_{\mathscr{F}}^2\leqslant c\delta\sup_{f\in\mathscr{F}}\|f\|_2\int_{\Pi}|L_n(x)K_n(-x)|^2\,\mathrm{d}x$$
$$+c\max_{i=1,\ldots,m}\left|\int_{\Pi}f_i(x)C(x)L_n(x)K_n(-x)\,\mathrm{d}x\right|^2.$$

Since $\sup_{f\in\mathscr{F}}\|f\|_2<\infty$ it suffices to show that, for each $f\in\mathscr{F}$,

$$I(f)=n^{-1/2}\left|\int_{\Pi}f(x)C(x)L_n(x)K_n(-x)\,\mathrm{d}x\right|=\mathrm{o_P}(1), \tag{5.14}$$

$$I=n^{-1}\int_{\Pi}|L_n(x)K_n(-x)|^2\,\mathrm{d}x=\mathrm{O_P}(1). \tag{5.15}$$

We have by the Cauchy–Schwartz inequality and by orthogonality of the cosine functions

$$I^2\leqslant\int_{\Pi}I_{n,Z}^2(x)\,\mathrm{d}x\int_{\Pi}|K_n(-x)|^4\,\mathrm{d}x\leqslant c\sum_{t=0}^{n-1}\gamma_{n,Z}^2(t)\int_{\Pi}|K_n(-x)|^4\,\mathrm{d}x.$$

But $E\sum_{t=0}^{n-1}\gamma_{n,Z}^2(t)\leqslant c$, hence $\sum_{t=0}^{n-1}\gamma_{n,Z}^2(t)=\mathrm{O_P}(1)$, and $\int_{\Pi}|K_n(-x)|^4\,\mathrm{d}x=\mathrm{O_P}(1)$ in view of (5.12). This proves (5.15).

Now we turn to (5.14). We have

$$E|I(f)|^2\leqslant cn^{-1}E\left|\sum_{t=1}^{n}\sum_{s=0}^{\infty}Z_tZ_{-s}\sum_{j=s+1}^{\infty}c_j\int_{\Pi}f(x)C(x)\mathrm{e}^{-\mathrm{i}x(s+t-j)}\,\mathrm{d}x\right|^2$$
$$+cn^{-1}E\left|\sum_{t=1}^{n}\sum_{s=-n}^{\infty}Z_tZ_{-s}\sum_{j=n+s+1}^{\infty}c_j\int_{\Pi}f(x)C(x)\mathrm{e}^{-\mathrm{i}x(s+t-j)}\,\mathrm{d}x\right|^2$$
$$\leqslant cn^{-1}\sum_{t=1}^{n}\sum_{s=0}^{\infty}\left|\sum_{j=s+1}^{\infty}c_j\int_{\Pi}f(x)C(x)\mathrm{e}^{-\mathrm{i}x(s+t-j)}\,\mathrm{d}x\right|^2$$
$$+cn^{-1}E\left|\sum_{t=1}^{n}\sum_{s=1}^{n}Z_tZ_s\sum_{j=n-s+1}^{\infty}c_j\int_{\Pi}f(x)C(x)\mathrm{e}^{-\mathrm{i}x(t-j-s)}\,\mathrm{d}x\right|^2$$

$$+ cn^{-1}E\left|\sum_{t=1}^{n}\sum_{s=0}^{\infty} Z_t Z_{-s} \sum_{j=n+s+1}^{\infty} c_j \int_{\Pi} f(x)C(x)e^{-ix(t-j+s)}dx\right|^2$$

$$= c(I_1 + I_2 + I_3).$$

Using the Cauchy–Schwartz inequality we obtain for some $\varepsilon > 0$

$$\begin{aligned}
I_1 &\leqslant n^{-1}\sum_{t=1}^{n}\sum_{s=0}^{\infty}\left[\left(\sum_{j_1=s+1}^{\infty} c_{j_1}^2 j_1^{1/2+\varepsilon}\right)\left(\sum_{j_2=s+1}^{\infty}\frac{1}{j_2^{1/2+\varepsilon}}\left|\int_{\Pi} f(x)C(x)e^{-ix(s+t-j_2)}dx\right|^2\right)\right]\\
&\leqslant c\left(\sum_{s=1}^{\infty} c_s^2 s^{3/2+\varepsilon}\right)\left(n^{-1}\sum_{t=1}^{n}\sum_{j=1}^{\infty}\frac{1}{j^{1/2+\varepsilon}}\left|\int_{\Pi} f(x)C(x)e^{-ix(t-j)}dx\right|^2\right)\\
&\leqslant cn^{-1}\sum_{t=0}^{n-1}\left(\left|\int_{\Pi} f(x)C(x)e^{-ixt}dx\right|^2 + \left|\int_{\Pi} f(x)C(x)e^{ixt}dx\right|^2\right)\sum_{j=1}^{\infty}\frac{1}{j^{1+2\varepsilon}}\\
&\quad + cn^{-1}\sum_{t=1}^{n}\sum_{j=n-t+1}^{\infty}(j+t)^{-1/2-\varepsilon}\left|\int_{\Pi} f(x)C(x)e^{ixj}dx\right|^2 \qquad (5.16)\\
&\leqslant cn^{-1}\sum_{t=0}^{n-1}\left(\left|\int_{\Pi} f(x)C(x)e^{-ixt}dx\right|^2 + \left|\int_{\Pi} f(x)C(x)e^{ixt}dx\right|^2\right)\\
&\quad + cn^{-1}\sum_{t=1}^{n}\sum_{j=t}^{\infty}\left|\int_{\Pi} f(x)C(x)e^{ixj}dx\right|^2.
\end{aligned}$$

The right-hand side converges to zero as $n \to \infty$ since, by Parseval's formula, by the Cauchy–Schwartz inequality and since $\sup_{f\in\mathscr{F}}\|f\|_2 < \infty$,

$$\sum_{j=-\infty}^{\infty}\left|\int_{\Pi} f(x)C(x)e^{ixj}dx\right|^2 = 2\pi\int_{\Pi} f^2(x)|C(x)|^2 dx \leqslant 2\pi\|f\|_2\|f\|_{2,C} \leqslant c < \infty.$$

The estimation of I_3 follows the same patterns and is therefore omitted.

The term I_2 can be bounded as follows:

$$\begin{aligned}
I_2 &\leqslant cn^{-1}E\left|\sum_{t=1}^{n}(Z_t^2 - EZ_1^2)\sum_{j=n-t+1}^{\infty} c_j\int_{\Pi} f(x)C(x)e^{ixj}dx\right|^2\\
&\quad + cn^{-1}\left|\sum_{t=1}^{n}\sum_{j=n-t+1}^{\infty} c_j\int_{\Pi} f(x)C(x)e^{ixj}dx\right|^2\\
&\quad + cn^{-1}E\left|\sum_{1\leqslant t\neq s\leqslant n} Z_t Z_s \sum_{j=n-s+1}^{\infty} c_j\int_{\Pi} f(x)C(x)e^{-ix(t-j-s)}dx\right|^2.
\end{aligned}$$

Then similar moment estimates and multiple use of the Cauchy–Schwartz inequality yield that $I_2 \leqslant c$. This proves (5.15) and concludes the proof of the lemma. □

Proof of Theorem 3.1. Let $\mathscr{F}_C = \{f|C|^2 : f\in\mathscr{F}\}$. Then $N(\varepsilon, \mathscr{F}_C, d_2) = N(\varepsilon, \mathscr{F}, d_{2,C})$ for each $\varepsilon > 0$. Hence, by Proposition 5.1, we have

$$\sqrt{n}(J_{n,Z}(f|C|^2) - J(f|C|^2))_{f\in\mathscr{F}} \xrightarrow{d} (G(f|C|^2))_{f\in\mathscr{F}} = (G_C(f))_{f\in\mathscr{F}}.$$

Now the statement of Theorem 3.1 follows from the representation (5.10) and from Lemma 5.7. □

Proof of Theorem 3.2. In view of decompositions (5.8) and (2.1) for $X = Z$, by Lemma 5.7, we may conclude that

$$\begin{aligned}\mu_n(f) &= \sqrt{n}\left(J_{n,X}(f) - \frac{T_n}{\sigma^2}J_C(f)\right)\\ &= \sqrt{n}\left(J_{n,Z}(f|C|^2) - \frac{1}{2\pi}J_{n,Z}(1)\frac{1}{\sigma^2}J(f|C|^2)\right)\\ &\quad + \frac{1}{\sqrt{n}}\int_\Pi f(x)R_n(x)\,\mathrm{d}x + \frac{J(f|C|^2)}{2\pi\sigma^2\sqrt{n}}\int_\Pi \frac{R(x)}{|C(x)|^2}\,\mathrm{d}x\\ &= 2\sqrt{n}\sum_{t=1}^{n-1} a_{t,C}(f)\gamma_{n,Z}(t) + o_P(1)\end{aligned}$$

uniformly for $f \in \mathscr{F}$. Now the same arguments as for the proof of Theorem 3.1 apply showing that $\mu_n \xrightarrow{d} \tilde{G}_C$ in $\mathscr{C}_u(\mathscr{F})$. □

6. Proof of Proposition 3.3

We first formulate and prove Proposition 3.3 for $X = Z$:

Proposition 6.1. *Assume that Z is a sequence of iid random variables with $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2 < \infty$. Let $\mathscr{F} \subset \mathscr{L}^2(\Pi)$ be totally bounded. Then $J_{n,Z}$ satisfies the WLLN in $\mathscr{C}_u(\mathscr{F})$, i.e. $\|J_{n,Z} - J\|_{\mathscr{F}} \xrightarrow{P} 0$.*

For ease of notation we always assume $\sigma^2 = 1$.

Proof. By (2.1) for $X = Z$, we have for each $f \in \mathscr{F}$

$$|J_{n,Z}(f) - J(f)| \leqslant \sqrt{2\pi}\sup_{f\in\mathscr{F}}\|f\|_2\,|\gamma_{n,Z}(0) - 1| + |Q_{n,Z}(f)|.$$

Hence, by the SLLN for $\gamma_{n,Z}(0)$, it is sufficient to show that, for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} P(\|Q_{n,Z}\|_{\mathscr{F}} > \varepsilon) \leqslant \varepsilon. \tag{6.1}$$

Fix $\varepsilon > 0$. Since $\{f|C|^2 : f \in \mathscr{F}\}$ is totally bounded there exist functions $f_1, \dots, f_m \in \mathscr{F}$ such that

$$\sup_{f\in\mathscr{F}}\ \min_{i=1,\dots,m} \|f - f_i\|_2 < \tfrac{1}{4}\varepsilon^{3/2}.$$

Therefore

$$P(\|Q_{n,Z}\|_{\mathscr{F}} > \varepsilon) \leqslant \sum_{i=1}^{m} P(|Q_{n,Z}(f_i)| > \tfrac{\varepsilon}{2}) + P\left(\sup_{\mathscr{F}(\varepsilon^{3/2}/4)} |Q_{n,Z}(f) - Q_{n,Z}(g)| > \frac{\varepsilon}{2}\right), \tag{6.2}$$

where $\mathscr{F}(\delta) = \{(f,g): f, g \in \mathscr{F}, \|f - g_2\|_2 < \delta\}$. By Čebyshev's inequality and by orthogonality of the $\gamma_{n,Z}(t)$ for each $i = 1, \ldots, m$

$$P(|Q_{n,Z}(f_i)| > \tfrac{\varepsilon}{2}) = P\left(\left|\sum_{t=1}^{n-1} \gamma_{n,Z}(t) a_t(f_i)\right| > \frac{\varepsilon}{4}\right)$$

$$\leqslant 4^2 \varepsilon^{-2} E\left(\sum_{t=1}^{n-1} \gamma_{n,Z}(t) a_t(f_i)\right)^2 = 4^2 n^{-2} \varepsilon^{-2} \sum_{t=1}^{n-1} a_t^2(f_i)(n-t)$$

$$\leqslant 4^2 n^{-1} \varepsilon^{-2} \|f_i\|_2^2. \tag{6.3}$$

In the last step we used Parseval's formula. Applying Markov's inequality we conclude that

$$P(\|Q_{n,Z}\|_{\mathscr{F}(\varepsilon^{3/2}/4)} > \tfrac{\varepsilon}{2}) = P\left(\sup_{\|f-g\|_2 < \varepsilon^{3/2}/4} \left|\sum_{t=1}^{n-1} \gamma_{n,Z}(t) a_t(f-g)\right| > \frac{\varepsilon}{4}\right)$$

$$\leqslant 4^2 \varepsilon^{-2} E \sup_{\|f-g\|_2 < \varepsilon^{3/2}/4} \left|\sum_{t=1}^{n-1} \gamma_{n,Z}(t) a_t(f-g)\right|^2. \tag{6.3'}$$

The right-hand side can be bounded by use of the Cauchy–Schwartz inequality and by Parseval's formula as follows:

$$\text{RHS of (6.3')} \leqslant 4^2 \varepsilon^{-2} \sup_{\|f-g\|_2 < \varepsilon^{3/2}/4} \left|\sum_{t=1}^{\infty} a_t^2(f-g)\right| E\left(\sum_{t=1}^{n-1} \gamma_{n,Z}^2(t)\right)$$

$$\leqslant 4^2 \varepsilon^{-2} \sup_{\|f-g\|_2 < \varepsilon^{3/2}/4} \|f-g\|_2^2 \sum_{t=1}^{n-1} E\gamma_{n,Z}^2(t)$$

$$= \varepsilon n^{-2} \sum_{t=1}^{n-1} (n-t) \leqslant \varepsilon. \tag{6.4}$$

Therefore (6.1) follows from (6.2)–(6.4). This proves the proposition. □

Proposition 3.3 follows now from Proposition 6.1, the decomposition (5.8) and the following lemma.

Lemma 6.2. *Assume that* $\sum_{j=0}^{\infty} c_j^2 j < \infty$. *Let* $\mathscr{F}$ *be a totally bounded subset of* $\mathscr{L}^2(\Pi, C)$ *with* $\sup_{f \in \mathscr{F}} \|f\|_2 < \infty$. *Then*

$$n^{-1} \sup_{f \in \mathscr{F}} \left| \int_{\Pi} R_n(x) f(x)\, \mathrm{d}x \right| \xrightarrow{P} 0.$$

Proof. Recall the decomposition (5.8). The Cauchy–Schwartz inequality yields

$$n^{-1}\left|\int_{\Pi} f(x)C(x)L_n(x)K_n(-x)\,\mathrm{d}x\right| \leqslant \left(\int_{\Pi}|f(x)|\,|C(x)|^2 I_{n,Z}(x)\,\mathrm{d}x\right)^{1/2}\left(n^{-1}\int_{\Pi}|f(x)|\,|K_n(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Since $\mathscr{F}$ is totally bounded we conclude from Proposition 6.1 that $(J_{n,Z}(|f|\,|C|^2))_{f\in\mathscr{F}}$ satisfies the WLLN. Hence it suffices to show that

$$n^{-1}\sup_{f\in\mathscr{F}}\left|\int_{\Pi}|f(x)|\,|K_n(x)|^2\,\mathrm{d}x\right| \xrightarrow{P} 0.$$

We have

$$\begin{aligned}
&n^{-1}\int_{\Pi}|f(x)|\,|K_n(x)|^2\,\mathrm{d}x\\
&\leqslant cn^{-1}\int_{\Pi}|f(x)|\left|\sum_{j=n+1}^{\infty}c_j\mathrm{e}^{-\mathrm{i}xj}\sum_{t=1}^{n}Z_t\mathrm{e}^{-\mathrm{i}xt}\right|^2\mathrm{d}x\\
&\quad+n^{-1}\int_{\Pi}|f(x)|\left|\sum_{j=n+1}^{\infty}c_j\mathrm{e}^{-\mathrm{i}xj}\sum_{t=1-j}^{n-j}Z_t\mathrm{e}^{-\mathrm{i}xt}\right|^2\mathrm{d}x\\
&\quad+cn^{-1}\int_{\Pi}|f(x)|\left|\sum_{j=1}^{n}c_j\mathrm{e}^{-\mathrm{i}xj}\sum_{t=1-j}^{0}Z_t\mathrm{e}^{-\mathrm{i}xt}\right|^2\mathrm{d}x\\
&\quad+cn^{-1}\int_{\Pi}|f(x)|\left|\sum_{j=1}^{n}c_j\mathrm{e}^{-\mathrm{i}xj}\sum_{t=n-j+1}^{n}Z_t\mathrm{e}^{-\mathrm{i}xt}\right|^2\mathrm{d}x\\
&=c(I_1+I_2+I_3+I_4).
\end{aligned}$$

We restrict ourselves to the estimation of I_3 in order to illustrate the method. We have

$$\begin{aligned}
I_3&=n^{-1}\int_{\Pi}|f(x)|\sum_{t=0}^{n-1}Z_t^2\left|\sum_{j=t+1}^{n}c_j\mathrm{e}^{-\mathrm{i}xj}\right|^2\mathrm{d}x\\
&\quad+n^{-1}\int_{\Pi}|f(x)|\sum_{0\leqslant s\neq t\leqslant n-1}Z_sZ_t\sum_{j_1=s+1}^{n}c_{j_1}\mathrm{e}^{-\mathrm{i}x(j_1-s)}\sum_{j_2=t+1}^{n}c_{j_2}\mathrm{e}^{\mathrm{i}x(j_2-t)}\,\mathrm{d}x\\
&=I_{31}+I_{32}.
\end{aligned}$$

Applying (5.13) and using the Cauchy–Schwartz inequality, Parseval's formula and the assumption $\sum_{j=1}^{\infty}c_j^2 j<\infty$, we obtain

$$\begin{aligned}
EI_{31}&\leqslant cn^{-1}E\sum_{t=0}^{n-1}Z_t^2\left(\sum_{j_1=0}^{n-t-1}a_{j_1}^2(|f|)\right)^{1/2}\left(\sum_{j_2=0}^{n-t-1}\left(\sum_{r=t+1}^{n-j}c_rc_{r+j_2}\right)^2\right)^{1/2}\\
&\leqslant c\|f\|_2 n^{-1}\sum_{t=0}^{n-1}\left(\sum_{j=0}^{n-t-1}\sum_{r_1=t+1}^{n-j}c_{r_1}^2\sum_{r_2=t+1}^{n-j}c_{r_2+j}^2\right)^{1/2}\\
&\leqslant c\|f\|_2\left(\sum_{r=1}^{\infty}c_r^2 r\right)^{1/2}n^{-1}\sum_{t=0}^{n-1}\left(\sum_{r=t+1}^{\infty}c_r^2\right)^{1/2}=\mathrm{o}(1).
\end{aligned}$$

In the last step we used that $\sup_{f\in\mathscr{F}}\|f\|_2 < \infty$. Applying the Cauchy–Schwartz inequality, we have

$$EI_{32}^2 \leqslant c\|f\|_2^2 n^{-2} E\int_{\Pi}\left|\sum_{0\leqslant s\neq t\leqslant n-1} Z_s Z_t \sum_{j_1=s+1}^{n} c_{j_1}\mathrm{e}^{-\mathrm{i}x(j_1-s)}\sum_{j_2=t+1}^{n} c_{j_2}\mathrm{e}^{\mathrm{i}x(j_2-t)}\right|^2 \mathrm{d}x$$

$$\leqslant c\|f\|_2^2 n^{-2}\sum_{0\leqslant s\neq t\leqslant n-1}\int_{\Pi}\left|\sum_{j_1=s+1}^{n} c_{j_1}\mathrm{e}^{-\mathrm{i}xj_1}\sum_{j_2=t+1}^{n} c_{j_2}\mathrm{e}^{\mathrm{i}xj_2}\right|^2 \mathrm{d}x.$$

Representation (5.13), the orthogonality of the cosine functions and the Cauchy–Schwartz inequality lead to the estimate

$$EI_{32}^2 \leqslant c\|f\|_2^2 n^{-2}\sum_{0\leqslant s<t\leqslant n}\sum_{l=0}^{n-s}\left|\sum_{j_1=s+1}^{n-l} c_{j_1}c_{j_1+l}\sum_{j_2=t+1}^{n-l} c_{j_2}c_{j_2+l}\right|$$

$$\leqslant c\|f\|_2^2\left(n^{-1}\sum_{s=0}^{n}\left(\sum_{k=s}^{\infty} c_k^2\right)^{1/2}\right)^2\left(\sum_{l=0}^{\infty}\sum_{j=l}^{\infty} c_j^2\right)$$

$$= \mathrm{o}\left(\sum_{l=1}^{\infty} c_l^2 l\right) = \mathrm{o}(1).$$

This concludes the proof of the lemma. □

7. Proof of Theorem 3.4

We first formulate and prove Theorem 3.4 for $X = Z$:

Proposition 7.1. *Assume Z is a sequence of iid random variables with $EZ_1 = 0$, $\mathrm{var}(Z_1) = \sigma^2$ and $EZ_1^4 < \infty$. Let $\mathscr{F} \subset \mathscr{L}^2(\Pi, C)$ be totally bounded. Then $J_{n,Z}$ satisfies the SLLN in $\mathscr{C}_u(\mathscr{F})$, i.e.*

$$\|J_{n,Z} - J\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0.$$

We first give two tail estimates for the maximum of the norms of $\mathscr{C}_u(\mathscr{F})$-valued quadratic forms.

Lemma 7.2. *Let (Z_t) be iid random variables and $\mathscr{F} \subset \mathscr{L}^2(\Pi)$ be totally bounded. Assume that $EZ_1 = 0$ and $EZ_1^2 = 1$. Let $Z' = (Z_t')$ be an independent copy of Z. For every $x > 0$ and $\delta \in (0, 1)$ there exist a constant $c = c(\delta, x) > 0$ and an integer n_0 such that the following relation holds:*

$$\delta P\left(\frac{1}{n}\max_{k\leqslant n}\|kQ_k(Z, Z')\|_{\mathscr{F}} > x\right)$$

$$= q_1 P(q_2\|Q_n(Z - Z', Z - Z')\|_{\mathscr{F}} > x) + \frac{c}{n} + 5P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^2 > 2\right), \quad n \geqslant n_0,$$

where $q_1, q_2 > 0$ are absolute constants.

Proof. Fix $x > 0$ and $\delta \in (0, 1)$. Since $\mathscr{F}$ is totally bounded, for each $\gamma \in (0, 1)$ there exist functions $f_1, \ldots, f_m$ such that

$$\sup_{f\in\mathscr{F}} \min_{i=1,\ldots,m} d_2(f,f_i) < \gamma. \tag{7.1}$$

First we bound the conditional probability

$$I_1 = P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z,Z')\|_{\mathscr{F}} > \frac{x}{2}\,\middle|\, Z'\right).$$

In view of (7.1), I_1 is bounded by the sum of the probabilities

$$I_2 = P\left(\frac{1}{n}\max_{k\leqslant n}\max_{i=1,\ldots,m} |kQ_k(Z,Z')(f_i)| > \frac{x}{4}\,\middle|\, Z'\right),$$

$$I_3 = P\left(\frac{1}{n}\max_{k\leqslant n}\max_{i=1,\ldots,m}\sup_{d_2(f,f_i)<\gamma} |kQ_k(Z,Z')(f-f_i)| > \frac{x}{4}\,\middle|\, Z'\right).$$

By Doob's submartingale maximal inequality, we have

$$\begin{aligned} I_2 &\leqslant \sum_{i=1}^m P\left(\frac{1}{n}\max_{k\leqslant n}|kQ_k(Z,Z')(f_i)| > \frac{x}{4}\,\middle|\, Z'\right) \\ &\leqslant 4\frac{4^2}{x^2}\sum_{i=1}^m E((Q_n(Z,Z')(f_i))^2 | Z') \\ &= \frac{4^3}{x^2}\sum_{i=1}^m \left(\frac{1}{n^2}\sum_{t=1}^n\left(\sum_{s=1,s\neq t}^n a_{t-s}(f_i)Z'_s\right)^2\right). \end{aligned} \tag{7.2}$$

Again using Doob's inequality for Banach space valued martingales (cf. Proposition 4.1.1 in Kwapień and Woyczyński, 1992) we obtain

$$\begin{aligned} I_3 &\leqslant 4\frac{4^2}{x^2} E\left(\max_{i=1,\ldots,m}\sup_{d_2(f,f_i)<\gamma} |Q_n(Z,Z')(f-f_i)|^2\,\middle|\, Z'\right) \\ &= \frac{4^3}{x^2n^2} E\left(\max_{i=1,\ldots,m}\sup_{d_2(f,f_i)<\gamma}\left|\sum_{t=1}^{n-1} a_t(f-f_i)\sum_{s=1}^{n-t}(Z_sZ'_{s+t} + Z'_sZ_{s+t})\right|^2\,\middle|\, Z'\right). \end{aligned}$$

An application of the Cauchy–Schwartz inequality and of Parseval's formula yields

$$\begin{aligned} I_3 &\leqslant \frac{4^3\gamma^2}{x^2n^2}\sum_{t=1}^{n-1} E\left(\left(\sum_{s=1}^{n-t}(Z_sZ'_{s+t} + Z'_sZ_{s+t})\right|^2\,\middle|\, Z'\right) \\ &\leqslant \frac{4^4\gamma^2}{x^2n^2}\sum_{t=1}^{n-1}\sum_{s=1}^{n}(Z'_s)^2 \leqslant \left(\frac{4^4\gamma^2}{x^2}\right)\left(\frac{1}{n}\sum_{s=1}^n (Z'_s)^2\right). \end{aligned} \tag{7.3}$$

We introduce the two events

$$A_1 = \left\{\frac{1}{n^2}\sum_{t=1}^n\left(\sum_{s=1,s\neq t}^n a_{t-s}(f_i)Z'_s\right)^2 \leqslant \frac{\gamma_1}{m},\ i = 1,\ldots,m\right\},$$

$$A_2 = \left\{\frac{1}{n}\sum_{s=1}^n (Z'_s)^2 \leqslant 2\right\}$$

for an arbitrary $\gamma_1 > 0$. Choosing γ and γ_1 in such a way that

$$\frac{4^3\gamma_1}{x^2} + \frac{4^5\gamma^2}{x^2} \leqslant 1 - \sqrt{\delta},$$

we conclude from (7.2) and (7.3) that the relation $I_1 \leqslant 1 - \sqrt{\delta}$ holds on $A_1 \cap A_2$. Let (Z_t'') be an independent copy of Z' and of Z and set $\tilde{Z}_t = Z_t - Z_t''$ for all t. Then we obtain on $A_1 \cap A_2$

$$\begin{aligned}
&\sqrt{\delta}\, P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z, Z')\|_{\mathscr{F}} > x \,\middle|\, Z'\right)\\
&\quad\leqslant P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z, Z')\|_{\mathscr{F}} > x \,\middle|\, Z'\right) P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z'', Z')\|_{\mathscr{F}} \leqslant \frac{x}{2} \,\middle|\, Z'\right)\\
&\quad\leqslant P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(\tilde{Z}, Z')\|_{\mathscr{F}} > \frac{x}{2} \,\middle|\, Z'\right)\\
&\quad\leqslant 2P\left(\|Q_n(\tilde{Z}, Z')\|_{\mathscr{F}} > \frac{x}{2} \,\middle|\, Z'\right).
\end{aligned}$$

In the last step we used Lévy's inequality for sums of independent symmetric random variables, conditionally on Z'. Taking expectations with respect to Z' we finally obtain the unconditional bound

$$\begin{aligned}
&\delta P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z, Z')\|_{\mathscr{F}} > x\right)\\
&\quad\leqslant \sqrt{\delta}\,2P\left(\|Q_n(\tilde{Z}, Z')\|_{\mathscr{F}} > \frac{x}{2}\right) + \sqrt{\delta}P(A_1^c) + \sqrt{\delta}P(A_2^c),
\end{aligned} \tag{7.4}$$

where A^c is the complement of the event A. Conditioning on $\tilde{Z} = (\tilde{Z}_t)$, the same arguments as above together with (7.4) lead to the estimate

$$\begin{aligned}
&\delta P\left(\frac{1}{n}\max_{k\leqslant n} \|kQ_k(Z, Z')\|_{\mathscr{F}} > x\right)\\
&\quad\leqslant 2P\left(\|Q_n(\tilde{Z}, \tilde{Z}')\|_{\mathscr{F}} > \frac{x}{4}\right) + P(A_1^c) + P(A_2^c) + 2P(A_3^c) + 2P(A_4^c),
\end{aligned}$$

where $\tilde{Z}' = (\tilde{Z}_t')$ is an independent copy of $\tilde{Z}$ and

$$A_3 = \left\{\frac{1}{n^2}\sum_{t=1}^{n}\left(\sum_{s=1,\, s\neq t}^{n} a_{t-s}(f_i)\tilde{Z}_s\right)^2 \leqslant \frac{\gamma_1}{m},\ i = 1, \dots, m\right\},$$

$$A_4 = \left\{\frac{1}{n}\sum_{s=1}^{n} \tilde{Z}_s^2 \leqslant 4\right\}.$$

Notice that by Markov's inequality

$$\begin{aligned}
P(A_1^c) &\leqslant \frac{1}{n^2}\frac{m^2}{\gamma_1}\max_{i=1,\dots,m}\sum_{t=1}^{n} E\left(\sum_{s=1,\, s\neq t}^{n} a_{t-s}(f_i) Z_s\right)^2\\
&\leqslant \frac{1}{n}\left(\frac{1}{n}\frac{m^2}{\gamma_1}\left\|\sum_{t=1}^{n}\sum_{s=1,\, s\neq t}^{n} a_{t-s}^2\right\|_{\mathscr{F}}\right).
\end{aligned}$$

Notice that the right-hand side can be bounded by c/n for a constant $c = c(x, \delta)$. The probability $P(A_3^c)$ can be treated in the same way. Finally, observe that

$$P(A_4^c) \leqslant 2P(A_2^c).$$

The final statement of the lemma now follows by an application of a coupling argument for quadratic forms (see de la Peña and Montgomery-Smith, 1995). This concludes the proof of the lemma. □

Lemma 7.3. *Assume $Z = (Z_t)$ is a sequence of iid symmetric random variables with $EZ_1^4 < \infty$. There exist positive constants q_1, q_2 and an integer n_0 such that for all sufficiently small $x > 0$ and for $n \geqslant n_0$*

$$P\left(\frac{1}{n}\max_{k\leqslant n}\|kQ_{k,Z}\|_{\mathscr{F}} > x\right)$$
$$\leqslant q_1\left(e^{-q_2\sqrt{n}x^2} + P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^4 > 2EZ_1^4\right) + P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^2 > 2EZ_1^2\right) + \frac{1}{n}\right).$$

Proof. In view of Lévy's maximal inequality for quadratic forms (Kwapień and Woyczyński, 1992, Theorem 6.2.1) we have, for some absolute constant $q > 0$ and all $x > 0$,

$$P\left(\frac{1}{n}\max_{k\leqslant n}\|kQ_{k,Z}\|_{\mathscr{F}} > x\right) \leqslant qP(\|Q_{n,Z}\|_{\mathscr{F}} > x). \tag{7.5}$$

Let $\varepsilon = (\varepsilon_t)$ be a Rademacher sequence independent of Z and write $\varepsilon Z = (\varepsilon_t Z_t)$. To estimate the right-hand side of (7.5) we replace Z by εZ, condition on Z and apply the exponential tail estimate from part A of Lemma 4.3. Define

$$c_0 = \left\|\sum_{t=1}^{\infty} a_t^2\right\|_{\mathscr{F}} \leqslant \sup_{f\in\mathscr{F}}\|f\|_2^2 < \infty$$

and

$$\sigma_n^2 = \frac{1}{n^2}\left\|\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2 Z_t^2 Z_s^2\right\|_{\mathscr{F}}.$$

We introduce the events

$$A_1 = \left\{\frac{1}{n}\sum_{t=1}^{n} Z_t^4 \leqslant 2EZ_1^4\right\}, \qquad A_2 = \left\{\frac{1}{n}\sum_{t=1}^{n} Z_t^2 \leqslant 2EZ_1^2\right\},$$

$$A_3 = \left\{\frac{c_0}{2}\frac{(EZ_1^2)^2}{n} \leqslant \sigma_n^2 \leqslant (c_0 + c_1)\frac{1}{n}\right\},$$

for a constant $c_1 > 0$ to be chosen later. Then we have

$$P(\|Q_{n,Z}\|_{\mathscr{F}} > x) = P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > x)$$
$$\leqslant E_Z(P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > x\,|\,Z)I_{A_1\cap A_2\cap A_3}) + P(A_1^c) + P(A_2^c) + P(A_3^c).$$

We have

$$P(A_3^c) \leqslant P\left(\sigma_n^2 < \frac{c_0}{2}\frac{(EZ_1^2)^2}{n}\right) + P\left(\sigma_n^2 > (c_0 + c_1)\frac{1}{n}\right) = I_1 + I_2.$$

By definition of c_0, we find an $f_0 \in \mathscr{F}$ such that for sufficiently large n

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0) > \frac{3c_0}{4}. \tag{7.6}$$

For such an f_0, by Čebyshev's inequality and by (7.6),

$$\begin{aligned} I_1 &\leqslant P\left(\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0)Z_t^2Z_s^2 < \frac{c_0(EZ_1^2)^2}{2}\right) \\ &\leqslant P\left(\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0)(Z_t^2 - EZ_1^2)(Z_s^2 - EZ_1^2)\right. \\ &\qquad \left. + \frac{2}{n}EZ_1^2\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0)(Z_t^2 - EZ_1^2) < -\frac{c_0(EZ_1^2)^2}{4}\right) \\ &\leqslant \frac{c}{n^2}\left(\sum_{t=1}^{n}\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0) + \sum_{t=1}^{n}\left(\sum_{s=1,s\neq t}^{n} a_{t-s}^2(f_0)\right)^2\right) \\ &\leqslant \frac{c}{n} \end{aligned}$$

for sufficiently large n. Moreover, by the Cauchy–Schwartz inequality, it follows

$$\begin{aligned} I_2 &= P\left(\frac{2}{n}\left\|\sum_{t=1}^{n-1} a_t^2 \sum_{s=1}^{n-t} Z_s^2 Z_{s+t}^2\right\|_{\mathscr{F}} > c_0 + c_1\right) \\ &\leqslant P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^4 > \frac{1}{2} + \frac{1}{2}\frac{c_1}{c_0}\right). \end{aligned}$$

Choose c_1 such that $1 + c_1/c_0 > 4EZ_1^4$. Then

$$P(A_3^c) \leqslant c\left(\frac{1}{n} + P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^4 > 2EZ_1^4\right)\right). \tag{7.7}$$

We also obtain

$$P(A_1^c) \leqslant P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^4 > 2EZ_1^4\right), \qquad P(A_2^c) \leqslant P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^2 > 2EZ_1^2\right)\Bigg). \tag{7.8}$$

From (7.7) and (7.8) we thus obtain

$$\begin{aligned} &P(A_1^c) + P(A_2^c) + P(A_3^c) \\ &\quad \leqslant c\left(\frac{1}{n} + P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^4 > 2EZ_1^4\right) + P\left(\frac{1}{n}\sum_{t=1}^{n} Z_t^2 > 2EZ_1^2\right)\right). \end{aligned} \tag{7.9}$$

Next using part (A) of Lemma 4.3 we estimate the conditional probability

$$P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > x \mid Z) I_{A_1 \cap A_2 \cap A_3}.$$

Given Z we set $b_{ts} = n^{-1} a_{t-s} Z_t Z_s$. We have to find m, $M > 0$ as required in Lemma 4.3, i.e. we must find m, M such that

$$P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > M \mid Z) < \tfrac{1}{64} \tag{7.10}$$

and

$$J_1 = P\left(\frac{1}{n^2}\left\| \sum_{t=1}^{n} Z_t^2 \left(\sum_{s=1, s \neq t}^{n} a_{t-s} Z_s \varepsilon_s \right)^2 \right\|_{\mathscr{F}} > m^2 \,\middle|\, Z \right) < \frac{1}{16} \tag{7.11}$$

hold on $A_1 \cap A_2 \cap A_3$. We assume that $x \in (0, 1)$. We will apply Lemma 4.3 for $y > 0$ such that $x \geqslant M + my + \sigma_n y^2$. We choose

$$M = \frac{x}{3}, \quad y^2 = \frac{x^2}{c_3^2 \sigma_n}, \quad m = \sqrt{\sigma_n} c_4,$$

and c_3, $c_4 > 0$ such that $c_3^2 > 3$ and $c_4/c_3 \leqslant \frac{1}{3}$. Then it is immediate that $M + my + \sigma_n y^2 \leqslant x$ and, on $A_1 \cap A_2 \cap A_3$, part A of Lemma 4.3 yields the estimate

$$\begin{aligned} &P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > x \mid Z) \\ &\quad \leqslant c \exp\{-cy^2\} = c \exp\{-cx^2/(c_3^2 \sigma_n)\} \leqslant c \exp\{-cx^2 \sqrt{n}\}. \end{aligned}$$

After taking expectations with respect to Z the latter estimate together with (7.9) concludes the proof.

Thus it remains to find m, M as in (7.10) and (7.11). We start with M. Using the same arguments as in the proof of Proposition 6.1, for every $\varepsilon > 0$ we find an n_ε such that for all $n \geqslant n_\varepsilon$ and $\omega \in A_2$

$$P(\|Q_{n,\varepsilon Z}\|_{\mathscr{F}} > \varepsilon \mid Z) < \varepsilon.$$

Thus we may choose $M > 0$ arbitrarily small, in particular, we may set $M = x/3$. Next we show that we may choose $m^2 = c_4^2 \sigma_n$. We have

$$\begin{aligned} J_1 &\leqslant I_{\left\{ n^{-2} \left\| \sum_{t=1}^{n} Z_t^2 \sum_{s=1, s \neq t}^{n} a_{t-s}^2 Z_s^2 \right\|_{\mathscr{F}} > c_4^2 \sigma_n / 2 \right\}} \\ &\quad + P\left(\frac{1}{n^2} \left\| \sum_{t=1}^{n} Z_t^2 \sum_{s_1 s_2 = 1, s_1 \neq t, s_2 \neq t, s_1 \neq s_2}^{n} a_{t-s_1} a_{t-s_2} Z_{s_1} Z_{s_2} \varepsilon_{s_1} \varepsilon_{s_2} \right\|_{\mathscr{F}} > \frac{c_4^2 \sigma_n}{2} \,\middle|\, Z \right) \\ &= J_2 + J_3. \end{aligned}$$

Notice that, for $\omega \in A_3$,

$$J_2 = I_{\{\sigma_n > c_4^2/2\}} = 0$$

for $n \geqslant n_0$ where n_0 does not depend on ω. Moreover, by the Cauchy–Schwartz inequality and by the orthogonality of the cosine functions

$$\left\| \sum_{t=1}^{n} Z_t^2 \sum_{s_1 s_2 = 1, s_1 \neq t, s_2 \neq t, s_1 \neq s_2}^{n} a_{t-s_1} a_{t-s_2} Z_{s_1} Z_{s_2} \varepsilon_{s_1} \varepsilon_{s_2} \right\|_{\mathscr{F}}$$

$$\leqslant c \left\| \left(\int_{\Pi} \left(\sum_{t=1}^{n-1} \cos(xt) D_t \right)^2 \mathrm{d}x \right)^{1/2} \right\|_{\mathscr{F}} \leqslant c \left\| \left(\sum_{t=1}^{n-1} D_t^2 \right)^{1/2} \right\|_{\mathscr{F}},$$

where

$$D_t = \sum_{s=1}^{n-t} \left(\tilde{Z}_s^2 Z_{s+t} \varepsilon_{s+t} \sum_{s_2 = 1, s_2 \neq s, s_2 \neq s-t}^{n} a_{s-s_2} Z_{s_2} \varepsilon_{s_2} \right.$$

$$\left. + Z_{s+t}^2 Z_s \varepsilon_s \sum_{s_2 = 1, s_2 \neq s, s_2 \neq s+t}^{n} a_{t+s-s_s} Z_{s_2} \varepsilon_{s_2} \right).$$

The same arguments, now applied to the coefficients a_{s-s_2}, a_{t+s-s_2}, lead to an estimate of the right-hand side of the type

$$c \left(\sum_t \sum_{s_1} \left(\sum_{s_2} Z_t^2 Z_{s_1} \varepsilon_{s_1} Z_{s_2} \varepsilon_{s_2} \right)^2 \right)^{1/2},$$

where each of the sums $\sum_t, \sum_{s_1}, \sum_{s_2}$ ranges over at most $2n$ summands, and summation in $\sum_{s_2}$ is such that all indices s_1, s_2, t are different. The latter fact and Markov's second moment inequality yield on $A_1 \cap A_2$

$$J_3 \leqslant \frac{c}{c_4^4 n^4 \sigma_n^2} E\left(\sum_t \sum_{s_1} \left(\sum_{s_2} Z_t^2 Z_{s_1} \varepsilon_{s_1} Z_{s_2} \varepsilon_{s_2} \right)^2 \middle| Z \right)$$

$$\leqslant \frac{c_5}{c_4^4 n \sigma_n^2}$$

for some constant c_5. Thus we have for all $\omega \in A_1 \cap A_2 \cap A_3$

$$J_1 \leqslant J_2 + J_3 \leqslant \frac{2c_5}{c_4^4 c_0}.$$

Choosing c_4 large enough, we achieve that the estimate $J_1 < 1/16$ holds on $A_1 \cap A_2 \cap A_3$, just as required by (7.11). Recall that we also required that $c_4/c_3 \leqslant 1/3$ which is possible since we may choose c_3 arbitrarily large. This concludes the proof of the lemma. □

Proof of Proposition 7.1. The following identity holds for iid Z, Z':

$$Q_n(Z, Z) + Q_n(Z', Z') = Q_n(Z - Z', Z - Z') + 2Q_n(Z, Z').$$

An application of Lemmas 7.2 and 7.3 along the subsequence $n = 2^k$ together with a Borel–Cantelli argument show that

$$\|Q_n(Z - Z', Z - Z')\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad \|Q_n(Z, Z')\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0.$$

Hence $\|Q_{n,Z} + Q_{n,Z'}\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0$. Having in mind the WLLN of Proposition 6.1, an application of Lemma 4.4 yields $Q_{n,Z} \xrightarrow{\text{a.s.}} 0$. From decomposition (2.1) and by the SLLN for $\gamma_{n,Z}(0)$ and $Q_{n,Z}$ we obtain

$$\|J_{n,Z} - J\|_{\mathscr{F}} \leqslant \sqrt{2\pi} \sup_{f\in\mathscr{F}} \|f\|_2 |\gamma_{n,Z}(0) - 1| + \|Q_{n,Z}\|_{\mathscr{F}} \xrightarrow{\text{a.s.}} 0.$$

This concludes the proof. □

The proof of Theorem 3.4 is a consequence of Proposition 7.1, the decomposition (5.8) and the following lemma.

Lemma 7.4. *Assume that* $\sum_{j=1}^{\infty} c_j^2 j^{3/2} < \infty$. *Then*

$$\frac{1}{n} \sup_{f\in\mathscr{F}} \left| \int_{\Pi} R_n(x) f(x)\,\mathrm{d}x \right| \xrightarrow{\text{a.s.}} 0.$$

Proof. We follow the lines of the proof of Lemma 5.7 with the normalisation $1/n$ instead of $1/\sqrt{n}$. Instead of (5.14) and (5.15) we show, for any $f\in\mathscr{F}$,

$$I(f) = n^{-1} \int_{\Pi} f(x) C(x) L_n(x) K_n(-x)\,\mathrm{d}x = \mathrm{o}(1) \quad \text{a.s.} \tag{7.12}$$

$$I = n^{-2} \int_{\Pi} |L_n(x) K_n(-x)|^2\,\mathrm{d}x = \mathrm{o}(1) \quad \text{a.s.} \tag{7.13}$$

By the Cauchy–Schwartz inequality we have

$$I^2 \leqslant \int_{\Pi} I_{n,Z}^2(x)\,\mathrm{d}x \left(n^{-2} \int_{\Pi} |K_n(x)|^4\,\mathrm{d}x \right).$$

By (5.12) and using a Borel–Cantelli argument we have

$$n^{-2} \int_{\Pi} |K_n(x)|^4\,\mathrm{d}x = \mathrm{o}(1) \quad \text{a.s.}$$

Notice that

$$\int_{\Pi} I_{n,Z}^2(x)\,\mathrm{d}x \leqslant c \sum_{t=0}^{n-1} \gamma_{n,Z}^2(t),$$

$\gamma_{n,Z}^2(0)$ converges a.s. by the SLLN and

$$A_n = n^2 \sum_{t=1}^{n-1} \gamma_{n,Z}^2(t) - \sum_{t=1}^{n-1} (n-t) Z_t^2, \quad n \geqslant 2,$$

is a martingale. Doob's submartingale maximal inequality yields

$$P\left(\max_{n\leqslant 2^k} |A_n| > \varepsilon 2^{2k} \right) \leqslant c\varepsilon^{-2} 2^{-4k} E A_{2^k}^2 \leqslant c\varepsilon^{-2} 2^{-k}$$

and a Borel–Cantelli argument shows that $n^{-2} A_n \xrightarrow{\text{a.s.}} 0$ which proves (7.13). Thus it remains to show (7.12). We follow the lines of the proof of (5.14) and we restrict ourselves to show that (cf. (5.16) with $\varepsilon = 0$)

$$L_1 = \sum_{n=1}^{\infty} n^{-2} \ln n \sum_{t=0}^{n-1} \left| \int_{\Pi} f(x) C(x) \mathrm{e}^{-\mathrm{i}xt} \, \mathrm{d}x \right|^2 < \infty,$$

$$L_2 = \sum_{n=1}^{\infty} n^{-2} \sum_{t=1}^{n} \sum_{j=t+1}^{\infty} j^{-1/2} \left| \int_{\Pi} f(x) C(x) \mathrm{e}^{\mathrm{i}xj} \, \mathrm{d}x \right|^2 < \infty$$

which, together with a Borel–Cantelli argument, proves that (7.12) holds. Notice that $L_1 + L_2 < \infty$ follows by change of summation and by applications of Parseval's formula. □

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