J. Differential Equations 248 (2010) 1345–1375

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Vector fields with the oriented shadowing property

Sergei Yu. Pilyugin [a], Sergey B. Tikhomirov [b,*,1]

[a] Faculty of Mathematics and Mechanics, St. Petersburg State University, University av. 28, 198504 St. Petersburg, Russia
[b] Department of Mathematics, National Taiwan University, No. 1, Section 4, Roosevelt Road, Taipei 106, Taiwan

ARTICLE INFO

Article history:
Received 26 March 2009
Revised 25 September 2009

MSC:
37C50
37D20

Keywords:
Vector fields
Oriented shadowing
Structural stability

ABSTRACT

We give a description of the $\mathbf{C}^1$-interior ($\mathrm{Int}^1(\mathrm{OrientSh})$) of the set of smooth vector fields on a smooth closed manifold that have the oriented shadowing property. A special class $\mathcal{B}$ of vector fields that are not structurally stable is introduced. It is shown that the set $\mathrm{Int}^1(\mathrm{OrientSh}\setminus\mathcal{B})$ coincides with the set of structurally stable vector fields. An example of a field of the class $\mathcal{B}$ belonging to $\mathrm{Int}^1(\mathrm{OrientSh})$ is given. Bibliography: 18 titles.

1. Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) in dynamical systems is now well developed (see, for example, the monographs [1,2]). At the same time, the problem of complete description of systems having the shadowing property seems unsolvable. We have no hope to characterize systems with the shadowing property in terms of the theory of structural stability (such as hyperbolicity and transversality) since the shadowing property is preserved under homeomorphisms of the phase space (at least in the compact case), while the above-mentioned properties are not.

The situation changes completely when we pass from the set of smooth dynamical systems having the shadowing property (or some of its analogs) to its $\mathbf{C}^1$-interior. It was shown by Sakai [3] that the $\mathbf{C}^1$-interior of the set of diffeomorphisms with the shadowing property coincides with the set of

* Corresponding author.
E-mail addresses: sp@sp1196.spb.edu (S.Yu. Pilyugin), sergey.tikhomirov@gmail.com (S.B. Tikhomirov).

1 Research of the author is supported by NSC (Taiwan) 98-2811-M-002-061.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.09.024

structurally stable diffeomorphisms. Later, a similar result was obtained for the set of diffeomorphisms with the orbital shadowing property [4].

In this context, there is a real difference between the cases of discrete dynamical systems generated by diffeomorphisms and systems with continuous time (flows) generated by smooth vector fields. This difference is due to the necessity of reparametrizing shadowing trajectories in the latter case. One of the main goals of the present paper is to show that this difference is crucial, and the results for flows are essentially different from those for diffeomorphisms.

Let us pass to the main definitions and results. Let M be a smooth closed (i.e., compact and boundaryless) manifold with Riemannian metric dist and let $n = \dim M$. Consider a smooth ($\mathbf{C}^1$) vector field on X and denote by ϕ the flow of X. We denote by

$$O(x,\phi) = \{\phi(t,x)\colon\ t \in \mathbb{R}\}$$

the trajectory of a point x in the flow ϕ; $O^+(x,\phi)$ and $O^-(x,\phi)$ are the positive and negative semi-trajectories, respectively.

Fix a number $d > 0$. We say that a mapping $g:\mathbb{R} \to M$ (not necessarily continuous) is a d-pseudotrajectory (both for the field X and flow ϕ) if

$$\operatorname{dist}\big(g(\tau+t), \phi\big(t, g(\tau)\big)\big) < d \quad \text{for } \tau \in \mathbb{R},\ t \in [0,1]. \tag{1}$$

A reparametrization is an increasing homeomorphism h of the line $\mathbb{R}$; we denote by Rep the set of all reparametrizations.

For $a > 0$, we denote

$$\operatorname{Rep}(a) = \left\{h \in \operatorname{Rep}\colon \left|\frac{h(t)-h(s)}{t-s} - 1\right| < a,\ t, s \in \mathbb{R},\ t \neq s\right\}.$$

In this paper, we consider the following three shadowing properties (and the corresponding sets of dynamical systems).

We say that a vector field X has the standard shadowing property ($X \in \mathrm{StSh}$) if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory $g(t)$ of X there exist a point $p \in M$ and a reparametrization $h \in \operatorname{Rep}(\varepsilon)$ such that

$$\operatorname{dist}\big(g(t), \phi\big(h(t), p\big)\big) < \varepsilon \quad \text{for } t \in \mathbb{R}. \tag{2}$$

We say that a vector field X has the oriented shadowing property ($X \in \mathrm{OrientSh}$) if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory of X there exist a point $p \in M$ and a reparametrization $h \in \operatorname{Rep}$ such that inequalities (2) hold (thus, it is not assumed that the reparametrization h is close to identity).

Finally, we say that a vector field X has the orbital shadowing property ($X \in \mathrm{OrbitSh}$) if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory of X there exists a point $p \in M$ such that

$$\operatorname{dist}_H\big(\operatorname{Cl} O(p,\phi), \operatorname{Cl}\{g(t)\colon\ t \in \mathbb{R}\}\big) < \varepsilon,$$

where dist_H is the Hausdorff distance.

Let us note that the standard shadowing property is equivalent to the strong pseudo orbit tracing property (POTP) in the sense of Komuro [5]; the oriented shadowing property was called the normal POTP by Komuro [5] and the POTP for flows by Thomas [6].

We consider the following $\mathbf{C}^1$ metric on the space of smooth vector fields: If X and Y are vector fields of class $\mathbf{C}^1$, we set

$$\rho_1(X,Y) = \max_{x\in M}\left(\big|X(x) - Y(x)\big| + \left\|\frac{\partial X}{\partial x}(x) - \frac{\partial Y}{\partial x}(x)\right\|\right),$$

where $|.|$ is the norm on the tangent space T_xM generated by the Riemannian metric dist, and $\|.\|$ is the corresponding operator norm for matrices.

For a set A of vector fields, $\mathrm{Int}^1(A)$ denotes the interior of A in the $\mathbf{C}^1$ topology generated by the metric ρ_1.

Let us denote by $\mathbf{S}$ and $\mathbf{N}$ the sets of structurally stable and nonsingular vector fields, respectively.

The only result in the problem under study was recently published by Lee and Sakai [7]: $\mathrm{Int}^1(\mathrm{StSh} \cap \mathbf{N}) \subset \mathbf{S}$.

To formulate our main results, we need one more definition.

Let us say that a vector field X belongs to the class $\mathcal{B}$ if X has two hyperbolic rest points p and q (not necessarily different) with the following properties:

(1) The Jacobi matrix $DX(q)$ has two complex conjugate eigenvalues $\mu_{1,2} = a_1 \pm ib_1$ of multiplicity one with $a_1 < 0$ such that if $\lambda \neq \mu_{1,2}$ is an eigenvalue of $DX(q)$ with $\mathrm{Re}\,\lambda < 0$, then $\mathrm{Re}\,\lambda < a_1$;
(2) the Jacobi matrix $DX(p)$ has two complex conjugate eigenvalues $\nu_{1,2} = a_2 \pm ib_2$ with $a_2 > 0$ of multiplicity one such that if $\lambda \neq \nu_{1,2}$ is an eigenvalue of $DX(p)$ with $\mathrm{Re}\,\lambda > 0$, then $\mathrm{Re}\,\lambda > a_2$;
(3) the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ have a trajectory of nontransverse intersection.

Condition (1) above means that the "weakest" contraction in $W^s(q)$ is due to the eigenvalues $\mu_{1,2}$ (condition (2) has a similar meaning).

Theorem 1. $\mathrm{Int}^1(\mathrm{OrientSh} \setminus \mathcal{B}) = \mathbf{S}$.

Let us note that Theorem 1 was stated (without a proof) in the author's short note [8]. Let us also note that if $\dim M \leqslant 3$, then $\mathrm{Int}^1(\mathrm{OrientSh}) = \mathbf{S}$ (which also was stated in [8] and proved by the second author in [9]; in [9], it was also shown that if LipSh is the set of vector fields that have an analog of the standard shadowing property with ε replaced by Ld, then $\mathrm{Int}^1(\mathrm{LipSh}) = \mathbf{S}$).

Theorem 2. $\mathrm{Int}^1(\mathrm{OrientSh}) \cap \mathcal{B} \neq \emptyset$.

Theorem 3. $\mathrm{Int}^1(\mathrm{OrbitSh} \cap \mathbf{N}) \subset \mathbf{S}$.

Let us note that Theorem 3 generalizes the above-mentioned result by Lee and Sakai.

The structure of the paper is as follows: In Section 2, we prove Theorem 1 and discuss the proof of Theorem 3; in Section 3, we prove Theorem 2.

2. Proof of Theorem 1

First we introduce some notation.

We denote by $B(a, A)$ the a-neighborhood of a set $A \subset M$.

The term "transverse section" will mean a smooth open disk in M of codimension 1 that is transverse to the flow ϕ at any of its points.

Let $\mathrm{Per}(X)$ denote the set of rest points and closed orbits of a vector field X.

Let us recall that X is called a Kupka–Smale field ($X \in \mathrm{KS}$) if

(KS1) any trajectory in $\mathrm{Per}(X)$ is hyperbolic;
(KS2) stable and unstable manifolds of trajectories from $\mathrm{Per}(X)$ are transverse.

The proof of Theorem 1 is based on the following result (see [10]): $\mathrm{Int}^1(\mathrm{KS}) = \mathbf{S}$.

Let $\mathcal{T}$ denote the set of vector fields X that have property (KS1). Our first lemma is applied in the proofs of both Theorems 1 and 3; for this purpose, we formulate and prove it for the set OrbitSh.

Lemma 1.

$$\mathrm{Int}^1(\mathrm{OrbitSh}) \subset \mathcal{T}. \tag{3}$$

Proof. To get a contradiction, let us assume that there exists a vector field $X \in \mathrm{Int}^1(\mathrm{OrbitSh})$ that does not have property (KS1), i.e., the set $\mathrm{Per}(X)$ contains a trajectory p that is not hyperbolic.

Let us first consider the case where p is a rest point. Identify M with $\mathbb{R}^n$ in a neighborhood of p. Applying an arbitrarily $\mathbf{C}^1$-small perturbation of the field X, we can find a field $Y \in \mathrm{Int}^1(\mathrm{OrbitSh})$ that is linear in a neighborhood U of p (we also assume that p is the origin of U).

(Here and below in the proof of Lemma 1, all the perturbations are $\mathbf{C}^1$-small perturbations that leave the field in $\mathrm{Int}^1(\mathrm{OrbitSh})$; we denote the perturbed fields by the same symbol X and their flows by ϕ.)

Then trajectories of X in U are governed by a differential equation

$$\dot{x} = Px, \tag{4}$$

where the matrix P has an eigenvalue λ with $\mathrm{Re}\,\lambda = 0$.

Consider first the case where $\lambda = 0$. We perturb the field X (and change coordinates, if necessary) so that, in Eq. (4), the matrix P is block-diagonal,

$$P = \mathrm{diag}(0, P_1), \tag{5}$$

and P_1 is an $(n-1)\times(n-1)$ matrix.

Represent coordinate x in U as $x = (y, z)$ with respect to (5); then

$$\phi\bigl(t, (y, z)\bigr) = \bigl(y, \exp(P_1 t) z\bigr)$$

in U.

Take $\varepsilon > 0$ such that $B(4\varepsilon, p) \subset U$. To get a contradiction, assume that $X \in \mathrm{OrbitSh}$; let d correspond to the chosen ε.

Fix a natural number m and consider the following mapping from $\mathbb{R}$ into U:

$$g(t) = \begin{cases} y = -2\varepsilon, \quad z = 0; \quad t \leqslant 0, \\ y = -2\varepsilon + t/m, \quad z = 0; \quad 0 < t < 4m\varepsilon, \\ y = 2\varepsilon, \quad z = 0; \quad 4m\varepsilon < t. \end{cases}$$

Since the mapping g is continuous, piecewise differentiable, and either $\dot{y} = 0$ or $\dot{y} = 1/m$, g is a d-pseudotrajectory for large m.

Any trajectory of X in U belongs to a plane $y = \mathrm{const}$; hence,

$$\mathrm{dist}_H\bigl(\mathrm{Cl}\bigl(O(q, \phi)\bigr), \mathrm{Cl}\bigl(\bigl\{g(t)\colon t \in \mathbb{R}\bigr\}\bigr)\bigr) \geqslant 2\varepsilon$$

for any q. This completes the proof in the case considered.

Similar reasoning works if p is a rest point and the matrix P in (4) has a pair of eigenvalues $\pm ib$, $b \neq 0$.

Now we assume that p is a nonhyperbolic closed trajectory. In this case, we perturb the vector field X in a neighborhood of the trajectory p using the perturbation technique developed by Pugh and Robinson in [11]. Let us formulate their result (which will be used below several times).

Pugh–Robinson perturbation. *Assume that r_1 is not a rest point of a vector field X. Let $r_2 = \phi(\tau, r_1)$, where $\tau > 0$. Let Σ_1 and Σ_2 be two small transverse sections such that $r_i \in \Sigma_i$, $i = 1, 2$. Let σ be the local Poincaré transformation generated by these transverse sections.*

Consider a point $r' = \phi(\tau', r_1)$, where $\tau' \in (0, \tau)$, and let U be an arbitrary open set containing r'.

Fix an arbitrary C^1-neighborhood F of the field X.

There exist positive numbers ε_0 and Δ_0 with the following property: if σ' is a local diffeomorphism from the Δ_0-neighborhood of r_1 in Σ_1 into Σ_2 such that

$$\mathrm{dist}_{C^1}(\sigma, \sigma') < \varepsilon_0,$$

then there exists a vector field $X' \in F$ such that

(1) *$X' = X$ outside U;*
(2) *σ' is the local Poincaré transformation generated by the sections Σ_1 and Σ_2 and trajectories of the field X'.*

Let ω be the least positive period of the nonhyperbolic closed trajectory p. We fix a point $\pi \in p$, local coordinates in which π is the center, and a hyperplane Σ of codimension 1 transverse to the vector $F(\pi)$. Let y be coordinate in Σ.

Let σ be the local Poincaré transformation generated by the transverse section Σ; denote $P = D\sigma(0)$. Our assumption implies that the matrix P is not hyperbolic. In an arbitrarily small neighborhood of the matrix P, we can find a matrix P' such that P' either has a real eigenvalue with unit absolute value of multiplicity 1 or a pair of complex conjugate eigenvalues with unit absolute value of multiplicity 1. In both cases, we can choose coordinates $y = (v, w)$ in Σ in which

$$P' = \mathrm{diag}(Q, P_1), \tag{6}$$

where Q is a 1×1 or 2×2 matrix such that $|Qv| = |v|$ for any v.

Now we can apply the Pugh–Robinson perturbation (taking $r_1 = r_2 = \pi$ and $\Sigma_1 = \Sigma_2 = \Sigma$) that modifies X in a small neighborhood of the point $\phi(\omega/2, \pi)$ and such that, for the perturbed vector field X', the local Poincaré transformation generated by the transverse section Σ is given by $y \mapsto P'y$.

Clearly, in this case, the trajectory of π in the field X' is still closed (with some period ω'). As was mentioned, we assume that X' has the orbital shadowing property (and write X, ϕ, ω instead of X', ϕ', ω').

We introduce in a neighborhood of the point π coordinates $x = (x', y)$, where x' is one-dimensional (with axis parallel to $X(\pi)$), and y has the above-mentioned property.

Of course, the new coordinates generate a new metric, but this new metric is equivalent to the original one; thus, the corresponding shadowing property (or its absence) is preserved.

We need below one more technical statement.

LE (local estimate). *There exist a neighborhood W of the origin in Σ and constants $l, \delta_0 > 0$ with the following property: if $z_1 \in \Sigma \cap W$ and $|z_2 - z_1| < \delta < \delta_0$, then we can represent z_2 as $\phi(\tau, z_2')$ with $z_2' \in \Sigma$ and*

$$|\tau|, \left|z_2' - z_1\right| < l\delta. \tag{7}$$

This statement is an immediate corollary of the theorem on local rectification of trajectories (see, for example, [12]): In a neighborhood of a point that is not a rest point, the flow of a vector field of class C^1 is diffeomorphic to the family of parallel lines along which points move with unit speed (and it is enough to note that a diffeomorphic image of Σ is a smooth submanifold transverse to lines of the family).

We may assume that the neighborhood W in LE is so small that for $y \in \Sigma \cap W$, the function $\alpha(y)$ (the time of first return to Σ) is defined, and that the point $\phi(\alpha(v, w), (0, v, w))$ has coordinates $(Qv, P_1 w)$ in Σ.

Let us take a neighborhood U of the trajectory p such that if $r \in U$, then the first point of intersection of the positive semitrajectory of r with Σ belongs to W.

Take $a > 0$ such that the $4a$-neighborhood of the origin in Σ is a subset of W. Fix

$$\varepsilon < \min\left(\delta_0, \frac{a}{4l}\right),$$

where δ_0 and l satisfy the LE. Let d correspond to this ε (in the definition of the orbital shadowing property).

Take $y_0 = (v_0, 0)$ with $|v_0| = a$. Fix a natural number N and set

$$\alpha_k = \alpha\left(\left(\frac{k}{N} Q^k v_0, 0\right)\right), \quad k \in [0, N-1),$$

$$\beta_0 = 0, \qquad \beta_k = \alpha_1 + \cdots + \alpha_k,$$

and

$$g(t) = \begin{cases} \phi(t, (0,0,0)), & t < 0; \\ \phi(t - \beta_k, (0, \frac{k}{N} Q^k v_0, 0)), & \beta_k \leqslant t < \beta_{k+1},\ k \in [0, N-1); \\ \phi(t - \beta_N, (0, Q^N v_0, 0)), & t \geqslant \beta_N. \end{cases}$$

Note that for any point $y = (v, 0)$ of intersection of the set $\{g(t)\colon t \in \mathbb{R}\}$ with Σ, the inequality $|v| \leqslant a$ holds. Hence, we can take a so small that

$$B\big(2a, \mathrm{Cl}\big(\{g(t)\colon t \in \mathbb{R}\}\big)\big) \subset U.$$

Since

$$\left|\frac{k}{N} Q^{k+1} v_0 - \frac{k+1}{N} Q^{k+1} v_0\right| = \frac{a}{N} \to 0, \quad N \to \infty,$$

$g(t)$ is a d-pseudotrajectory for large N.

Assume that there exists a point q such that

$$\mathrm{dist}_H\big(\mathrm{Cl}\big(O(q,\phi)\big), \mathrm{Cl}\big(\{g(t)\colon t \in \mathbb{R}\}\big)\big) < \varepsilon.$$

In this case, $O(q,\phi) \subset U$, and there exist points $q_1, q_2 \in O(q,\phi)$ such that

$$|q_1| = \big|q_1 - (0,0,0)\big| < \varepsilon$$

and

$$\big|q_2 - \big(0, Q^N v_0, 0\big)\big| < \varepsilon.$$

By the choice of ε, there exist points $q_1', q_2' \in O(q,\phi) \cap \Sigma$ such that

$$\big|q_1'\big| < l\varepsilon < a/4 \quad \text{and} \quad \big|q_2' - Q^N v_0\big| < l\varepsilon < a/4.$$

Let $q_1' = (0, v_1, w_1)$ and $q_2' = (0, v_2, w_2)$. Since these points belong to the same trajectory that is contained in U, $|v_1| = |v_2|$. At the same time,

$$|v_1| < a/4, \quad \big|v_2 - Q^N v_0\big| < a/4, \quad \text{and} \quad \big|Q^N v_0\big| = a,$$

and we get a contradiction which proves our lemma. $\square$

To complete the proof of Theorem 1, we show that any vector field

$$X \in \operatorname{Int}^1(\text{OrientSh} \setminus \mathcal{B})$$

has property (KS2).

To get a contradiction, let us assume that there exist trajectories $p, q \in \operatorname{Per}(X)$ for which the unstable manifold $W^u(q)$ and the stable manifold $W^s(p)$ have a point r of nontransverse intersection. We have to consider separately the following two cases.

Case (B1): p and q are rest points of the flow ϕ.
Case (B2): either p or q is a closed trajectory.

Case (B1). Since $X \notin \mathcal{B}$, we may assume (after an additional perturbation, if necessary) that the eigenvalues $\lambda_1, \dots, \lambda_u$ with $\operatorname{Re}\lambda_j > 0$ of the Jacobi matrix $DX(p)$ have the following property:

$$\operatorname{Re}\lambda_j > \lambda_1 > 0, \quad j = 2, \dots, u$$

(where u is the dimension of $W^u(p)$). This property means that there exists a one-dimensional "direction of weakest expansion" in $W^u(p)$.

If this is not the case, then our assumption that $X \notin \mathcal{B}$ implies that the eigenvalues $\mu_1, \dots, \mu_s$ with $\operatorname{Re}\mu_j < 0$ of the Jacobi matrix $DX(q)$ have the following property:

$$\operatorname{Re}\mu_j < \mu_1 < 0, \quad j = 2, \dots, s$$

(where s is the dimension of $W^s(q)$). If this condition holds, we reduce the problem to the previous case by passing from the field X to the field $-X$ (clearly, the fields X and $-X$ have the oriented shadowing property simultaneously).

Making a perturbation (in this part of the proof, we always assume that the perturbed field belongs to the set $\text{OrientSh} \setminus \mathcal{B}$), we may "linearize" the field X in a neighborhood U of the point p; thus, trajectories of X in U are governed by a differential equation

$$\dot{x} = Px,$$

where

$$P = \operatorname{diag}(P_s, P_u), \quad P_u = \operatorname{diag}(\lambda, P_1), \quad \lambda > 0, \tag{8}$$

P_1 is a $(u-1) \times (u-1)$ matrix for which there exist constants $K > 0$ and $\mu > \lambda$ such that

$$\left\| \exp(-P_1 t) \right\| \leqslant K^{-1} \exp(-\mu t), \quad t \geqslant 0, \tag{9}$$

and $\operatorname{Re}\lambda_j < 0$ for the eigenvalues λ_j of the matrix P_s.

Let us explain how to perform the above-mentioned perturbations preserving the nontransversality of $W^u(q)$ and $W^s(p)$ at the point r (we note that a similar reasoning can be used in "replacement" of a component of intersection of $W^u(q)$ with a transverse section Σ by an affine space, see the text preceding Lemma 2 below).

Consider points $r^* = \phi(\tau, r)$, where $\tau > 0$, and $r' = \phi(\tau', r)$, where $\tau' \in (0, \tau)$. Let Σ and Σ^* be small transverse sections that contain the points r and r^*. Take small neighborhoods V and U' of p and r', respectively, so that the set V does not intersect the "tube" formed by pieces of trajectories through points of U' whose endpoints belong to Σ and Σ^*. In this case, if we perturb the vector field X in V and apply the Pugh–Robinson perturbation in U', these perturbations are "independent."

We perturb the vector field X in V obtaining vector fields X' that are linear in small neighborhoods $V' \subset V$ and such that the values $\rho_1(X, X')$ are arbitrarily small.

Let γ_s and γ_s^* be the components of intersection of the stable manifold $W^s(p)$ (for the field X) with Σ and Σ^* that contain the points r and r^*, respectively.

Since the stable manifold of a hyperbolic rest point depends (on its compact subsets) C^1-smoothly on C^1-small perturbations, the stable manifolds $W^s(p)$ (for the perturbed fields X') contain components γ_s' of intersection with Σ^* that converge (in the C^1 metric) to γ_s^*.

Now we apply the Pugh–Robinson perturbation in U' and find a field X' in an arbitrary C^1-neighborhood of X such that the local Poincaré transformation generated by the field X' and sections Σ and Σ^* takes γ_s' to γ_s (which means that the nontransversality at r is preserved).

We introduce in U coordinates $x = (y; v, w)$ according to (8): y is coordinate in the s-dimensional "stable" subspace (denoted E^s); (v, w) are coordinates in the u-dimensional "unstable" subspace (denoted E^u). The one-dimensional coordinate v corresponds to the eigenvalue λ (and hence to the one-dimensional "direction of weakest expansion" in E^u).

In the neighborhood U,

$$\phi\big(t, (y, v, w)\big) = \big(\exp(P_s t)y; \exp(\lambda t)v, \exp(P_1 t)w\big),$$

and it follows from (9) that

$$\big|\exp(P_1 t)w\big| \geqslant K \exp(\mu t)|w|, \quad t \geqslant 0. \tag{10}$$

Denote by E_1^u the one-dimensional invariant subspace corresponding to λ.

We naturally identify $E^s \cap U$ and $E^u \cap U$ with the intersections of U with the corresponding local stable and unstable manifolds of p, respectively.

Let us construct a special transverse section for the flow ϕ. We may assume that the point r of nontransverse intersection of $W^u(q)$ and $W^s(p)$ belongs to U. Take a hyperplane Σ' in E^s of dimension $s-1$ that is transverse to the vector $X(r)$. Set $\Sigma = \Sigma' + E^u$; clearly, Σ is transverse to $X(r)$.

By a perturbation of the field X outside U, we may get the following: in a neighborhood of r, the component of intersection $W^u(q) \cap \Sigma$ containing r (for the perturbed field) has the form of an affine space $r + L$, where L is the tangent space, $L = T_r(W^u(q) \cap \Sigma)$, of the intersection $W^u(q) \cap \Sigma$ at the point r for the unperturbed field (compare, for example, with [7]).

Let Σ_r be a small transverse disk in Σ containing the point r. Denote by γ the component of intersection of $W^u(q) \cap \Sigma_r$ containing r.

Lemma 2. *There exists $\varepsilon > 0$ such that if $x \in \Sigma_r$ and*

$$\operatorname{dist}\big(\phi(t, x), O^-(r, \phi)\big) < \varepsilon, \quad t \leqslant 0, \tag{11}$$

then $x \in \gamma$.

Proof. To simplify presentation, let us assume that q is a rest point; the case of a closed trajectory is considered using a similar reasoning.

By the Grobman–Hartman theorem, there exists $\varepsilon_0 > 0$ such that the flow of X in $B(2\varepsilon_0, q)$ is topologically conjugate to the flow of a linear vector field.

Denote by A the intersection of the local stable manifold of q, $W^s_{loc}(q)$, with the boundary of the ball $B(2\varepsilon_0, q)$.

Take a negative time T such that if $s = \phi(T, r)$, then

$$\phi(t, s) \in B(\varepsilon_0, q), \quad t \leqslant 0. \tag{12}$$

Clearly, if ε_0 is small enough, then the compact sets A and

$$B = \big\{\phi(t, r)\colon T \leqslant t \leqslant 0\big\}$$

are disjoint. There exists a positive number $\varepsilon_1 < \varepsilon_0$ such that the ε_1-neighborhoods of the sets A and B are disjoint as well.

Take $\varepsilon_2 \in (0, \varepsilon_1)$. There exists a neighborhood V of the point s with the following property: if $y \in V \setminus W^u_{loc}(q)$, then the first point of intersection of the negative semitrajectory of y with the boundary of $B(2\varepsilon_0, q)$ belongs to the ε_2-neighborhood of the set A (this statement is obvious for a neighborhood of a saddle rest point of a linear vector field; by the Grobman–Hartman theorem, it holds for X as well).

Clearly, there exists a small transverse disk Σ_s containing s and such that if $y \in \Sigma_s \cap W^u_{loc}(q)$, then the first point of intersection of the positive semitrajectory of y with the disk Σ_r belongs to γ (in addition, we assume that Σ_s belongs to the chosen neighborhood V).

There exists $\varepsilon \in (0, \varepsilon_1 - \varepsilon_2)$ such that the flow of X generates a local Poincaré transformation

$$\sigma : \Sigma_r \cap B(\varepsilon, r) \to \Sigma_s.$$

Let us show that this ε has the desired property. It follows from our choice of Σ_s and (11) with $t = 0$ that if $x \notin \gamma$, then

$$y := \sigma(x) \in \Sigma_s \setminus W^u_{loc}(q);$$

in this case, there exists $\tau < 0$ such that the point $z = \phi(\tau, y)$ belongs to the intersection of $B(\varepsilon_2, A)$ with the boundary of $B(2\varepsilon_0, q)$. By (12),

$$\operatorname{dist}\big(z, \phi(t, s)\big) > \varepsilon_0, \quad t \leqslant 0. \tag{13}$$

At the same time,

$$\operatorname{dist}\big(z, \phi(t, r)\big) > \varepsilon_1 - \varepsilon_2, \quad T \leqslant t \leqslant 0. \tag{14}$$

Inequalities (13) and (14) contradict condition (11). Our lemma is proved. □

Now let us formulate the property of nontransversality of $W^u(q)$ and $W^s(p)$ at the point r in terms of the introduced objects.

Let Π^u be the projection to E^u parallel to E^s.

The transversality of $W^u(q)$ and $W^s(p)$ at r means that

$$T_r W^u(q) + T_r W^s(p) = \mathbb{R}^n.$$

Since Σ is a transverse section to the flow ϕ at r, the above equality is equivalent to the equality

$$L + E^s = \mathbb{R}^n.$$

Thus, the nontransversality means that

$$L + E^s \neq \mathbb{R}^n,$$

which implies that

$$L' := \Pi^u L \neq E^u. \tag{15}$$

We claim that there exists a linear isomorphism J of Σ for which the norm $\|J - \mathrm{Id}\|$ is arbitrarily small and such that

$$\Pi^u J L \cap E^u_1 = \{0\}. \tag{16}$$

Let e be a unit vector of the line E^u_1. If $e \notin L'$, we have nothing to prove (take $J = \mathrm{Id}$). Thus, we assume that $e \in L'$. Since $L' \neq E^u$, there exists a vector $v \in E^u \setminus L'$.

Fix a natural number N and consider a unit vector v_N that is parallel to $Ne + v$. Clearly, $v_N \to e$ as $N \to \infty$. There exists a sequence T_N of linear isomorphisms of E^u such that $T_N v_N = e$ and

$$\|T_N - \mathrm{Id}\| \to 0, \quad N \to \infty.$$

Note that $T_N^{-1}e$ is parallel to v_N; hence, $T_N^{-1}e$ does not belong to L', and

$$T_N \Pi^u L \cap E^u_1 = \{0\}. \tag{17}$$

Define an isomorphism J_N of Σ by

$$J_N(y, z) = (y, T_N z)$$

and note that

$$\|J_N - \mathrm{Id}\| \to 0, \quad N \to \infty.$$

Let $L_N = J_N L$. Equality (17) implies that

$$\Pi^u L_N \cap E^u_1 = \{0\}. \tag{18}$$

Our claim is proved.

First we consider the case where $\dim E^u \geqslant 2$. Since $\dim L' < \dim E^u$ by (15) and $\dim E^u_1 = 1$, our reasoning above (combined with a Pugh–Robinson perturbation) shows that we may assume that

$$L' \cap E^u_1 = \{0\}. \tag{19}$$

For this purpose, we take a small transverse section Σ' containing the point $r' = \phi(-1, r)$, denote by γ the component of intersection of $W^u(q)$ with Σ' containing r', and note that the local Poincaré transformation σ generated by Σ' and Σ takes γ to the linear space L (in local coordinates of Σ). The mapping $\sigma_N = J_N \sigma$ is C^1-close to σ for large N and takes γ to L_N for which equality (18) is valid. Thus, we get equality (19) for the perturbed vector field.

This equality implies that there exists a constant $C > 0$ such that if $(y; v, w) \in r + L$, then

$$|v| \leqslant C|w|. \tag{20}$$

Fix $a > 0$ such that $B(4a, p) \subset U$. Take a point $\alpha = (0; a, 0) \in E^u_1$ and a positive number T and set $\alpha_T = (r_y; a\exp(-\lambda T), 0)$, where r_y is the y-coordinate of r. Construct a pseudotrajectory as follows:

$$g(t) = \begin{cases} \phi(t, r), & t \leqslant 0, \\ \phi(t, \alpha_T), & t > 0. \end{cases}$$

Since

$$|r - \alpha_T| = a\exp(-\lambda T) \to 0$$

as $T \to \infty$, for any d there exists T such that g is a d-pseudotrajectory.

Lemma 3. *Assume that $b \in (0, a)$ satisfies the inequality*

$$\log K - \log C + \left(\frac{\mu}{\lambda} - 1\right)\left(\log \frac{a}{2} - \log b\right) \geqslant 0.$$

Then for any $T > 0$, reparametrization h, and a point $s \in r + L$ such that $|r - s| < b$ there exists $\tau \in [0, T]$ such that

$$\left|\phi\big(h(\tau), s\big) - g(\tau)\right| \geqslant \frac{a}{2}.$$

Proof. To get a contradiction, assume that

$$\left|\phi\big(h(\tau), s\big) - g(\tau)\right| < \frac{a}{2}, \quad \tau \in [0, T]. \tag{21}$$

Let $s = (y_0; v_0, w_0) \in r + L$. Since $|r - s| < b$,

$$|v_0| < b. \tag{22}$$

By (21),

$$\phi\big(h(\tau), s\big) \in U, \quad \tau \in [0, T].$$

Take $\tau = T$ in (21) to show that

$$|v_0| \exp\big(\lambda h(T)\big) > \frac{a}{2}.$$

It follows that

$$h(T) > \lambda^{-1}\left(\log \frac{a}{2} - \log |v_0|\right). \tag{23}$$

Set $\theta(\tau) = |\exp(P_1 h(\tau)) w_0|$; then $\theta(0) = |w_0|$. By (20),

$$|v_0| \leqslant C\theta(0). \tag{24}$$

By (10),

$$\theta(T) \geqslant K \exp\big(\mu h(T)\big)\theta(0). \tag{25}$$

We deduce from (22)–(25) that

$$\begin{aligned}
\log\left(\frac{2\theta(T)}{a}\right) &\geqslant \log \theta(T) - \log\left|v_0 \exp\big(\lambda h(T)\big)\right| \\
&\geqslant \log K + \log \theta(0) - \log |v_0| + (\mu - \lambda) h(T) \\
&\geqslant \log K - \log C + \left(\frac{\mu}{\lambda} - 1\right)\left(\frac{a}{2} - \log |v_0|\right) \\
&\geqslant \log K - \log C + \left(\frac{\mu}{\lambda} - 1\right)\left(\frac{a}{2} - \log b\right) \geqslant 0.
\end{aligned}$$

We get a contradiction with (21) for $\tau = T$ since the norm of the w-coordinate of $\phi(h(T), s)$ equals $\theta(T)$, while the w-coordinate of $g(T)$ is 0. The lemma is proved. $\square$

Let us complete the proof of Theorem 1 in case (B1). Assume that $l, \delta_0 > 0$ are chosen for Σ so that the LE holds.

Take $\varepsilon \in (0, \min(\delta_0, \varepsilon_0, a/2))$ so small that if $|y - r| < \varepsilon$, then $\phi(t, y)$ intersects Σ at a point s such that

$$\operatorname{dist}\bigl(\phi(t, s), r\bigr) < \varepsilon_0, \quad |t| \leqslant l\varepsilon. \tag{26}$$

Consider the corresponding d and a d-pseudotrajectory g described above.

Assume that

$$\operatorname{dist}\bigl(\phi\bigl(h(t), x\bigr), g(t)\bigr) < \varepsilon, \quad t \in \mathbb{R}, \tag{27}$$

for some point x and reparametrization h and set $y = \phi(h(0), x)$.

Then $|y - r| < \varepsilon$, and there exists a point $s = \phi(\tau, y) \in \Sigma$ with $|\tau| < l\varepsilon$.

If $-l\varepsilon \leqslant t \leqslant 0$, then

$$\operatorname{dist}\bigl(\phi(t, s), O^-(r, \phi)\bigr) \leqslant \varepsilon_0$$

by (26).

If $t < -l\varepsilon$, then $h(0) + \tau + t < h(0)$, and there exists $t' < 0$ such that $h(t') = h(0) + \tau + t$. In this case,

$$\phi(t, s) = \phi\bigl(h(0) + \tau + t, x\bigr) = \phi\bigl(h(t'), x\bigr),$$

and

$$\operatorname{dist}\bigl(\phi(t, s), O^-(r, \phi)\bigr) \leqslant \operatorname{dist}\bigl(\phi\bigl(h(t'), x\bigr), \phi(t', r)\bigr) \leqslant \varepsilon_0.$$

By Lemma 2, $s \in r + L$. If ε is small enough, then $|s - r| < b$, where b satisfies the condition of Lemma 3, whose conclusion contradicts (27).

This completes the consideration of case (B1) for $\dim W^u(p) \geqslant 2$. If $\dim W^u(p) = 1$, then the nontransversality of $W^u(q)$ and $W^s(p)$ implies that $L \subset E^s$. This case is trivial since any shadowing trajectory passing close to r must belong to the intersection $W^u(q) \cap W^s(p)$, while we can construct a pseudotrajectory “going away” from p along $W^u(p)$. If $\dim W^u(p) = 0$, $W^u(q)$ and $W^s(p)$ cannot have a point of nontransverse intersection.

Case (B2). Passing from the vector field X to $-X$, if necessary, we may assume that p is a closed trajectory. We “linearize” X in a neighborhood of p as described in the proof of Lemma 1 so that the local Poincaré transformation of transverse section Σ is a linear mapping generated by a matrix P with the following properties: With respect to some coordinates in Σ,

$$P = \operatorname{diag}(P_s, P_u), \tag{28}$$

where $|\lambda_j| < 1$ for the eigenvalues λ_j of the matrix P_s, and $|\lambda_j| > 1$ for the eigenvalues λ_j of the matrix P_u, every eigenvalue has multiplicity 1, and P is in a Jordan form.

The same reasoning as in case (B1) shows that it is possible to perform such a “linearization” (and other perturbations of X performed below) so that the nontransversality of $W^u(q)$ and $W^s(p)$ is preserved.

Consider an eigenvalue λ of P_u such that $|\lambda| \leqslant |\mu|$ for the remaining eigenvalues μ of P_u.

We treat separately the following two cases.

Case (B2.1): $\lambda \in \mathbb{R}$.
Case (B2.2): $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Case (B2.1). Applying a perturbation, we may assume that

$$P_u = \operatorname{diag}(\lambda, P_1),$$

where $|\lambda| < |\mu|$ for the eigenvalues μ of the matrix P_1 (thus, there exists a one-dimensional direction of "weakest expansion" in $W^u(p)$). In this case, we apply precisely the same reasoning as that applied to treat case (B1) (we leave details to the reader).

Case (B2.2). Applying one more perturbation of X, we may assume that

$$\lambda = \nu + i\eta = \rho \exp\left(\frac{2\pi m_1 i}{m}\right),$$

where m_1 and m are relatively prime natural numbers, and

$$P_u = \operatorname{diag}(Q, P_1),$$

where

$$Q = \begin{pmatrix} \nu & -\eta \\ \eta & \nu \end{pmatrix}$$

with respect to some coordinates (y, v, w) in Σ, where $\rho = |\lambda| < |\mu|$ for the eigenvalues μ of the matrix P_1.

Denote

$$E^s = \{(y, 0, 0)\}, \qquad E^u = \{(0, v, w)\}, \qquad E_1^u = \{(0, v, 0)\}.$$

Thus, E^s is the "stable subspace," E^u is the "unstable subspace," and E_1^u is the two-dimensional "unstable subspace of the weakest expansion."

Geometrically, the Poincaré transformation $\sigma : \Sigma \to \Sigma$ (extended as a linear mapping to E_1^u) acts on E_1^u as follows: the radius of a point is multiplied by ρ, while $2\pi m_1/m$ is added to the polar angle.

As in the proof of Lemma 1, we take a small neighborhood W of the origin of the transverse section Σ so that, for points $x \in W$, the function $\alpha(x)$ (the time of first return to Σ) is defined.

We assume that the point r of nontransverse intersection of $W^u(q)$ and $W^s(p)$ belongs to the section Σ. Similarly to case (B1), we perturb X so that, in a neighborhood of r, the component of intersection of $W^u(q) \cap \Sigma$ containing r has the form of an affine space, $r + L$.

Let Π^u be the projection in Σ to E^u parallel to E^s, and let Π_1^u be the projection to E_1^u; thus,

$$\Pi^u(y, u, v) = (0, u, v) \quad \text{and} \quad \Pi_1^u(y, u, v) = (0, u, 0).$$

The nontransversality of $W^u(q)$ and $W^s(p)$ at r means that

$$L' = \Pi^u L \neq E^u$$

(see case (B1)). Applying a reasoning similar to that in case (B1), we perturb X so that if $L'' = L' \cap E_1^u$, then

$$\dim L'' < \dim E_1^u = 2.$$

Hence, either $\dim L'' = 1$ or $\dim L'' = 0$. We consider only the first case, the second one is trivial.

Denote by A the line L''. Images of A under degrees of σ (extended to the whole plane E_1^u) are m different lines in E_1^u.

In what follows, we refer to an obvious geometric statement (given without a proof).

Proposition 1. *Consider Euclidean space $\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$. Let $x' = (x_1, x_2)$, $x'' = (x_3, \dots, x_n)$, and let G be the plane of coordinate x'. Let D be a hyperplane in $\mathbb{R}^n$ such that*

$$D \cap G = \{x_2 = 0\}.$$

For any $b > 0$ there exists $c > 0$ such that if $x = (x', x'') \in D$ and $x' = (x'_1, x'_2)$, then either $|x'_2| \leqslant b|x'_1|$ or $|x''| \geqslant c|x'|$.

Take $a > 0$ such that the $2a$-neighborhood of the origin in Σ belongs to W. We may assume that if $v = (v_1, v_2)$, then the line A is $\{v_2 = 0\}$.

Take $b > 0$ such that the images of the cone

$$C = \bigl\{v\colon |v_2| \leqslant b|v_1|\bigr\}$$

in E_1^u under degrees of σ intersect only at the origin (denote these images by $C_1, \dots, C_m$).

We apply Proposition 1 to find a number $c > 0$ such that if $(0, v, w) \in L'$, then either $(0, v, 0) \in C$ or

$$|w| \geqslant c|v|. \tag{29}$$

Take a point $\beta = (0, v, 0) \in \Sigma$, where $|v| = a$, such that $\beta \notin C_1 \cup \dots \cup C_m$.

For a natural number N, set $\beta_N = (r_y, P_u^{-N}(v, 0)) \in \Sigma$ (we recall that equality (28) holds), where r_y is the y-coordinate of r. We naturally identify β and β_N with points of M and consider the following pseudotrajectory:

$$g(t) = \begin{cases} \phi(t, r), & t \leqslant 0; \\ \phi(t, \beta_N), & t > 0. \end{cases}$$

The following statement (similar to Lemma 2) holds: there exists $\varepsilon_0 > 0$ such that if

$$\operatorname{dist}\bigl(\phi(t, s), O^-(r, \phi)\bigr) < \varepsilon_0, \quad t \leqslant 0,$$

for some point $s \in \Sigma$, then $s \in r + L$.

Since β does not belong to the closed set $C_1 \cup \dots \cup C_m$, we may assume that the disk in E_1^u centered at β and having radius ε_0 does not intersect the set $C_1 \cup \dots \cup C_m$.

Define numbers

$$\alpha_1(N) = \alpha(\beta_N), \qquad \alpha_2(N) = \alpha_1(N) + \alpha\bigl(\sigma(\beta_N)\bigr), \quad \dots, \quad \alpha_N(N) = \alpha_{N-1}(N) + \alpha\bigl(\sigma^{N-1}(\beta_N)\bigr).$$

Take δ_0 and l for which LE holds for the neighborhood W (reducing W, if necessary). Take $\varepsilon < \min(\varepsilon_0/l, \delta_0)$ and assume that there exists the corresponding d (from the definition of the class OrientSh). Take N so large that g is a d-pseudotrajectory.

Let h be a reparametrization; assume that

$$\bigl|\phi\bigl(h(t), p_0\bigr) - g(t)\bigr| < \varepsilon, \quad 0 \leqslant t \leqslant \alpha_N(N),$$

for some point $p_0 \in \Sigma$.

Since $g(\alpha_k(N)) \in \Sigma$ for $0 \leqslant k \leqslant N$ by construction, there exist numbers χ_k such that

$$\left|\sigma^{\chi_k}(p_0) - g\big(\alpha_k(N)\big)\right| < \varepsilon_0, \quad 0 \leqslant k \leqslant N.$$

To complete the proof of Theorem 1, let us show that for any $p_0 \in r + L$ and any reparametrization h there exists $t \in [0, \alpha_N(N)]$ such that

$$\operatorname{dist}\big(\phi\big(h(t), p_0\big), g(t)\big) \geqslant \varepsilon.$$

Assuming the contrary, we see that

$$\left|\sigma^{\chi_k}(p_0) - g\big(\alpha_k(N)\big)\right| < \varepsilon_0, \quad 0 \leqslant k \leqslant N,$$

where the numbers χ_k were defined above.

We consider two possible cases.

If

$$\Pi_1^u p_0 \in C$$

(C is the cone defined before estimate (29)), then

$$\Pi_1^u \sigma^{\chi_k}(p_0) \in C_1 \cup \dots \cup C_m.$$

By construction, $\Pi_1^u g(\alpha_N(N))$ is β. Hence,

$$\left|\Pi_1^u \sigma^{\chi_N}(p_0) - \Pi_1^u g\big(\alpha_N(N)\big)\right| > \varepsilon_0,$$

and we get the desired contradiction.

If

$$\Pi_1^u p_0 \notin C$$

and $p_0 = (y_0, v_0, w_0)$, then $(0, v_0, w_0) \in L'$, and it follows from (29) that $|w_0| \geqslant c|v_0|$. In this case, decreasing ε_0, if necessary, we apply the reasoning similar to Lemma 3.

Thus, we have shown that

$$\operatorname{Int}^1(\mathrm{OrientSh} \setminus \mathcal{B}) \subset \operatorname{Int}^1(\mathrm{KS}) = \mathbf{S}. \tag{30}$$

It was shown in [13] that $\mathbf{S} \subset \mathrm{StSh}$; since the set $\mathbf{S}$ is $\mathbf{C}^1$-open and $\mathbf{S} \cap \mathcal{B} = \emptyset$,

$$\mathbf{S} \subset \operatorname{Int}^1(\mathrm{StSh} \setminus \mathcal{B}) \subset \operatorname{Int}^1(\mathrm{OrientSh} \setminus \mathcal{B}). \tag{31}$$

Inclusions (30) and (31) prove Theorem 1.

By Lemma 1, if $X \in \operatorname{Int}^1(\mathrm{OrbitSh})$, then $X \in \operatorname{Int}^1(\mathcal{T})$. For nonsingular flows, the latter inclusion implies that X is Ω-stable [14] (note that this is not the case for flows with rest points [15]). Now, based on the second part of the proof of Theorem 1, one easily proves Theorem 3 following the same lines as in [4, Theorem 4].

3. Proof of Theorem 2

Consider a vector field X^* on the manifold $M = S^2 \times S^2$ that has the following properties (F1)–(F3) (ϕ^* denotes the flow generated by X^*).

(F1) The nonwandering set of ϕ^* is the union of four rest points p^*, q^*, s^*, u^*.
(F2) For some $\delta > 0$ we can introduce coordinates in the neighborhoods $B(\delta, p^*)$ and $B(\delta, q^*)$ such that

$$X^*(x) = J_p^*(x - p^*), \quad x \in B(\delta, p^*), \quad \text{and} \quad X^*(x) = J_q^*(x - q^*), \quad x \in B(\delta, q^*),$$

where

$$J_p^* = -J_q^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(F3) The point s^* is an attracting hyperbolic rest point. The point u^* is a repelling hyperbolic rest point. The following condition holds:

$$W^u(p^*) \setminus \{p^*\} \subset W^s(s^*), \qquad W^s(q^*) \setminus \{q^*\} \subset W^u(u^*). \tag{32}$$

The intersection of $W^s(p^*) \cap W^u(q^*)$ consists of a single trajectory α^*, and for any $x \in \alpha^*$, the condition

$$\dim T_x W^s(p^*) \oplus T_x W^u(q^*) = 3 \tag{33}$$

holds.

These conditions imply that the two-dimensional manifolds $W^s(p^*)$ and $W^u(q^*)$ intersect along a one-dimensional curve in the four-dimensional manifold M. Thus, $W^s(p^*)$ and $W^u(q^*)$ are not transverse; hence, $X^* \in \mathcal{B}$.

A construction of such a vector field is given in Appendix A.

To prove Theorem 2, we show that $X^* \in \text{Int}^1(\text{OrientSh})$.

The vector field X^* satisfies Axiom A and the no-cycle condition; hence, X^* is Ω-stable. Thus, there exists a neighborhood V of X^* in the C^1-topology such that for any field $X \in V$, its nonwandering set consists of four hyperbolic rest points p, q, s, u which belong to small neighborhoods of p^*, q^*, s^*, u^*, respectively. We denote by ϕ the flow of any $X \in V$ and by $W^s(p)$, $W^u(p)$, etc. the corresponding stable and unstable manifolds.

Note that if the neighborhood V is small enough, then there exists a number $c > 0$ (the same for all $X \in V$) such that

$$B(c, s^*) \subset W^s(s) \quad \text{and} \quad B(c, u^*) \subset W^u(u).$$

Consider the set $\Theta = W^u(p^*) \cap \partial B(\delta, p^*)$ (where ∂A is the boundary of a set A). Condition (32) implies that there exist a neighborhood U_Θ of Θ and a number $T > 0$ such that

$$\phi^*(T, x) \in B(c/2, s^*), \quad x \in U_\Theta.$$

Reducing V, if necessary, we may assume that

$$W^u(p) \cap \partial B(\delta, p) \subset U_\Theta \quad \text{and} \quad \phi(T, x) \in B(c, s^*), \quad x \in U_\Theta.$$

Hence, $W^u(p) \setminus \{p\} \subset W^s(s)$, and

$$W^u(p) \cap W^s(q) = \emptyset. \tag{34}$$

Similarly, we may assume that $W^s(q) \setminus \{q\} \subset W^u(u)$.

The following two cases are possible for $X \in V$.

(S1) $W^s(p) \cap W^u(q) = \emptyset$.
(S2) $W^s(p) \cap W^u(q) \neq \emptyset$.

In case (S1), X is a Morse–Smale field; hence, $X \in \mathbf{S}$. Since $\mathbf{S} \subset \mathrm{StSh}$ (see [13]), $X \in \mathrm{OrientSh}$.

Remark 1. In fact, it is shown in [13] that if a vector field $X \in \mathbf{S}$ does not have closed trajectories (as in our case), then X has the Lipschitz shadowing property without reparametrization of shadowing trajectories: there exists $L > 0$ such that if $g(t)$ is a d-pseudotrajectory with small d, then there exists a point x such that

$$\operatorname{dist}\big(g(t), \phi(t, x)\big) \leqslant Ld, \quad t \in \mathbb{R}.$$

We refer to this fact below.

Thus, in the rest of the proof of Theorem 2, we consider case (S2). Our goal is to show that if the neighborhood V is small enough, then $X \in \mathrm{OrientSh}$.

Lemma 4. *If the neighborhood V is small enough, then the intersection $W^s(p) \cap W^u(q)$ consists of a single trajectory.*

Proof. Denote $x_p^* = \alpha^* \cap \partial B(\delta, p^*)$ and $x_q^* = \alpha^* \cap \partial B(\delta, q^*)$.

Consider sections Q_p and Q_q transverse to α at the points x_p^* and x_q^*, respectively, and the corresponding Poincaré map $F^*: Q_q \to Q_p$. Consider the curves $\xi_p^* = W^s(p^*) \cap Q_p \cap B(\delta/2, x_p^*)$ and $\xi_q^* = W^s(q^*) \cap Q_q \cap B(\delta/2, x_q^*)$. Note that ξ_p^* and $F^*(\xi_q^*)$ intersect at a single point x_p^*.

Let $\xi_p = W^s(p) \cap Q_p \cap B(\delta/2, x_p^*)$ and $\xi_q = W^u(q) \cap Q_q \cap B(\delta/2, x_q^*)$. Let F be the Poincaré transformation for X from Q_q to Q_p similar to F^*.

If the neighborhood V is small enough, then the curves ξ_p, ξ_q, and $F(\xi_q)$ are $\mathbf{C}^1$-close to ξ_p^*, ξ_q^*, and $F^*(\xi_q^*)$, respectively (hence, the intersection of ξ_p and $F(\xi_q)$ contains not more than one point).

The same reasoning as in the proof of (34) shows that if the neighborhood V is small enough, $x \in W^s(p) \setminus \{p\}$, and the trajectory of x does not intersect ξ_p, then $x \in W^u(u)$.

Thus, any trajectory in $W^s(p) \cap W^u(q)$ must intersect ξ_p; similarly, it must intersect ξ_q as well as $F(\xi_q)$.

It follows that the intersection $W^s(p) \cap W^u(q)$ (which is nonempty since we consider case (S2)) consists of a single trajectory containing the unique point x_p of intersection of ξ_p and $F(\xi_q)$ (we denote this trajectory by α). This completes the proof of Lemma 4. $\square$

Remark 2. Let us note an important property of intersection of $W^s(p)$ and $W^u(q)$ along α (see (36) below).

Let $x_q = F^{-1}(x_p)$; denote by i_p and i_q unit tangent vectors to the curves ξ_p and ξ_q at x_p and x_q, respectively. Our reasoning above and condition (33) show that if the neighborhood V is small enough, then the vectors i_p and $DF(x_q)i_q$ are not parallel:

$$DF(x_q)i_q \nparallel i_p. \tag{35}$$

Take any two points $y_p = \phi(t_1, x_p)$ and $y_q = \phi(t_2, x_q)$ with $t_1 \geqslant 0$, $t_2 \leqslant 0$; let S_p and S_q be smooth transversals to α at these points. Let e_p and e_q be tangent vectors of $S_p \cap W^s(p)$ and $S_q \cap W^u(q)$ at

y_p and y_q, respectively. Denote by $f: S_q \to S_p$, $H_p: Q_p \to S_p$, and $H_q: S_q \to Q_q$ the corresponding Poincaré transformations for X. Then $f = H_p \circ F \circ H_q$,

$$e_p \parallel DH_p(x_p)i_p, \quad \text{and} \quad e_q \parallel DH_q^{-1}(x_q)i_q.$$

Hence, $Df(y_q)e_q \parallel DH_p \circ DF(x_q)i_q$, and it follows from (35) that

$$Df(y_q)e_q \nparallel e_p. \tag{36}$$

Now it remains to show that if V is small enough and $X \in V$, then $X \in \text{OrientSh}$ (recall that we consider case (S2)). This proof is rather complicated, and we first describe its scheme.

We fix two points $y_p, y_q \in \alpha$ in small neighborhoods U_p and U_q of p and q, respectively (the choice of U_p and U_q is specified later). We consider special pseudotrajectories (of type Ps): the "middle" part of such a pseudotrajectory is the part of α between y_q and y_p, while its "negative" and "positive" tails are parts of trajectories that start near y_q and y_p, respectively. We show that our shadowing problem is reduced to shadowing of pseudotrajectories of type Ps.

The key part of the proof is a statement "on four balls." It is shown that if $B_1, \dots, B_4$ are small balls such that B_1 and B_4 are centered at points of $W^s(q)$ and $W^u(p)$, while B_2 and B_3 are centered at y_q and y_p, respectively, then there exists an exact trajectory that intersects $B_1, \dots, B_4$ successfully as time grows. This statement (and its analog) allows us to prove that pseudotrajectories of type Ps can be shadowed.

Let us fix points $y_p, y_q \in \alpha$ (everywhere below, we assume that $y_p = \alpha(T_p)$ and $y_q = \alpha(T_q)$ with $T_p > T_q$) and a number $\delta > 0$. We say that $g(t)$ is a pseudotrajectory of type $\text{Ps}(\delta)$ if

$$g(t) = \begin{cases} \phi(t - T_p, x_p), & t > T_p, \\ \phi(t - T_q, x_q), & t < T_q, \\ \alpha(t), & t \in [T_q, T_p], \end{cases} \tag{37}$$

for some points

$$x_p \in B(\delta, y_p) \quad \text{and} \quad x_q \in B(\delta, y_q).$$

Fix an arbitrary $\varepsilon > 0$. We prove the following two statements (Propositions 2 and 3). In these statements, we say that a pseudotrajectory $g(t)$ can be ε-shadowed if there exist a reparametrization h and a point p such that (2) holds.

An Ω-stable vector field has a continuous Lyapunov function that strictly decreases along wandering trajectories (see [16]). Hence, there exist small neighborhoods U_p and U_q of points p and q, respectively, such that

$$\phi(t, x) \notin U_q, \quad x \in U_p,\ t \geqslant 0. \tag{38}$$

Proposition 2. *For any $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ there exists $d > 0$ such that if $g(t)$ is a d-pseudotrajectory of X, then either $g(t)$ can be ε-shadowed or there exists a pseudotrajectory $g^*(t)$ of type* $\text{Ps}(\delta)$ *with these y_p and y_q such that* $\text{dist}(g(t), g^*(t)) < \varepsilon/2$, $t \in \mathbb{R}$.

Proposition 3. *There exist $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$ such that any pseudotrajectory of type* $\text{Ps}(\delta)$ *with these y_p and y_q can be $\varepsilon/2$-shadowed.*

Clearly, Propositions 2 and 3 imply that $X \in \text{OrientSh}$.

To prove Proposition 2, we need an auxiliary statement.

Lemma 5. *For any $x \in \alpha$ and $\varepsilon, \varepsilon_1 > 0$ there exists $d > 0$ such that if*

$$\{g(t)\colon\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset, \tag{39}$$

for a d-pseudotrajectory $g(t)$, then one can find $x_0 \in M$ and $h(t) \in \mathrm{Rep}$ such that

$$\operatorname{dist}\big(g(t), \phi\big(h(t), x_0\big)\big) < \varepsilon, \quad t \in \mathbb{R}.$$

Proof. Take $\Delta < \varepsilon_1/2$ such that if $a_p = \phi(1, x)$ and $a_q = \phi(-1, x)$, then $a_p, a_q \notin B(\Delta, x)$. Let S_p and S_q be three-dimensional transversals to α at a_p and a_q, respectively. Let $f\colon\ S_q \to S_p$ be the corresponding Poincaré mapping. Note that the intersections $W^u(q) \cap S_q$ and $W^s(p) \cap S_p$ near a_q and a_p are one-dimensional, hence the curves $f(W^u(q) \cap S_q)$ and $W^s(p) \cap S_p$ in S_p are nontransverse.

It is shown in [11,17] that there exists an arbitrarily small perturbation of the field X supported in $B(\Delta, x)$ and such that the Poincaré mapping $\tilde{f}: S_q \to S_p$ of the perturbed field $\tilde{X}$ satisfies the condition

$$\tilde{f}\big(W^u(q) \cap S_q\big) \cap \big(W^s(p) \cap S_p\big) = \emptyset.$$

Similarly to case (S1), we conclude that we can find $\tilde{X} \in \mathbf{S}$.

Set $\varepsilon_2 = \min(\varepsilon, \varepsilon_1/2)$ and find $d > 0$ such that any d-pseudotrajectory of the field $\tilde{X}$ can be ε_2-shadowed. We assume, in addition, that

$$\Delta + d < \varepsilon_1. \tag{40}$$

Consider an arbitrary d-pseudotrajectory $g(t)$ of X for which (39) holds. By (40), $g(t)$ is a d-pseudotrajectory of the field $\tilde{X}$. Due to the choice of d, there exist $x_0 \in M$ and $h(t) \in \mathrm{Rep}$ such that

$$\operatorname{dist}\big(g(t), \tilde{\phi}\big(h(t), x_0\big)\big) < \varepsilon_2,$$

where $\tilde{\phi}$ is the flow of $\tilde{X}$. Hence, $\{\tilde{\phi}(h(t), x_0),\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset$; it follows that $\tilde{\phi}(h(t), x_0) = \phi(h(t), x_0)$, which proves Lemma 5. □

Proof of Proposition 2. Take $\delta > 0$, $y_p \in \alpha \cap U_p$, and $y_q \in \alpha \cap U_q$. Let $y_q = \alpha(T_q)$ and $y_p = \alpha(T_p)$. There exists $\delta_1 \in (0, \min(\delta, \varepsilon))$ such that $B(\delta_1, y_p) \subset U_p$, $B(\delta_1, y_q) \subset U_q$, and if $x_p \in B(\delta_1, y_p)$ and $x_q \in B(\delta_1, y_q)$, then

$$g^*(t) = \begin{cases} \phi(t - T_p, x_p), & t > T_p, \\ \alpha(t), & t \in [T_q, T_p], \\ \phi(t - T_q, x_q), & t < T_q, \end{cases} \tag{41}$$

is a pseudotrajectory of type $\mathrm{Ps}(\delta)$.

Take $x = \alpha(T)$, where $T \in (T_q, T_p)$. Applying Lemma 5, we can find $\varepsilon_1 > 0$ such that if d is small enough, then for any d-pseudotrajectory $g(t)$, one of the following two cases holds (after a shift of time):

(A1) $$\{g(t),\ t \in \mathbb{R}\} \cap B(\varepsilon_1, x) = \emptyset,$$

and $g(t)$ can be ε-shadowed;

(A2) $$g(T_p) \in B(\delta_1/2, y_p), \qquad g(T_q) \in B(\delta_1/2, y_q),$$

and

$$\operatorname{dist}\big(g(t), \alpha(t)\big) < \varepsilon/2, \quad t \in [T_q, T_p].$$

To prove Proposition 2, it remains to consider case (A2).

Apply the same reasoning as in Lemma 5 to construct a field $\tilde{X} \in \mathbf{S}$ that coincides with X outside $B(\delta_1/2, y_q)$; let $\tilde{\phi}$ be the flow of $\tilde{X}$.

Note that $\tilde{X}$ does not have closed trajectories. Reducing d, if necessary, we may assume that any d-pseudotrajectory of $\tilde{X}$ can be $\delta_1/2$-shadowed in the sense of Remark 1.

Consider the mapping

$$\tilde{g}_p(t) = \begin{cases} \tilde{\phi}(t - T_p, g(T_p)), & t < T_p, \\ g(t), & t \in [T_p, T], \\ \tilde{\phi}(t - T, g(T)), & t > T, \end{cases}$$

where

$$T = \inf\{t > T_p\colon\ g(t) \in B(\delta_1, y_q)\}$$

(if $\{t > T_p\colon\ g(t) \in B(\delta_1, y_q)\} = \emptyset$, we set $T = +\infty$). Since

$$B(\delta_1/2, g(t)) \cap B(\delta_1/2, y_q) = \emptyset$$

for $t \in [T_p, T)$, $\tilde{g}_p(t)$ is a d-pseudotrajectory of $\tilde{X}$. Hence, there exists a point x_p such that

$$\operatorname{dist}(\tilde{g}_p(t), \tilde{\phi}(t - T_p, x_p)) < \delta_1/2, \quad t \in \mathbb{R}.$$

The first inclusion in (A2) implies that $x_p \in B(\delta, y_p)$.

Since trajectories of X and $\tilde{X}$ coincide outside $B(\delta_1/2, y_q)$, we deduce from (38) that $T = +\infty$; hence,

$$\operatorname{dist}(g(t), \phi(t - T_p, x_p)) < \delta_1/2, \quad t \geqslant T_p.$$

Similarly (reducing d, if necessary), we find $x_q \in B(\delta, y_q)$ such that

$$\operatorname{dist}(g(t), \phi(t - T_q, x_q)) < \delta_1/2, \quad t \leqslant T_q.$$

Clearly, the mapping (41) is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ such that

$$\operatorname{dist}(g(t), g^*(t)) < \varepsilon/2, \quad t \in \mathbb{R}.$$

This completes the proof of Proposition 2. □

In the remaining part of the paper, we prove Proposition 3. Let us recall that we consider a vector field X in a small neighborhood V of X^* for which $W^s(p) \cap W^u(q) \neq \emptyset$.

Without loss of generality, we may assume that

$$O^+(B(\varepsilon/2, s), \phi) \subset B(\varepsilon, s) \quad \text{and} \quad O^-(B(\varepsilon/2, u), \phi) \subset B(\varepsilon, u).$$

Take $m \in (0, \varepsilon/8)$ such that $B(m, p) \subset U_p$, $B(m, q) \subset U_q$ and the flow of the vector field X in the neighborhoods $B(2m, p)$ and $B(2m, q)$ is conjugate by a homeomorphism to the flow of a linear vector field.

We take points $y_p = \alpha(T_p) \in B(m/2, p) \cap \alpha$ and $y_q = \alpha(T_q) \in B(m/2, q) \cap \alpha$. Then $O^+(y_p, \phi) \subset B(m, p)$ and $O^-(y_q, \phi) \subset B(m, q)$. Take $\delta > 0$ such that if $g(t)$ is a pseudotrajectory of type $\mathrm{Ps}(\delta)$ (with y_p and y_q fixed above), $t_0 \in \mathbb{R}$, and $x_0 \in B(2\delta, g(t_0))$, then

$$\operatorname{dist}\bigl(\phi(t-t_0,x_0),g(t)\bigr)<\varepsilon/2,\quad |t-t_0|\leqslant T+1, \tag{42}$$

where $T=T_p-T_q$.

Consider a number $\tau>0$ such that if $x\in W^u(p)\setminus B(m/2,p)$, then $\phi(\tau,x)\in B(\varepsilon/8,s)$. Take $\varepsilon_1\in(0,m/4)$ such that if two points $z_1,z_2\in M$ satisfy the inequality $\operatorname{dist}(z_1,z_2)<\varepsilon_1$, then

$$\operatorname{dist}\bigl(\phi(t,z_1),\phi(t,z_2)\bigr)<\varepsilon/8,\quad |t|\leqslant\tau.$$

In this case, for any $y\in B(\varepsilon_1,x)$ (recall that we consider $x\in W^u(p)\setminus B(m/2,p)$), the following inequalities hold:

$$\operatorname{dist}\bigl(\phi(t,x),\phi(t,y)\bigr)<\varepsilon/4,\quad t\geqslant 0. \tag{43}$$

Reducing ε_1, if necessary, we may assume that if $x'\in W^s(q)\setminus B(m/2,q)$ and $y'\in B(\varepsilon_1,x')$, then

$$\operatorname{dist}\bigl(\phi(t,x'),\phi(t,y')\bigr)<\varepsilon/4,\quad t\leqslant 0.$$

Let $g(t)$ be a pseudotrajectory of type $\mathrm{Ps}(\delta)$, where δ, y_p, and y_q satisfy the above-formulated conditions. We claim that if δ is small enough, then $g(t)$ can be $\varepsilon/2$-shadowed (in fact, we have to reduce δ and to impose additional conditions on y_p and y_q). Below we denote $W^u_{loc}(p,m)=W^u(p)\cap B(m,p)$, etc.

Additionally decreasing δ, we may assume that for any points $z_p\in W^u_{loc}(p,m)$, $x_0\in B(\delta,y_p)$, and $s>0$ such that $\phi(s,x_0)\in B(\delta,z_p)$, the following inclusions hold:

$$\phi(t,x_0)\in B(2m,p),\quad t\in[0,s]. \tag{44}$$

Let us consider several possible cases.

Case (P1): $x_p\notin W^s(p)$ and $x_q\notin W^u(q)$. Let

$$T'=\inf\bigl\{t\in\mathbb{R}\colon\ \phi(t,x_p)\notin B(p,3m/4)\bigr\}.$$

If δ is small enough, then $\operatorname{dist}(\phi(T',x_p),W^u(p))<\varepsilon_1$. In this case, there exists a point $z_p\in W^u_{loc}(p,m)\setminus B(m/2,p)$ such that

$$\operatorname{dist}\bigl(\phi(T',x_p),z_p\bigr)<\varepsilon_1. \tag{45}$$

Applying a similar reasoning in a neighborhood of q (and reducing δ, if necessary), we find a point $z_q\in W^s_{loc}(q,m)\setminus B(m/2,q)$ and a number $T''<0$ such that $\operatorname{dist}(\phi(T'',x_q),z_q)<\varepsilon_1$.

Let us formulate a key lemma which we prove later (precisely this lemma is the above-mentioned statement "on four balls").

Lemma 6. *There exists $m>0$ such that for any points*

$$y_p\in B(m,p)\cap\alpha,\qquad z_p\in W^u_{loc}(p,m)\setminus\{p\},$$

$$y_q\in B(m,q)\cap\alpha,\qquad z_q\in W^s_{loc}(q,m)\setminus\{q\},$$

and for any number $m_1>0$ there exists a trajectory of the vector field X that intersects successively the balls $B(m_1,z_q)$, $B(m_1,y_q)$, $B(m_1,y_p)$, and $B(m_1,z_p)$ as time grows.

We reduce m to satisfy Lemma 6 and apply this lemma with $m_1 = \min(\delta, \varepsilon_1)$. Find a point x_0 and numbers $t_1 < t_2 < t_3 < t_4$ such that

$$\phi(t_1, x_0) \in B(m_1, z_q), \qquad \phi(t_2, x_0) \in B(m_1, y_q),$$

$$\phi(t_3, x_0) \in B(m_1, y_p), \qquad \phi(t_4, x_0) \in B(m_1, z_p).$$

Inequalities (42) imply that if δ is small enough, then

$$\operatorname{dist}\bigl(\phi(t_3 + t, x_0), g(T_p + t)\bigr) < \varepsilon/2, \quad t \in [T_q - T_p, 0]. \tag{46}$$

Define a reparametrization $h(t)$ as follows:

$$h(t) = \begin{cases} h(T_q + T'' + t) = t_1 + t, & t < 0, \\ h(T_p + T' + t) = t_4 + t, & t > 0, \\ h(T_p + t) = t_3 + t, & t \in [T_q - T_p, 0], \\ h(t) \text{ increases}, & t \in [T_p, T_p + T'] \cup [T_q + T'', T_q]. \end{cases}$$

If $t \geqslant T_p + T'$, then inequality (43) implies that

$$\operatorname{dist}\bigl(\phi\bigl(h(t), x_0\bigr), \phi\bigl(t - (T_p + T'), z_p\bigr)\bigr) < \varepsilon/4$$

and

$$\operatorname{dist}\bigl(\phi(t - T_p, x_p), \phi\bigl(t - (T_p + T'), z_p\bigr)\bigr) < \varepsilon/4.$$

Hence, if $t \geqslant T_p + T'$, then

$$\operatorname{dist}\bigl(\phi\bigl(h(t), x_0\bigr), g(t)\bigr) < \varepsilon/2. \tag{47}$$

Inclusion (44) implies that for $t \in [T_p, T_p + T']$ the inclusions $\phi(h(t), x_0), g(t) \in B(m, p)$ hold, and inequality (47) holds for these t as well.

A similar reasoning shows that inequality (47) holds for $t \leqslant T_q$. If $t \in [T_q, T_p]$, then inequality (47) follows from (46). This completes the proof in case (P1).

[Case (P2):] $x_p \in W^s(p)$ and $x_q \notin W^u(q)$. In this case, Lemma 6 is replaced by the following statement.

Lemma 7. *There exists $m > 0$ such that for any points*

$$y_p \in B(m, p) \cap \alpha, \qquad y_q \in B(m, q) \cap \alpha, \qquad z_q \in W^s_{loc}(q, m) \setminus \{q\},$$

and a number $m_1 > 0$ there exists a trajectory of the vector field X that intersects successively the balls $B(m_1, z_q)$, $B(m_1, y_q)$, and $B(m_1, y_p) \cap W^s_{loc}(p, m)$ as time grows.

The rest of the proof uses the same reasoning as in case (P1).

Case (P3): $x_p \notin W^s(p)$ and $x_q \in W^u(q)$. This case is similar to case (P2).

Case (P4): $x_p \in W^s(p)$ and $x_q \in W^u(q)$. In this case, we take α as the shadowing trajectory; the reparametrization is constructed similarly to case (P1).

Thus, to complete the consideration of case (S2), it remains to prove Lemmas 6 and 7.

To prove Lemma 6, we first fix proper coordinates in small neighborhoods of the points p and q. Let us begin with the case of the point p.

Taking a small neighborhood V of the vector field X^*, we may assume that the Jacobi matrix $J_p = DX(p)$ is as close to J_p^* as we want.

Thus, we assume that $p = 0$ in coordinates $u_1 = (x_1, x_2)$, $u_2 = (x_3, x_4)$, and $J_p = \mathrm{diag}(A_p, B_p)$, where

$$A_p = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \qquad B_p = \begin{pmatrix} a_p & -b_p \\ b_p & a_p \end{pmatrix}, \tag{48}$$

and

$$\lambda_1, \lambda_2, a_p, b_p > 4g, \tag{49}$$

where g is a small positive number to be chosen later (and a similar notation is used in U_q).

Then we can represent the field X in a small neighborhood U of the point p in the form

$$X(u_1, u_2) = \begin{pmatrix} A_p & 0 \\ 0 & B_p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} X_{12}(u_1, u_2) \\ X_{34}(u_1, u_2) \end{pmatrix}, \tag{50}$$

where

$$X_{12}, X_{34} \in \mathbf{C}^1, \qquad |X_{12}|_{\mathbf{C}^1}, |X_{34}|_{\mathbf{C}^1} < g, \qquad X_{12}(0,0) = X_{34}(0,0) = (0,0). \tag{51}$$

Under these assumptions, $p = 0$ is a hyperbolic rest point whose two-dimensional unstable manifold in the neighborhood U is given by $u_2 = G(u_1)$, where $G : \mathbb{R}^2 \to \mathbb{R}^2$, $G \in \mathbf{C}^1$. We can find $g > 0$ such that if the functions X_{12} and X_{34} satisfy relations (51), then

$$\|DG(u_1)\| < 1 \quad \text{while } \big(u_1, G(u_1)\big) \in U. \tag{52}$$

We introduce new coordinates in U by $v(u_1, u_2) = (u_1, u_2 - G(u_1))$ and use a smooth cut-off function to extend v to a $\mathbf{C}^1$ diffeomorphism w of M such that $w(x) = x$ outside a larger neighborhood U' of p. Denote by Y the resulting vector field in the new coordinates.

Remark 3. Note that Y is continuous but not necessary $\mathbf{C}^1$. Nevertheless, the following holds. Let S_1 and S_2 be small smooth three-dimensional disks transverse to a trajectory of Y and let f_Y be the corresponding Poincaré transformation generated by the vector field Y. Consider smooth disks $w^{-1}(S_1)$ and $w^{-1}(S_2)$ and let $f_X : w^{-1}(S_1) \to w^{-1}(S_2)$ be the corresponding Poincaré transformation. Since $f_X \in \mathbf{C}^1$ and $f_Y = w \circ f_X \circ w^{-1}$, we conclude that $f_Y \in \mathbf{C}^1$. We will use this fact below.

If $(v_1, v_2) = v(u_1, u_2)$, then

$$u_1 = v_1, \qquad u_2 = v_2 + G(v_1). \tag{53}$$

Let $Y(v_1, v_2) = (Y_1(v_1, v_2), Y_2(v_1, v_2))$. Since the surface $u_2 = G(u_1)$ is a local stable manifold of the rest point 0 of the field X, the surface $v_2 = 0$ is a local stable manifold of the rest point 0 of the vector field Y. Hence,

$$Y_2(v_1, 0) = 0 \quad \text{for } (v_1, 0) \in v(U).$$

Lemma 8. *The inequalities*

$$\big|Y_2(v_1, v_2) - \big(Y_2(v_1, 0) + B_p v_2\big)\big| \leqslant 2g|v_2|, \quad (v_1, v_2) \in v(U), \tag{54}$$

hold.

Proof. Substitute equalities (53) into (50) to show that

$$Y_2(v_1, v_2) = B_p\big(v_2 + G(v_1)\big) + X_{34}\big(v_1, v_2 + G(v_1)\big) - DG(v_1)\big(A_p v_1 + X_{12}\big(v_1, v_2 + G(v_1)\big)\big).$$

Relations (51) and (52) imply that

$$\big|X_{34}\big(v_1, v_2 + G(v_1)\big) - X_{34}\big(v_1, G(v_1)\big)\big| \leqslant g|v_2|$$

and

$$\big|DG(v_1)\big(A_p v_1 + X_{12}\big(v_1, v_2 + G(v_1)\big)\big) - DG(v_1)\big(A_p v_1 + X_{12}\big(v_1, G(v_1)\big)\big)\big| \leqslant g|v_2|.$$

Hence,

$$\begin{aligned}&\big|X_{34}\big(v_1, v_2 + G(v_1)\big) - X_{34}\big(v_1, G(v_1)\big)\\&\quad - \big(DG(v_1)\big(A_p v_1 + X_{12}\big(v_1, v_2 + G(v_1)\big)\big) - DG(v_1)\big(A_p v_1 + X_{12}\big(v_1, G(v_1)\big)\big)\big)\big| \leqslant 2g|v_2|.\end{aligned}$$

The left-hand side of the above inequality equals $|Y_2(v_1, v_2) - (Y_2(v_1, 0) + B_p v_2)|$, which proves inequality (54). □

Note that if y_p, y_q, z_p, z_q, and $m_1 > 0$ are fixed, then there exists $m^* > 0$ such that if a trajectory β^* of the vector field Y intersects successfully the balls $B(m^*, v(z_q))$, $B(m^*, v(y_q))$, $B(m^*, v(y_p))$, and $B(m^*, v(z_p))$, then the trajectory $w^{-1}(\beta^*)$ of X has the property described in Lemma 6.

Thus, it is enough to prove Lemma 6 for the vector field Y. Since the mapping w is smooth, the vector field Y satisfies condition (36).

To simplify presentation, denote Y by X and its flow by ϕ. In this notation, there exists a neighborhood U_p of $p = 0$ in which

$$X(x) = \begin{pmatrix} A_p & 0 \\ 0 & B_p \end{pmatrix} x + X_p(x), \tag{55}$$

where $X_p \in \mathbf{C}^0$, and if $(x_1, x_2, x_3, x_4) \in U_p$, then

$$\big|P^p_{34} X_p(x_1, x_2, x_3, x_4)\big| < 2g \max\big(|x_3|, |x_4|\big) \quad \text{and} \quad P^p_{34} X_p(x_1, x_2, 0, 0) = 0 \tag{56}$$

(where we denote by P^p_{34} the projection in U_p to the plane of variables x_3, x_4 parallel to the plane of variables x_1, x_2). Conditions (56) imply that the plane $x_3 = x_4 = 0$ is a local stable manifold for the vector field X.

Introduce polar coordinates r, φ in the plane of variables x_3, x_4. In what follows (if otherwise is not stated explicitly), we use coordinates (x_1, x_2, r, φ). For $i \in \{1, 2, 3, 4, r, \varphi\}$, we denote by $P^p_i x$ the ith coordinate of a point $x \in U_p$.

Since the surface $W^u(p)$ is smooth and transverse to the plane $x_3 = x_4 = 0$, there exist numbers $K > 0$ and $m_2 > 0$ such that if points $x \in W^u_{loc}(p, m_2)$ and $y \in B(m_2, p)$ satisfy the equality $P^p_{34} x = P^p_{34} y$, then

$$\operatorname{dist}(x, y) \leqslant K \operatorname{dist}\big(y, W^u_{loc}(p, m_2)\big). \tag{57}$$

We reduce the neighborhood U_p so that $U_p \subset B(m_2, p)$.

Lemma 9. *Let $x(t) = (x_1(t), x_2(t), r(t), \varphi(t))$ be a trajectory of the vector field X. The relations*

$$\frac{\mathrm{d}}{\mathrm{d}t} r \in \big((a_p - 4g)r, (a_p + 4g)r\big) \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi \in (b_p - 4g, b_p + 4g) \tag{58}$$

hold while $x(t) \in U_p$.

Proof. Let $x_3(t) = P_3^p x(t)$ and $x_4(t) = P_4^p x(t)$. Relations (48), (55) and (56) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t} x_3(t) = a_p x_3(t) - b_p x_4(t) + \Delta_3(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} x_4(t) = b_p x_3(t) + a_p x_4(t) + \Delta_4(t),$$

where

$$\big|\Delta_3(t)\big|, \big|\Delta_4(t)\big| < 2gr(t). \tag{59}$$

Since $x_3(t) = r(t)\cos\varphi(t)$ and $x_4(t) = r(t)\sin\varphi(t)$, we obtain the equalities

$$r\frac{\mathrm{d}}{\mathrm{d}t}\varphi = rb_p + \Delta_4(t)\cos\varphi - \Delta_3(t)\sin\varphi$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} r = a_p r + \Delta_3(t)\cos\varphi + \Delta_4(t)\sin\varphi.$$

Inequalities (59) imply that

$$b_p - 4g < \frac{\mathrm{d}}{\mathrm{d}t}\varphi < b_p + 4g$$

and

$$(a_p - 4g)r < \frac{\mathrm{d}}{\mathrm{d}t} r < (a_p + 4g)r,$$

which proves our lemma. □

A similar reasoning shows that there exists a neighborhood U_q of the point q in which we can introduce (after a smooth change of variables) coordinates (y_1, y_2, y_3, y_4) (and the corresponding polar coordinates (r, φ) in the plane of variables y_3, y_4) such that

$$W^u_{loc}(q, m) \subset \{y_3 = y_4 = 0\}$$

and for any trajectory $y(t) = (y_1(t), y_2(t), r(t), \varphi(t))$ of the vector field X, the relations

$$\frac{\mathrm{d}}{\mathrm{d}t} r \in \big((a_q - 4g)r, (a_q + 4g)r\big) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\varphi \in (-b_q - 4g, -b_q + 4g)$$

hold while $y(t) \in U_q$.

Let us continue the proof of Lemma 6.

Let $S_p \subset U_p$ and $S_q \subset U_q$ be smooth three-dimensional disks that are transverse to the vector field X and contain the points y_p and y_q, respectively. Denote by $f : S_q \to S_p$ the corresponding Poincaré transformation (generated by the field X). We note that $f \in \mathbf{C}^1$ (see Remark 3) and $f(y_q) = y_p$.

Consider the lines $l_p = S_p \cap W^s_{loc}(p, m)$ and $l_q = S_q \cap W^u_{loc}(q, m)$ and unit vectors $e_p \in l_p$ and $e_q \in l_q$. Let P^p_{34} and P^q_{34} be the projections to the planes of variables x_3, x_4 and y_3, y_4 in the neighborhoods U_p and U_q, respectively. Relation (36) implies that

$$P^p_{34} Df(y_q)e_q \neq 0 \quad \text{and} \quad P^q_{34} Df^{-1}(y_p)e_p \neq 0. \tag{60}$$

Take $m_3 \in (0, m_1)$ such that

$$\phi(t, x) \in U_p, \quad x \in B(m_3, y_p),\ t \in \big(0, \tau_p(x)\big),$$

and

$$\phi(t, y) \in U_q, \quad y \in B(m_3, y_q),\ t \in \big(\tau_q(x), 0\big),$$

where

$$\tau_p(x) = \inf\big\{t > 0:\ P^p_r\big(\phi(t, x)\big) \geqslant P^p_r z_p\big\},$$

$$\tau_q(x) = \sup\big\{t < 0:\ P^q_r\big(\phi(t, y)\big) \geqslant P^q_r z_q\big\},$$

and z_p, z_q are the points mentioned in Lemma 6.

Consider the surface $L_p \subset S_p$ defined by

$$L_p = \big\{x + (y - y_p),\ x \in l_p,\ y \in f(l_q)\big\}.$$

Let $L_q = f^{-1}L_p \subset S_q$. The surfaces L_p and L_q are divided by the lines l_p and l_q into half-surfaces. Let L^+_p and L^+_q be any of these half-surfaces.

To any point $x \in L^+_p \cap f(L^+_q)$ there correspond numbers $r_p(x) = P^p_r x$ and $r_q(x) = P^q_r f^{-1}(x)$; consider the mapping $w : L^+_p \cap f(L^+_q) \to \mathbb{R}^2$ defined by $w(x) = (r_p(x), r_q(x))$. We claim that there exists a neighborhood $U_L \subset L^+_p \cap f(L^+_q)$ of the point y_p on which the mapping w is a homeomorphism onto its image.

Let r_0 and φ_0 be the polar coordinates of the vector $P^p_{34} Df(y_q)e_q$. Relation (60) implies that $r_0 \neq 0$. Hence, there exists a neighborhood V_q of the point y_q in S_q such that if $y \in V_q$, then

$$P^p_r Df(y)e_q \in [r_0/2, 2r_0] \quad \text{and} \quad P^p_\varphi Df(y)e_q \in [\varphi_0 - \pi/8, \varphi_0 + \pi/8]. \tag{61}$$

Take $c > 0$ such that $B(2c, y_q) \subset V_q$. Note that

$$f(y_q + \delta e_q) = f(y_q) + \int_0^\delta Df(y_q + s e_q)e_q\, ds, \quad \delta \in [0, c].$$

Conditions (61) imply that

$$P^p_\varphi\big(f(y_q + \delta e_q) - f(y_q)\big) \in \left[\varphi_0 - \frac{\pi}{8}, \varphi_0 + \frac{\pi}{8}\right], \quad \delta \in [0, c], \tag{62}$$

and the mapping $Q_p(\delta):[0,c] \to \mathbb{R}$ defined by $Q_p(\delta) = P_r^p f(y_q + \delta e_q)$ is a homeomorphism onto its image. Similarly (reducing g, if necessary), one can show that if $x \in B(g, y_p)$, then the mapping $Q_{q,x}(\delta):[0,g] \to \mathbb{R}$ defined by $Q_{q,x}(\delta) = P_r^q f^{-1}(x + \delta e_p)$ is a homeomorphism onto its image.

Take $\delta_p, \delta_q \in [0,c]$ and let $x = \delta_p e_p + f(y_q + \delta_q e_q)$. Then $r_p(x) = Q_p(\delta_q)$ and $r_q(x) = Q_{q,f(y_q+\delta_q e_q)}(\delta_p)$. It follows that the mapping w is a homeomorphism onto its image. Indeed, if $g_1 > 0$ is small enough, then the mapping $w^{-1}(\xi,\eta) = (x(\xi), Q^{-1}_{q,x(\xi)}(\eta))$, where $x(\xi) = f(y_q + Q_p^{-1}(\xi)e_q)$, is uniquely defined and continuous for $(\xi,\eta) \in [0,g_1] \times [0,g_1]$.

We reduce m_3 so that the following relations hold:

$$m_3 < c, \quad B(m_3, y_p) \cap L_p^+ \subset U_L, \quad \text{and} \quad B(m_3, y_q) \cap L_q^+ \subset f^{-1}U_L.$$

Let us prove a statement which we use below.

Lemma 10. *For any $m_1 > 0$ there exist numbers $r_1, r_2 \in (0, m_1)$ and $T_1, T_2 > 0$ with the following property: if $\gamma(s):[0,1] \to L_p^+$ is a curve such that*

$$P_r^p \gamma(0) = r_1, \qquad P_r^p \gamma(1) = r_2, \tag{63}$$

and

$$\gamma(s) \in L_p^+ \cap B(m_2, y_p), \quad s \in [0,1], \tag{64}$$

then there exist numbers $\tau \in [T_2, T_1]$ and $s \in [0,1]$ such that

$$\phi\big(\tau, \gamma(s)\big) \in B(m_1, z_p).$$

Proof. Let $r_p = P_r^p z_p$ and $\varphi_p = P_\varphi^p z_p$. For $r > 0$, denote

$$T_{\min}(r) = \frac{\log r_p - \log r}{a_p + 4g} \quad \text{and} \quad T_{\max}(r) = \frac{\log r_p - \log r}{a_p - 4g}.$$

Note that if $r < r_p$, then $T_{\max}(r) > T_{\min}(r)$ and that $T_{\min}(r) \to \infty$ as $r \to 0$. Take $T > 0$ such that if $\tau > T$, $x \in B(m_2, y_p)$, and

$$\phi(t,x) \subset U_p, \quad t \in [0,\tau],$$

then

$$\operatorname{dist}\big(W^u_{loc}(p,m), \phi(\tau,x)\big) < \frac{m_1}{2K}. \tag{65}$$

Take $r_1, r_2 \in (0, \min(m_2, r_p))$ such that

$$r_2 > r_1, \qquad T_{\min}(r_2) > T,$$

and

$$(b_p - 4g)T_{\min}(r_1) - (b_p + 4g)T_{\max}(r_2) > 4\pi. \tag{66}$$

Set $T_1 = T_{\max}(r_1)$ and $T_2 = T_{\min}(r_2)$. Since the function $\gamma(s)$ is continuous, inclusions (58) and inequalities (49) imply that there exists a uniquely defined continuous function $\tau(s): [0,1] \to \mathbb{R}$ such that

$$P_r^p \phi\big(\tau(s), \gamma(s)\big) = r_p.$$

It follows from inclusions (58) and equalities (63) that

$$\tau(0) \in \big[T_{\min}(r_1), T_{\max}(r_1)\big], \qquad \tau(1) \in \big[T_{\min}(r_2), T_{\max}(r_2)\big], \qquad \tau(s) \in [T_2, T_1].$$

Now we apply relations (49), (58), and (62) to show that

$$P_\varphi^p \phi\big(\tau(0), \gamma(0)\big) \geqslant (b_p - 4g) T_{\min}(r_1) + \varphi_0 - \pi/8$$

and

$$P_\varphi^p \phi\big(\tau(1), \gamma(1)\big) \leqslant (b_p + 4g) T_{\max}(r_2) + \varphi_0 + \pi/8.$$

Since the function $\tau(s)$ is continuous, the above inequalities and inequalities (66) imply the existence of $s \in [0,1]$ such that

$$P_\varphi^p \phi\big(\tau(s), \gamma(s)\big) = \varphi_p \mod 2\pi.$$

Hence, $P_{34}^p \phi(\tau(s), \gamma(s)) = P_{34}^p z_p$. It follows from this equality combined with relations (57), (65), and the inequality $\tau(s) > T$ that $\phi(\tau(s), \gamma(s)) \in B(m_1/2, z_p)$, which proves Lemma 10. $\square$

Let $r_1, r_2 \in (0, m_2)$ and $T_1, T_2 > 0$ be the numbers given by Lemma 10. Consider the set

$$A_p = \big\{\phi(t, x):\ t \in [-T_1, -T_2],\ x \in \operatorname{Cl} B(m_2/2, z_p)\big\} \cap L_p^+.$$

Note that A_p is a closed set that intersects any curve $\gamma(s)$ satisfying conditions (63) and (64).

We apply a similar reasoning in the neighborhood U_q to the vector field $-X$ to show that there exist numbers $r_1', r_2' \in (0, m_2)$ and $T_1', T_2' > 0$ such that the set

$$A_q = \big\{\phi(t, x):\ t \in \big[T_2', T_1'\big],\ x \in \operatorname{Cl} B(m_2/2, z_q)\big\} \cap L_q^+$$

is closed and intersects any curve $\gamma(s): [0,1] \to L_q^+ \cap B(m_2, y_q)$ such that

$$P_r^q \gamma(0) = r_1' \quad \text{and} \quad P_r^q \gamma(1) = r_2'.$$

We claim that

$$A_p \cap f(A_q) \neq \emptyset, \tag{67}$$

which proves Lemma 6.

Consider the set $K \subset L_p^+ \cap f(L_q^+)$ bounded by the curves $k_1 = L_p^+ \cap \{P_r^p x = r_1\}$, $k_2 = L_p^+ \cap \{P_r^p x = r_2\}$, $k_1' = f(L_q^+ \cap \{P_r^q y = r_1'\})$, and $k_2' = f(L_q^+ \cap \{P_r^q y = r_2'\})$. Since $w(x)$ is a homeomorphism, the set K is homeomorphic to the square $[0,1] \times [0,1]$.

The following statement was proved in [18].

Lemma 11. *Introduce in the square $I = [0, 1] \times [0, 1]$ coordinates (u, v). Assume that closed sets $A, B \subset I$ are such that any curve inside I that joins the segments $u = 0$ and $u = 1$ intersects the set A and any curve inside I that joins the segments $v = 0$ and $v = 1$ intersects the set B. Then $A \cap B \neq \emptyset$.*

The set A_p is closed. By Lemma 10, A_p intersects any curve in K that joins the sides k_1 and k_2. Similarly, the set A_q is closed and intersects any curve that belongs to $f^{-1}(K)$ and joins the sides $f^{-1}(k_1')$ and $f^{-1}(k_2')$. Thus, the set $f(A_q)$ intersects any curve in K that joins the sides k_1' and k_2'. By Lemma 11 inequality (67) holds. Lemma 6 is proved.

Proof of Lemma 7. Similarly to the proof of Lemma 6, let us consider the subspaces L_p^+ and L_q^+ and a number $m_2 \in (0, m_1)$ and construct the set $A_q \subset L_q^+$. Note that the set $f^{-1}(B(m_1, y_p) \cap W^s(p) \cap L_p^+)$ contains a curve that satisfies conditions (63) and (64). Hence, $B(m_1, y_p) \cap W^s(p) \cap f(A_q) \neq \emptyset$. For any point in this intersection, its trajectory is the desired shadowing trajectory. □

Acknowledgments

The authors are deeply grateful to the anonymous referee whose remarks helped us to significantly improve the presentation.

Appendix A. Construction of the vector field X^*

Consider two 2-dimensional spheres M_1 and M_2. Let us introduce coordinates (r_1, φ_1) and (r_2, φ_2) on M_1 and M_2, respectively, where $r_1, r_2 \in [-1, 1]$ and $\varphi_1, \varphi_2 \in \mathbb{R}/2\pi\mathbb{Z}$. We identify all points of the form $(-1, \cdot)$ as well as points of the form $(1, \cdot)$. Denote

$$M_1^+ = \{(r_1, \varphi_1),\ r_1 \geqslant 0\} \quad \text{and} \quad M_1^- = \{(r_1, \varphi_1),\ r_1 \leqslant 0\}.$$

Consider a smooth vector field X_1 defined on M_1^+ such that its trajectories $(r_1(t), \varphi_1(t))$ satisfy the following conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t} r_1 = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_1 = 0, \quad r_1 = 0;$$

$$\frac{\mathrm{d}}{\mathrm{d}t} r_1 > 0, \quad r_1 > 0;$$

$$\frac{\mathrm{d}}{\mathrm{d}t} r_1 = 0, \quad r_1 = 1.$$

We also assume that, in proper local coordinates in a neighborhood of the "North Pole" $(1, \cdot)$ of the sphere M_1, the vector field X_1 is linear, and

$$\mathrm{D}X_1(1, \cdot) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $(1, \cdot)$ is an attracting hyperbolic rest point of X_1, and every trajectory of X_1 in M_1^+ tends to $(1, \cdot)$ as time grows.

Consider a smooth vector field X_2 on M_2 such that its nonwandering set $\Omega(X_2)$ consists of two rest points: a hyperbolic attractor $s_2 = (0, \pi)$ and a hyperbolic repeller $u_2 = (0, 0)$. Assume that, in proper coordinates, the vector field X_2 is linear in neighborhoods of s_2 and u_2, and

$$\mathrm{D}X_2(s_2) = -\mathrm{D}X_2(u_2) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Consider the vector field X^+ defined on $M_1^+ \times M_2$ by the following formula

$$X^+(r_1, \varphi_1, r_2, \varphi_2) = \big(X_1(r_1, \varphi_1), r_1^2 X_2(r_2, \varphi_2)\big).$$

Consider infinitely differentiable functions $g_1 : M_1^+ \to \mathbb{R}$, $g_2, g_3 : [-1, 1] \to [-1, 1]$, and $g_4 : M_1^+ \to [0, 1]$ satisfying the following conditions:

$$g_1(0,0) = 0; \qquad g_1(r_1, \varphi_1) \in (0, 2\pi), \qquad (r_1, \varphi_1) \neq 0,$$

$$g_2'(r_2) \in (0, 2), \quad r_2 \in [-1, 1];$$

$$g_2(0) < 0, \qquad g_2(-1) = -1, \qquad g_2(1) = 1;$$

$$g_3(r_2) = 2r_2 - g_2(r_2), \quad r_2 \in [-1, 1];$$

$$g_4(0,0) = 1/2, \qquad \frac{\partial}{\partial \varphi_1} g_4(0,0) \neq 0.$$

Note that the functions g_2 and g_3 are monotonically increasing.

Consider a mapping $f^* : M_1^+ \times M_2 \to M_1^- \times M_2$ defined by the following formula:

$$f^*(r_1, \varphi_1, r_2, \varphi_2) = \big(-r_1, \varphi_1, g_4(r_1, \varphi_1) g_2(r_2) + \big(1 - g_4(r_1, \varphi_1)\big) g_3(r_2), \varphi_2 + g_1(r_1, \varphi_1)\big).$$

Clearly, f^* is surjective; the monotonicity of g_2 and g_3 implies that f^* is a diffeomorphism.

Using the standard technique with a "bump" function, one can construct a diffeomorphism $f : M_1^+ \times M_2 \to M_1^- \times M_2$ such that, for small neighborhoods $U_1 \subset U_2$ of $(1, \cdot, s_2)$, the following holds:

$$f(x) = f^*(x), \quad x \notin U_2,$$

and f is linear in U_1.

Consider the set $l = \{r_1 = 0,\ r_2 = 0,\ \varphi_2 = 0\}$. Simple calculations show that

$$f(l) \cap l = \big\{(0,0,0,0)\big\}, \tag{68}$$

and the tangent vectors to l and $f(l)$ at $(0,0,0,0)$ are parallel to the vectors $(0,1,0,0)$ and $(0, 1, (g_2(0) - g_3(0)) \frac{\partial}{\partial \varphi_1} g_4(0,0), \cdot)$, respectively. Hence,

$$\dim\big(T_{(0,0,0,0)} l \oplus T_{(0,0,0,0)} f(l)\big) = 2. \tag{69}$$

Define a vector field X^- on $M_1^- \times M_2$ by the formula

$$X^-(x) = -\mathrm{D}f\big(f^{-1}(x)\big) X^+\big(f^{-1}(x)\big)$$

(and note that $x(t)$ is a trajectory of X^+ if and only if $f(x(-t))$ is a trajectory of X^-).

Finally, we define the following vector field X^* on $M_1 \times M_2$:

$$X^*(x) = \begin{cases} X^+(x), & x \in M_1^+ \times M_2, \\ X^-(x), & x \in M_1^- \times M_2. \end{cases}$$

Let us check that the vector field X^* is well-defined on the set $\{r_1 = 0\}$. Indeed, $X^+(0, \varphi_1, r_2, \varphi_2) = (1,0,0,0)$ and $(\mathrm{D}f(0, \varphi_1, r_2, \varphi_2))^{-1}(1,0,0,0) = (-1,0,0,0)$. It is easy to see that $\mathrm{D}X^+(0, \varphi_1, r_2, \varphi_2) = \mathrm{D}X^-(0, \varphi_1, r_2, \varphi_2) = 0$. This implies that $X \in \mathbf{C}^1$.

Let us prove that the vector field X^* satisfies conditions (F1)–(F3). Let $(r_1(t), \varphi_1(t), r_2(t), \varphi_2(t))$ be a trajectory of X^*. The following inequalities hold:

$$\frac{\mathrm{d}}{\mathrm{d}t} r_1 > 0, \qquad r_1 \neq \pm 1. \tag{70}$$

This implies the inclusion $\Omega(X^*) \subset \{r_1 = \pm 1\}$. By the construction of X^+, $\Omega(X^*) \cap \{r_1 = 1\} = \{(1, \cdot, s_2), (1, \cdot, u_2)\}$. Similarly, $\Omega(X^*) \cap \{r_1 = -1\} = \{f(1, \cdot, s_2), f(1, \cdot, u_2)\}$. Denote $s^* = (1, \cdot, s_2)$, $p^* = (1, \cdot, u_2)$, $q^* = f(p)$, and $u^* = f(s)$. Clearly, s^*, u^*, p^*, q^* are hyperbolic rest points, s^* is an attractor, u^* is a repeller, $\mathrm{D}X(p^*) = J_p^*$, and $\mathrm{D}X(q^*) = J_q^*$. In addition, in small neighborhoods of p^* and q^*, the vector field X^* is linear.

It is easy to see that

$$W^s(p^*) \cap \{r_1 = 1\} = \{p^*\} \quad \text{and} \quad W^s(p^*) \cap \{r_1 = -1\} = \emptyset.$$

Inequality (70) implies that any trajectory in $W^s(p^*) \setminus \{p^*\}$ intersects the set $\{r_1 = 0\}$ at a single point. The definition of X^+ implies that $W^s(p^*) \cap \{r_1 = 0\} = l$. Similarly, any trajectory in $W^u(q^*) \setminus \{q^*\}$ intersects $\{r_1 = 0\}$ at a single point, and $W^u(q^*) \cap \{r_1 = 0\} = f(l)$. It follows from equality (68) that $W^s(p^*) \cap \{r_1 = 0\} \cap W^u(q^*)$ is a single point, and hence $W^s(p^*) \cap W^u(q^*)$ consists of a single trajectory.

Inequality (70) implies condition (32), and condition (69) implies (33).

References

[1] S.Yu. Pilyugin, Shadowing in Dynamical Systems, Lecture Notes in Math., vol. 1706, Springer-Verlag, 1999.
[2] K. Palmer, Shadowing in Dynamical Systems: Theory and Applications, Kluwer, 2000.
[3] K. Sakai, Pseudo orbit tracing property and strong transversality of diffeomorphisms on closed manifolds, Osaka J. Math. 31 (1994) 373–386.
[4] S.Yu. Pilyugin, A.A. Rodionova, K. Sakai, Orbital and weak shadowing properties, Discrete Contin. Dyn. Syst. 9 (2003) 287–308.
[5] M. Komuro, One-parameter flows with the pseudo orbit tracing property, Monatsh. Math. 98 (1984) 219–253.
[6] R.F. Thomas, Stability properties of one-parameter flows, Proc. London Math. Soc. 54 (1982) 479–505.
[7] K. Lee, K. Sakai, Structural stability of vector fields with shadowing, J. Differential Equations 232 (2007) 303–313.
[8] S.Yu. Pilyugin, S.B. Tikhomirov, Sets of vector fields with various shadowing properties of pseudotrajectories, Dokl. Math. 422 (2008) 30–31.
[9] S.B. Tikhomirov, Interiors of sets of vector fields with shadowing properties that correspond to some classes of reparametrizations, Vestnik St. Petersburg Univ. Math. 1 (2008) 90–97.
[10] S. Gan, Another proof for the C^1 stability conjecture for flows, Sci. China Ser. A 41 (1998) 1076–1082.
[11] C. Pugh, C. Robinson, The C^1-closing lemma, including Hamiltonians, Ergodic Theory Dynam. Systems 3 (1983) 261–313.
[12] V.I. Arnold, Ordinary Differential Equations, Universitext, Springer-Verlag, Berlin, 2006.
[13] S.Yu. Pilyugin, Shadowing in structurally stable flows, J. Differential Equations 140 (1997) 238–265.
[14] S. Gan, L. Wen, Nonsingular star flows satisfy Axiom A and the no-cycle condition, Invent. Math. 164 (2006) 279–315.
[15] R. Mane, A proof of the C^1 stability conjecture, Publ. Math. Inst. Hautes Etudes Sci. 66 (1987) 161–210.
[16] C. Pugh, M. Shub, The Ω-stability theorem for flows, Invent. Math. 11 (1971) 150–158.
[17] K. Moriyasu, K. Sakai, N. Sumi, Vector fields with topological stability, Trans. Amer. Math. Soc. 353 (2001) 3391–3408.
[18] S.Yu. Pilyugin, K. Sakai, C^0-transversality and shadowing properties, Proc. Steklov Inst. Math. 256 (2007) 290–305.